Isotopy classes of involutions of del Pezzo surfaces

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Abstract

Let $M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ for $0 \le n \le 8$ be the underlying smooth manifold of a degree $9-n$ del Pezzo surface. We prove three results about the mapping class group $\mathrm{Mod}(M_n) := \pi_0(\mathrm{Homeo}^+(M_n))$:

1. the classification of, and a structure theorem for, all involutions in $\mathrm{Mod}(M_n)$,
2. a positive solution to the smooth Nielsen realization problem for involutions of M_n, and
3. a purely topological characterization of three remarkable types of involutions on certain M_n coming from birational geometry: de Jonquiéres involutions, Geiser involutions, and Bertini involutions.

One main ingredient is the theory of hyperbolic reflection groups.

1 Introduction

A *del Pezzo surface* is a smooth projective algebraic surface with ample anticanonical divisor class. Any del Pezzo surface is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, or $\mathrm{Bl}_P\,\mathbb{CP}^2$ where P is a set of n points (with $1 \le n \le 8$) in general position (no three collinear points, no six coconic points, and no eight points on a cubic which is singular at any of the eight points); see [Dol12, Proposition 8.1.25]. The degree of the del Pezzo surface $\mathrm{Bl}_P\,\mathbb{CP}^2$ is $9 - |P|$, the degree of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is 8, and the degree of $\mathbb{CP}^2$ is 9.

The smooth 4-manifolds underlying del Pezzo surfaces are well-understood; we call such manifolds *del Pezzo manifolds*. The blowup of $\mathbb{CP}^2$ at a finite set P of n points is diffeomorphic to the smooth 4-manifold

$$M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}.$$

In particular, the smooth 4-manifold underlying a del Pezzo surface of degree $1 \le d \le 9$ is M_{9-d} if $d \neq 8$ and M_1 or $M_* := \mathbb{CP}^1 \times \mathbb{CP}^1$ if $d = 8$. Therefore, the manifolds M_n for $0 \le n \le 8$ and M_* make up the list of all del Pezzo manifolds.

In this paper we relate a property (which we call *irreducibility*) of elements of the mapping class group $\mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}^+(M))$ for all del Pezzo manifolds M to the classification of conjugacy classes of order 2 elements of the group of birational automorphisms of $\mathbb{CP}^2$. In doing so, we realize all order 2 mapping classes of del Pezzo manifolds by order 2 diffeomorphisms coming from a construction that we call *complex equivariant connected sums*. This yields an affirmative solution to the smooth Nielsen realization problem for involutions of del Pezzo manifolds, which is different from the solution for some other 4-manifolds; for example, Farb–Looijenga ([FL21]) study the Nielsen realization problem for K3 surfaces and show that not all order 2 mapping classes of K3 surfaces can be smoothly realized by involutions (or even by diffeomorphisms of finite order). See Remark 1.7 below.

Irreducibility of mapping classes. Let M be a del Pezzo manifold and let Q_M be the intersection form for M. If there exist (A_1, Q_1) and (A_2, Q_2) where A_i is a free $\mathbb{Z}$-module and Q_i is a symmetric bilinear form on A_i with an isometry

$$\iota : (A_1 \oplus A_2, Q_1 \oplus Q_2) \to (H_2(M;\mathbb{Z}), Q_M)$$

then there exists a natural induced inclusion

$$\mathrm{Aut}(A_1, Q_1) \times \mathrm{Aut}(A_2, Q_2) \hookrightarrow \mathrm{Aut}(H_2(M;\mathbb{Z}), Q_M).$$

By theorems of Freedman ([Fre82]) and Quinn ([Qui86]), there is an isomorphism $\Phi : \mathrm{Mod}(N) \to \mathrm{Aut}(H_2(N;\mathbb{Z}), Q_N)$ given by $\Phi([f]) = f_*$ for any closed, oriented, and simply connected 4-manifold N. Hence if (A_i, Q_i) for $i = 1, 2$ is of the form $(H_2(N_i, \mathbb{Z}), Q_{N_i})$ for such 4-manifolds N_i, there also exists a natural induced inclusion

$$\iota_* : \mathrm{Mod}(N_1) \times \mathrm{Mod}(N_2) \hookrightarrow \mathrm{Mod}(M).$$

Definition 1.1 (Irreducibility). Let M be a del Pezzo manifold and let $g \in \mathrm{Mod}(M)$. Suppose there exist a del Pezzo manifold N and some $k \in \mathbb{Z}_{>0}$ such that there is an isometry

$$\iota : (H_2(N;\mathbb{Z}) \oplus H_2(\#k\overline{\mathbb{CP}^2};\mathbb{Z}), Q_N \oplus Q_{\#k\overline{\mathbb{CP}^2}}) \to (H_2(M;\mathbb{Z}), Q_M)$$

and g is contained in the image of ι_*. Then g is called *reducible*. Otherwise, g is called *irreducible*.

