Finite-order mapping classes of del Pezzo surfaces

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Finite group actions on M

$$q : \mathrm{Homeo}^+(M) \to \mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}^+(M))$$

Question ((Cyclic) Nielsen realization problem)

For $g \in \mathrm{Mod}(M)$ of order $n < \infty$, does there exist $f \in \mathrm{Homeo}^+(M)$ of order n such that $[f] = g$?

Answer ($\dim_{\mathbb{R}} 2$)

Yes! (Nielsen '43) Also for $\dim_{\mathbb{C}} 1$.

What about for higher dimensional complex manifolds?

Example: order 2 diffeomorphisms on $\mathbb{CP}^2$

$$M = \mathbb{CP}^2.$$

Example

$$\sigma : [X : Y : Z] \mapsto [-X : Y : Z]$$

and $\sigma \in \mathrm{PGL}_3(\mathbb{C})$ which is connected so $[\sigma] = \mathrm{Id}$.

Example

$$\tau : [X : Y : Z] \mapsto [\overline{X} : \overline{Y} : \overline{Z}]$$
$$\tau_* = -\mathrm{Id} : H_2(M) \to H_2(M)$$

so $[\tau] \neq \mathrm{Id}$.

del Pezzo surfaces

Definition

A del Pezzo surface M is a complex surface

$$M = \mathbb{CP}^1 \times \mathbb{CP}^1 \qquad \textit{or} \qquad M = \mathrm{Bl}_P\,\mathbb{CP}^2$$

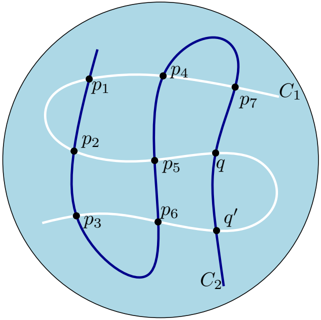
where $P \subseteq \mathbb{CP}^2$ is a finite set of points in general position with $0 \leq |P| \leq 8$.

Lemma

As smooth manifolds, $\mathrm{Bl}_P\,\mathbb{CP}^2$ is diffeomorphic to $M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ with $n = |P|$.

A classical example – Geiser involution

$\gamma \in \mathrm{Aut}(\mathrm{Bl}_P\,\mathbb{CP}^2)$ $\qquad P = \{7 \text{ points in general position}\}$



Let $q \in \mathbb{CP}^2 - P$.

Set of cubic curves through $P \cup \{q\}$ $= \mathbb{CP}^1$

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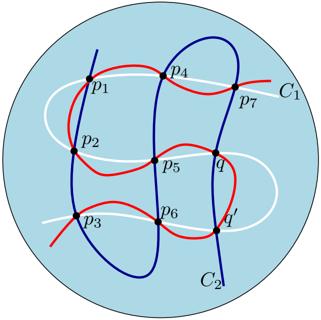

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$= \mathbb{CP}^1$

Cayley–Bacharach
$\implies$ ninth base-point q' of pencil

$\gamma(q) = q'$.

γ extends to an order two diffeomorphism of $\mathrm{Bl}_P\,\mathbb{CP}^2 \cong \mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$.

Mapping class groups of del Pezzo surfaces

Corollary (Special case of Freedman '82, Quinn '86)

$$\mathrm{Mod}(M_n) \to \mathrm{Aut}(H_2(M_n), Q_{M_n}) \cong \mathrm{O}(1,n)(\mathbb{Z})$$
$$[f] \mapsto f_* : H_2(M_n) \to H_2(M_n)$$

is an isomorphism of groups.

$$\mathrm{O}^+(1,n)(\mathbb{Z}) \le \mathrm{O}^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^n)$$

Vinberg: Hyperbolic reflection groups! $(2 \le n \le 9)$

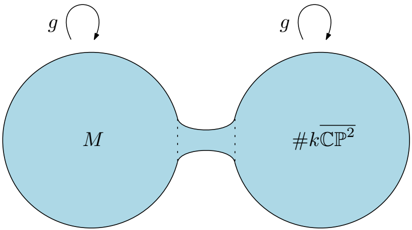
Two types of elements in $\mathrm{Mod}(M_n)$

1. (reducible) g preserves some

$$M_n \cong M \# k\overline{\mathbb{CP}^2}$$

on the level of homology

$$H_2(M_n) \cong H_2(M) \oplus H_2\left(\# k\overline{\mathbb{CP}^2}\right).$$



2. (irreducible) g does not preserve any such direct sum decomposition of $H_2(M_n)$.