Equivalently, g is reducible if there is some isometry ι as given above such that under this isometry, g preserves $H_2(N;\mathbb{Z})$ and $H_2(\#k\overline{\mathbb{CP}^2};\mathbb{Z})$ when considered as an automorphism of $H_2(M;\mathbb{Z})$. The restriction of g to $H_2(\#k\overline{\mathbb{CP}^2};\mathbb{Z})$ acts by an element of the finite group $\mathrm{O}(k)(\mathbb{Z}) := \mathrm{O}(k) \cap \mathrm{GL}(k, \mathbb{Z})$.

Involutions in the plane Cremona group. On the other hand, we consider the mapping classes of automorphisms of complex surfaces induced by involutions in the plane Cremona group. It is known that there are three types of order 2 conjugacy classes in the group of birational automorphisms of $\mathbb{CP}^2$; they are represented by de Jonquiéres involutions, Geiser involutions, and Bertini involutions. This classification was first given by Bertini in 1877 ([Ber77]) and proven later by Bayle–Beauville ([BB00]). The Geiser and Bertini involutions lift to complex automorphisms of del Pezzo surfaces of degree 2 and 1 respectively. The de Jonquiéres involutions lift to complex automorphisms of blowups of $\mathbb{CP}^2$ at finitely many points; because these points are not necessarily in general position, de Jonquiéres involutions do not generally lift to automorphisms of del Pezzo surfaces. We prove in Subsection 3.3 that the mapping classes of these involutions as diffeomorphisms of del Pezzo manifolds are irreducible.

Main results. Throughout, we say that an order 2 mapping class $g \in \mathrm{Mod}(M)$ of a del Pezzo manifold M is *realized* by an automorphism or anti-biholomorphism f of some complex surface X if f has order 2 and if there exists a diffeomorphism $\varphi : M \to X$ so that $[\varphi^{-1} \circ f \circ \varphi] = g$. This is a special case of a finite subgroup of $\mathrm{Mod}(M)$ realized by a *complex equivariant connected sum*; see Definition 4.1(1) and the subsequent remarks.

Our main result is a classification of irreducible mapping classes of order 2 of del Pezzo manifolds.

Theorem 1.2 (Characterizing de Jonquiéres-Geiser-Bertini). *All mapping classes of* $\mathrm{Mod}(M_0)$ *and* $\mathrm{Mod}(M_*)$ *are irreducible. For* $1 \le n \le 8$*, an order two element* $g \in \mathrm{Mod}(M_n)$ *is irreducible if and only if there exists a complex surface* $\mathrm{Bl}_P \mathbb{CP}^2$ *with* $|P| = n$ *such that*

1. *g is realized by a complex automorphism of $X = \mathrm{Bl}_P \mathbb{CP}^2$ induced by a de Jonquiéres involution of (algebraic) degree $d > 2$, a Geiser involution, or a Bertini involution,[1] where P is the set of its base points, or*
2. *g is realized by an order 2 anti-biholomorphism given by a composition $f \circ \tau$, where τ is an order 2 anti-biholomorphism of $\mathrm{Bl}_P \mathbb{CP}^2$ induced by complex conjugation on $\mathbb{CP}^2$ and f is an automorphism of $\mathrm{Bl}_P \mathbb{CP}^2$ induced by a de Jonquiéres involution of (algebraic) degree $d > 2$, a Geiser involution, or a Bertini involution, where P is the set of its base points.*

[1] See Section 3.3 for definitions of these involutions.

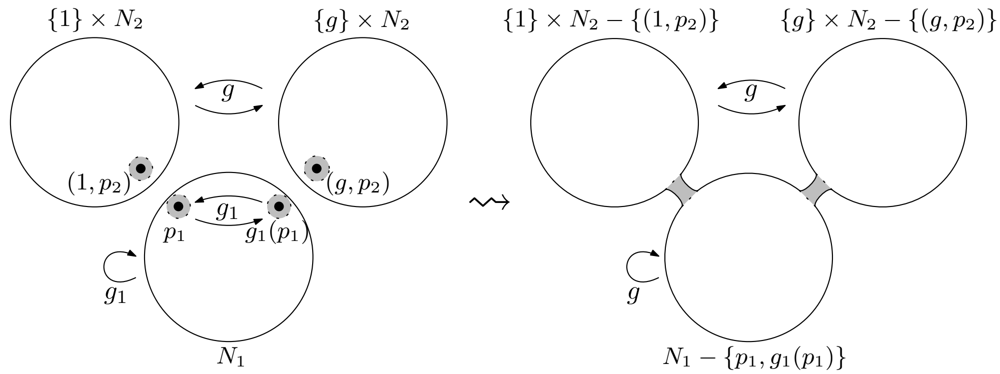

Figure 1: The equivariant connected sum $(N_1\#(G \times N_2), G)$ where $G = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$. If N_1 is a del Pezzo manifold and $N_2 \cong \overline{\mathbb{CP}^2}$, the action of G on $N_1\#(G \times N_2)$ induces a reducible mapping class.

There is an index 2 subgroup $\mathrm{Mod}^+(M_n)$ of $\mathrm{Mod}(M_n)$ for which the following simpler version of Theorem 1.2 holds; see Definition 2.7 for a precise description of $\mathrm{Mod}^+(M_n)$.

Theorem 1.3 (Irreducibility classification for $\mathrm{Mod}^+(M_n)$). *For $1 \leq n \leq 8$, an order two element $g \in \mathrm{Mod}^+(M_n)$ is irreducible if and only if g is realized by a complex automorphism of a complex surface $X = \mathrm{Bl}_P\,\mathbb{CP}^2$ induced by a de Jonquiéres involution of (algebraic) degree $d > 2$, a Geiser involution, or a Bertini involution, where P is the set of its base points.*

Using the theory of hyperbolic reflection groups and Carter's classification of conjugacy classes of Weyl groups ([Car72]), we enumerate the conjugacy classes of involutions in $\mathrm{Mod}^+(M_n)$, of which we study the irreducible ones to prove Theorem 1.3. We then extend Theorem 1.3 to Theorem 1.2 by exhibiting some birational involutions of $\mathbb{CP}^2$ that commute with complex conjugation.

In their classification of conjugacy classes of order 2 elements of $\mathrm{Cr}(2)$, Bayle–Beauville ([BB00]) study pairs (S, σ) where S is a rational surface and $\sigma \in \mathrm{Aut}(S)$ has order 2. Such a pair is called *minimal* if $f : S \to S'$ is a birational morphism such that there exists an involution $\sigma' \in \mathrm{Aut}(S')$ and $f \circ \sigma = \sigma' \circ f$ then f is an isomorphism. Bayle–Beauville classify all minimal pairs (S, σ) ([BB00, Theorem 1.4 and Proposition 1.7]); applying this classification yields the following simple reformulation of Theorem 1.3:

Corollary 1.4 (Minimal pairs). *Let M be a del Pezzo manifold. An order 2 mapping class $g \in \mathrm{Mod}^+(M)$ is irreducible if and only if it is realized by a minimal pair (S, σ) where S is diffeomorphic to M.*

Up to conjugacy, every mapping class of a del Pezzo manifold M is specified by an irreducible mapping class of some del Pezzo manifold and an involution in $\mathrm{O}(k)(\mathbb{Z})$ acting on $H_2(\#k\overline{\mathbb{CP}^2}; \mathbb{Z})$ for some $k \geq 0$. Theorem 1.2 shows that mapping classes of order 2 are built out de Jonquiéres, Geiser, and Bertini involutions and involutions of M_0 and M_*. In Section 4.1 we describe the construction of *complex equivariant connected sums* that builds smooth involutions representing reducible mapping classes out of biholomorphisms or anti-biholomorphisms of order 2 that represent irreducible mapping classes. See Figure 1 for an example of an equivariant connected sum. The smooth Nielsen realization problem for involutions then follows from Theorem 1.2.