A structure theorem for all involutions

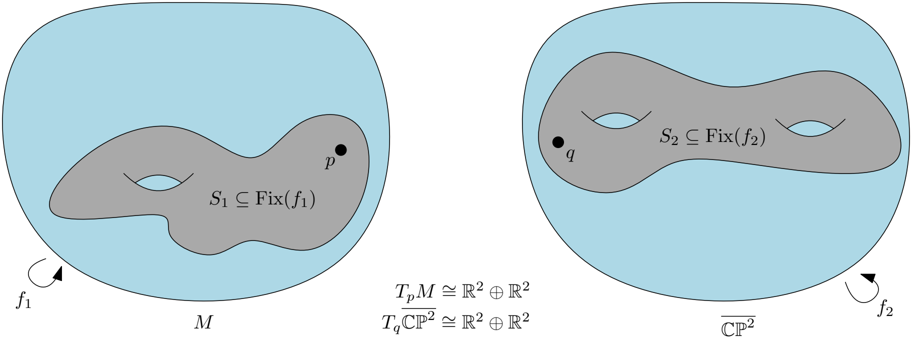
Birational geometry: three classical involutions of $\mathrm{Bl}_P\,\mathbb{CP}^2$:

1. Geiser involutions,
2. Bertini involutions, and
3. de Jonquiéres involutions.

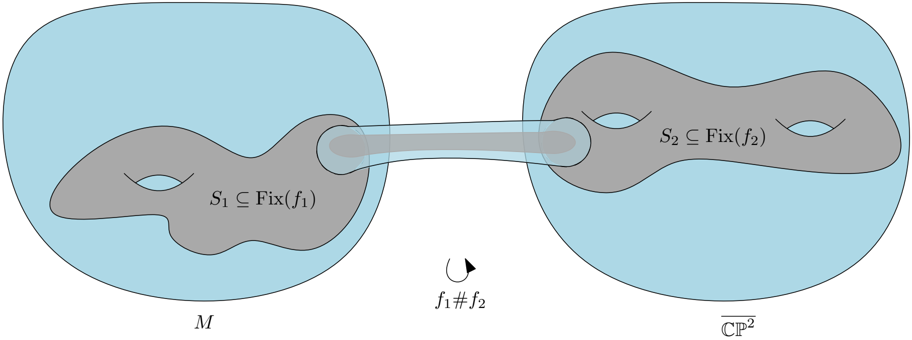
Theorem (L. '22)

A mapping class $g \in \mathrm{Mod}^+(M_n)$ of order 2 is irreducible if and only if g is realized by a de Jonquiéres, Geiser, or Bertini involution on $\mathrm{Bl}_P\,\mathbb{CP}^2$ for some $P \subseteq \mathbb{CP}^2$.

Making new actions out of old



Making new actions out of old



"Equivariant connected sum"

Nielsen realization for involutions

Corollary

Any element $g \in \mathrm{Mod}(M_n)$ *of order* 2 *is realized by a diffeomorphism* $f \in \mathrm{Diff}^+(M_n)$ *of order* 2.

Nielsen realization for M_2

Theorem (L. '21)

A finite subgroup $G \leq \mathrm{Mod}(M_2)$ *has a lift to* $\mathrm{Diff}^+(M_2) \leq \mathrm{Homeo}^+(M_2)$ *under* $q : \mathrm{Homeo}^+(M_2) \to \mathrm{Mod}(M_2)$ *if and only if* G *is realizable by an equivariant connected sum.*

Corollary

1. *If* $g \in \mathrm{Mod}(M_2)$ *has finite order* n *then there exists* $f \in \mathrm{Diff}^+(M_2)$ *with order* n *such that* $[f] = g \in \mathrm{Mod}(M_2)$.
2. *Some finite subgroups* $G \leq \mathrm{Mod}(M_2)$ *have no lift to* $\mathrm{Diff}^+(M_2)$.

Thank you!