Corollary 1.5 (Nielsen realization for involutions). *Let M be a del Pezzo manifold. Any order 2 element $g \in \mathrm{Mod}(M)$ is realized by a smooth involution. In fact, g is realized by a complex equivariant connected sum.*

Remark 1.6. The main results of [Lee21] show that finite subgroups $G \leq \mathrm{Mod}(M_2)$ and maximal finite subgroups $G \leq \mathrm{Mod}(M_3)$ have lifts to $\mathrm{Diff}^+(M_2)$ and $\mathrm{Diff}^+(M_3)$ respectively under the map $\pi : \mathrm{Diff}^+(M_n) \to \mathrm{Mod}(M_n)$ if and only if they are realized by a complex equivariant connected sum. Corollary 1.5 is an analogous statement for the case $G \cong \mathbb{Z}/2\mathbb{Z}$ and any del Pezzo manifold M.

The construction of complex equivariant connected sums is necessary in the solution for the smooth Nielsen realization problem. For all $n \geq 1$, there exist mapping classes $g \in \mathrm{Mod}(M_n)$ of order 2 that cannot be realized by complex automorphisms of any complex structure on M_n by [Lee21, Theorem 1.8] even though they can be realized by complex equivariant connected sums.

Remark 1.7. A special case of Corollary 1.5 says that for any Dehn twist T about a (-2)-sphere in any del Pezzo manifold M, there is an order 2 diffeomorphism of M (topologically) isotopic to T. (For the case $M = M_2$, this is the statement of [Lee21, Corollary 1.3].) In contrast, Farb–Looijenga ([FL21, Corollary 1.10]) shows that the (topological) isotopy class of any Dehn twist about a (-2)-sphere in a K3 surface is not represented by any finite order diffeomorphism.

Related work. This paper is a followup to [Lee21]. As described in Remark 1.6, we examine a similar phenomenon in [Lee21] in which finite subgroups of the mapping class groups of del Pezzo manifolds of high degree are realized by diffeomorphisms if and only if they are realized by complex equivariant connected sums.

As noted above, Bayle–Beauville ([BB00]) prove the classification of order 2 conjugacy classes of the plane Cremona group. Their proof involves studying minimal pairs (S, f) where S is a rational surface and $f \in \mathrm{Aut}(S)$ is an involution. We only invoke the classification (of minimal pairs or order 2 conjugacy classes in $\mathrm{Cr}(2)$) of Bayle–Beauville in the proof of Corollary 1.4.

Hambleton–Tanase ([HT04, Theorem A]) show that if $G = \mathbb{Z}/p\mathbb{Z}$ acts smoothly on $\#n\mathbb{CP}^2$ for $n \geq 1$ and p is an odd prime then there exists an equivariant connected sum of linear actions on $\mathbb{CP}^2$ with the same fixed-set data (see [HT04] for the exact description of this data) and the same induced action on $H_2(\#n\mathbb{CP}^2; \mathbb{Z})$. Corollary 1.5 of our paper is similar in flavor in that all involutions on $H_2(M; \mathbb{Z})$ for del Pezzo manifolds arise from a complex equivariant connected sum. However, our methods are much more elementary than those of Hambleton–Tanase ([HT04]) who utilize the theory of equivariant Yang–Mills moduli spaces; conversely, our methods do not yield as much information about the fixed sets of such involutions.

For some other examples of 4-manifolds, the existence of order 2 mapping classes of 4-manifolds that do not lift to an order 2 diffeomorphism was known; see Raymond–Scott ([RS77, Theorem 1]) for the case of certain nil-manifolds (in every dimension $d \geq 3$) and Baraglia–Konno ([BK19, Theorem 1.2]) for the case of the K3 manifold. The Nielsen realization problem for 4-manifolds was first studied by Farb–Looijenga in their recent paper [FL21]. Specifically, Farb–Looijenga study the case of K3 surfaces and solve the metric and complex Nielsen realization problem for all finite groups as well as the smooth Nielsen realization problem for $\mathbb{Z}/2\mathbb{Z}$. Their results show, in particular, that Dehn twists in the K3 manifold are not realized by finite-order diffeomorphisms ([FL21, Corollary 1.10]); this result was later extended to all smooth spin 4-manifolds with non-zero signature by Konno ([Kon22, Theorem 1.1]).

Outline of this paper. In Section 2 we outline the tools necessary to enumerate and study involutions in $\mathrm{Mod}(M)$ and to realize these mapping classes in $\mathrm{Diff}^+(M)$. Section 3 is dedicated to the proof of Theorem 1.2. More specifically, Sections 3.2 and 3.4 analyze involutions contained in some index 2 subgroup $\mathrm{Mod}^+(M) \leq \mathrm{Mod}(M)$ for each del Pezzo manifold M. Section 3.3 describes and examines the three types of conjugacy classes of involutions in the plane Cremona group. Finally, Section 3.5 extends the result for $\mathrm{Mod}^+(M)$ to $\mathrm{Mod}(M)$. Finally, Section 4 contains the proof of Corollary 1.5.

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2 Mapping class groups of del Pezzo manifolds

In this section we outline some tools used to study the mapping class groups of del Pezzo manifolds in this paper.

2.1 The mapping class group

The Mayer–Vietoris sequence implies that $H_2(M_n;\mathbb{Z}) = H_2(\mathbb{CP}^2;\mathbb{Z})\oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z})^{\oplus n}$ for any $0 \le n \le 8$ and gives a natural $\mathbb{Z}$-basis $\{H, E_1, \ldots, E_n\}$ with intersection form $Q_{M_n} \cong \langle 1\rangle \oplus n\langle -1\rangle$. The group $\mathrm{Aut}(H_2(M;\mathbb{Z}), Q_{M_n})$ is the indefinite orthogonal group $\mathrm{O}(1,n)(\mathbb{Z})$, i.e. by theorems of Freedman ([Fre82]) and Quinn ([Qui86]),

$$\mathrm{Mod}(M_n) \cong \mathrm{O}(1,n)(\mathbb{Z}).$$

Next, consider $M = M_* := \mathbb{CP}^1 \times \mathbb{CP}^1$. The lattice $(H_2(\mathbb{CP}^1 \times \mathbb{CP}^1;\mathbb{Z}), Q_{\mathbb{CP}^1\times\mathbb{CP}^1})$ has two isotropic generators S_1 and S_2 with $Q_{\mathbb{CP}^1\times\mathbb{CP}^1}(S_1, S_2) = 1$ coming from the factors of the product $\mathbb{CP}^1 \times \mathbb{CP}^1$. We will identify $\mathrm{Aut}(H_2(M;\mathbb{Z}), Q_M)$ and $\mathrm{Mod}(M)$ for all del Pezzo manifolds M in this paper.

Let $0 \le k < n$ and let $v_1, \ldots, v_{(n-k)}$ denote the orthogonal $\mathbb{Z}$-basis of $H_2(\#(n-k)\overline{\mathbb{CP}^2};\mathbb{Z})$. There is an isometry

$$\iota_k : (H_2(M_k;\mathbb{Z}), Q_{M_k}) \oplus (H_2(\#(n-k)\overline{\mathbb{CP}^2};\mathbb{Z}), Q_{\#(n-k)\overline{\mathbb{CP}^2}}) \to (H_2(M_n;\mathbb{Z}), Q_{M_n})$$

such that for $1 \le i \le k$ and $1 \le j \le n-k$,

$$\iota_k((H,0)) = H, \qquad \iota_k((E_i,0)) = E_i, \qquad \iota_k((0,v_j)) = E_{k+j}.$$

Moreover, there is an isometry

$$\iota : (H_2(M_*;\mathbb{Z}), Q_{M_*}) \oplus (H_2(\#(n-1)\overline{\mathbb{CP}^2};\mathbb{Z}), Q_{\#(n-1)\overline{\mathbb{CP}^2}}) \to (H_2(M_n;\mathbb{Z}), Q_{M_n})$$

such that for $i = 1,2$ and $2 \le j \le n-1$,

$$\iota((S_i,0)) = H - E_i, \qquad \iota((0,v_1)) = H - E_1 - E_2, \qquad \iota((0,v_j)) = E_{1+j},$$

where S_1 and S_2 denote the two isotropic generators of $H_2(M_*;\mathbb{Z})$ as above.

Definition 2.1. There exist induced inclusions

$$(\iota_k)_* : \mathrm{Mod}(M_k) \times \mathrm{Mod}(\#(n-k)\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}(M_n)$$

for $0 \le k < n$ and

$$\iota_* : \mathrm{Mod}(M_*) \times \mathrm{Mod}(\#(n-1)\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}(M_n)$$

by theorems of Freedman ([Fre82]) and Quinn ([Qui86]); see the discussion preceding Definition 1.1. The inclusions $(\iota_k)_*$ and ι_* are called *standard inclusions*.

Note that for $n \ge 2$, M_n is diffeomorphic to $(\mathbb{CP}^1\times\mathbb{CP}^1)\#(n-1)\overline{\mathbb{CP}^2}$. Applying [Wal64a, Theorem 2] to M_n with this diffeomorphism yields the following statement. (The same statement holds for M_0, M_*, and M_1; for example, see Lemma 4.3.)

Theorem 2.2 (Wall, [Wal64a, Theorem 2])**.** *For $M = M_*$ or M_n with $2 \le n \le 9$, the restriction of $\pi : \mathrm{Homeo}^+(M) \to \mathrm{Mod}(M)$ to the subgroup $\mathrm{Diff}^+(M) \le \mathrm{Homeo}^+(M)$ is surjective.*

Remark 2.3. Theorem 2.2 cannot be extended to manifolds M_n for $n \ge 10$; Friedman–Morgan ([FM88, Theorem 10]) show that the image of the quotient $\pi|_{\mathrm{Diff}^+(M_n)} : \mathrm{Diff}^+(M_n) \to \mathrm{Aut}(H_2(M_n;\mathbb{Z}), Q_{M_n})$ has infinite index in $\mathrm{Aut}(H_2(M_n;\mathbb{Z}), Q_{M_n})$ for all $n \ge 10$.

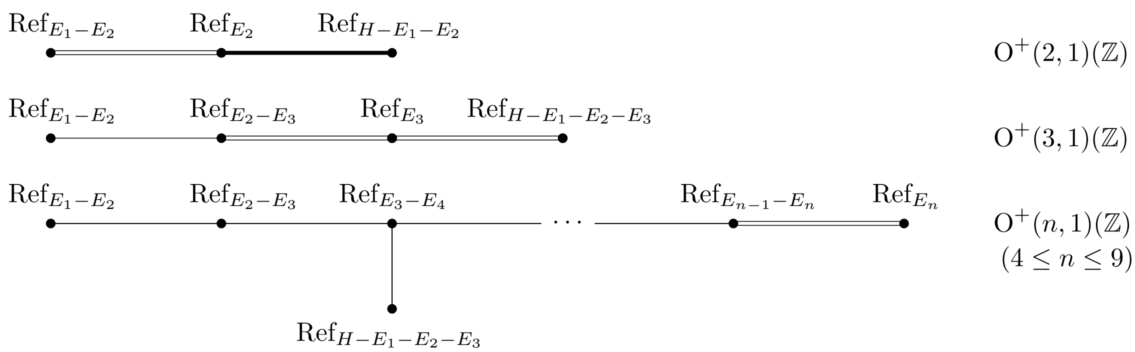

Figure 2: The Coxeter diagrams for $\mathrm{O}^+(n,1)(\mathbb{Z})$ for $2 \leq n \leq 9$. For fixed n, we refer to the specified Coxeter system as $(W(n), S(n))$.

2.2 Coxeter theory and the group $\mathrm{O}^+(n,1)(\mathbb{Z})$

Fix $2 \leq n \leq 9$ and consider the symmetric, bilinear form on $\mathbb{R}^{n+1}$ defined by

$$R_n((x_0, x_1, \ldots, x_n), (y_0, y_1, \ldots, y_n)) = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

We identify $\mathbb{R}^{n+1}$ with $H_2(M_n; \mathbb{Z}) \otimes \mathbb{R}$ such that the ordered $\mathbb{Z}$-basis $\{H, E_1, \ldots, E_n\}$ is identified the given ordered basis of $\mathbb{R}^{n+1}$. Then R_n is precisely the bilinear form $-Q_{M_n}$ extended $\mathbb{R}$-linearly. For any $v \in \mathbb{Z}^{n+1} \subseteq \mathbb{R}^{n+1}$ with $R_n(v,v) = \pm 1, \pm 2$, a *reflection* Ref_v about v defines an involution in $\mathrm{O}(n,1)(\mathbb{Z})$ by

$$\mathrm{Ref}_v(w) := w - \frac{2R_n(v,w)}{R_n(v,v)} v.$$

For any $n \geq 0$, let $\mathrm{O}^+(n,1)(\mathbb{Z})$ be the index 2 subgroup of $\mathrm{O}(n,1)(\mathbb{Z})$ defined

$$\mathrm{O}^+(n,1)(\mathbb{Z}) := \{A \in \mathrm{O}(n,1)(\mathbb{Z}) : A(H) = aH + b_1 E_1 + \cdots + b_n E_n,\ a > 0\}.$$

Wall gives explicit generators of $\mathrm{O}^+(n,1)(\mathbb{Z})$ for $2 \leq n \leq 9$ in terms of reflections:

Theorem 2.4 (Wall, [Wal64b, Theorems 1.5, 1.6])**.** *For $n = 2$,*

$$\mathrm{O}^+(2,1)(\mathbb{Z}) = \langle \mathrm{Ref}_{H+E_1+E_2}, \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2} \rangle.$$

For $3 \leq n \leq 9$,

$$\mathrm{O}^+(n,1)(\mathbb{Z}) = \langle \mathrm{Ref}_{H+E_1+E_2+E_3}, \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2-E_3}, \ldots, \mathrm{Ref}_{E_{n-1}-E_n}, \mathrm{Ref}_{E_n} \rangle.$$

Remark 2.5. Another way to phrase the first equality of Theorem 2.4 is that $\mathrm{O}^+(2,1)(\mathbb{Z})$ is the triangle group $\Delta(2, 4, \infty)$. This formulation is classical, shown by Fricke in [Fri91, p. 64-68].

It is straightforward to show that $\mathrm{O}^+(n,1)(\mathbb{Z})$ is the Coxeter group corresponding to the Coxeter system $(W(n), S(n))$, where

$$S(n) := \begin{cases} \{\mathrm{Ref}_{H-E_1-E_2}, \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2}\} & \text{if } n = 2, \\ \{\mathrm{Ref}_{H-E_1-E_2-E_3}, \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2-E_3}, \ldots, \mathrm{Ref}_{E_{n-1}-E_n}, \mathrm{Ref}_{E_n}\} & \text{if } 3 \leq n \leq 9. \end{cases}$$

The Coxeter diagrams for $(W(n), S(n))$ with $2 \leq n \leq 9$ are given in Figure 2.

Let V_n be the $\mathbb{R}$-span of $\{\alpha_s : s \in S(n)\}$ on which $\mathrm{O}^+(n,1)(\mathbb{Z})$ acts by the *geometric representation* of $(W(n), S(n))$ and let B_n be the standard symmetric bilinear form of V_n as defined in [Hum90, Section 5.3]. The signature of B_n is $(n, 1)$. There is an isometry $F_n : (V_n, B_n) \to (\mathbb{R}^{n+1}, R_n)$ given on the basis elements of V_n by $F_n(\alpha_{\mathrm{Ref}_v}) = R_n(v,v)^{-\frac{1}{2}}v$. One can check that $F_n(s\cdot v) = s\cdot F_n(v)$ for all $v \in V_n$ and $s \in S(n)$. Finally, the submanifold of $\mathbb{R}^{n+1}$ given by

$$\mathbb{H}^n = \{v = (v_0, \ldots, v_n) \in \mathbb{R}^{n+1} : v_0 > 0,\ R_n(v,v) = -1\}$$

with the metric induced by R_n is isometric to hyperbolic n-space (see [Thu97, Chapter 2]).

The fact that $\mathrm{O}^+(n,1)(\mathbb{Z})$ acts on $\mathbb{H}^n$ by isometries via the geometric representation of $(W(n), S(n))$ allows for an easy classification of involutions in $\mathrm{O}^+(n,1)(\mathbb{Z})$.

Lemma 2.6. *Fix $2 \le n \le 9$. Suppose $g \in \mathrm{O}^+(n,1)(\mathbb{Z})$ has finite order.*

1. *Up to conjugacy in $\mathrm{O}^+(n,1)(\mathbb{Z})$, the element g is contained in a subgroup $G_v := \langle s \in S(n) - \{\mathrm{Ref}_v\}\rangle$ for some $\mathrm{Ref}_v \in S(n) - \{\mathrm{Ref}_{E_1-E_2}\}$.*
2. *Suppose that there does not exist any isometries*

   $$\iota : (H_2(N;\mathbb{Z}), -Q_N) \oplus (\mathbb{Z}^k, k\langle 1\rangle) \to (\mathbb{Z}^{n+1}, R_n).$$

   where $k > 0$ and N is some del Pezzo manifold such that g preserves the images of each summand under ι. Then $g \in G_{E_n}$ up to conjugacy in $\mathrm{O}^+(n,1)(\mathbb{Z})$.

Proof.

1. The fundamental domain of the action of $\mathrm{O}^+(n,1)(\mathbb{Z})$ on $\mathbb{H}^n \subseteq (\mathbb{R}^{n+1}, R_n)$ is given by

   $$P := \bigcap_{\mathrm{Ref}_v \in S(n)} \{w \in \mathbb{H}^n : R_n(w,v) \le 0\}$$

   by [Vin72, Proposition 4, Table 4], after conjugating the generators $S(n)$ by the element of $\mathrm{O}^+(n,1)(\mathbb{Z})$ which negates each E_i and fixes H. If $U \subseteq V_n$ denotes the Tits cone of $(W(n), S(n))$ then $F_n^{-1}(P)$ is contained in $-U := \{u \in V_n : -u \in U\}$. Hence $F_n^{-1}(\mathbb{H}^n)$ is also contained in $-U$.

   The finite subgroup $\langle g\rangle$ acts on $\mathbb{H}^n$. The group $\langle g\rangle$ must fix a point $\mathbb{H}^n$ by [Thu97, Corollary 2.5.19]. Therefore it must fix a point $F_n^{-1}(\mathbb{H}^n) \subseteq -U$, and hence also a point p in the Tits cone U. By [Hum90, Theorem 5.13], the stabilizer of p in $\mathrm{O}^+(n,1)(\mathbb{Z})$ is

   $$W_I := \langle s \in I \subseteq S(n)\rangle$$

   for some $I \subseteq S(n)$, up to conjugation in $\mathrm{O}^+(n,1)(\mathbb{Z})$. If $I = S(n)$ then the only fixed point of W_I in V_n is 0, which is not contained in $F_n^{-1}(\mathbb{H}^n)$. If $I = S(n) - \{\mathrm{Ref}_{E_1-E_2}\}$, the fixed subspace of W_I in V_n is $F_n^{-1}(\mathbb{R}\{H - E_1\})$, which has empty intersection with $F_n^{-1}(\mathbb{H}^n)$. Therefore, $g \in G_v$ for some v such that $\mathrm{Ref}_v \in S(n) - \{\mathrm{Ref}_{E_1-E_2}\}$.

2. By the first part of this lemma, $g \in G_v$ up to conjugacy in $\mathrm{O}^+(n,1)(\mathbb{Z})$ for some $v \in S(n) - \{\mathrm{Ref}_{E_1-E_2}\}$. For all decompositions of $\mathbb{Z}^{n+1}$ given below, the restriction of R_n to the last summand is diagonal and positive definite.

   (a) If $v = E_2 - E_3$ then G_v preserves the summands in the decomposition

   $$\mathbb{Z}^{n+1} = \mathbb{Z}\{H - E_1,\ H - E_2\} \oplus \mathbb{Z}\{H - E_1 - E_2,\ E_3,\ \ldots,\ E_n\}.$$

   Note $(\mathbb{Z}\{H - E_1,\ H - E_2\}, R_n) \cong (H_2(M_*;\mathbb{Z}), -Q_{M_*})$.

   (b) If $v = E_3 - E_4$ then G_v preserves the summands in the decomposition

   $$\mathbb{Z}^{n+1} = \mathbb{Z}\{H,\ E_1,\ E_2,\ E_3\} \oplus \mathbb{Z}\{E_4,\ \ldots,\ E_n\}.$$

   Note $(\mathbb{Z}\{H,\ E_1,\ E_2,\ E_3\}, R_n) \cong (H_2(M_3;\mathbb{Z}), -Q_{M_3})$.

(c) If $v = H - E_1 - E_2 - E_3$ then G_v preserves the summands in the decomposition

$$\mathbb{Z}^{n+1} = \mathbb{Z}\{H\} \oplus \mathbb{Z}\{E_1, \ldots, E_n\}.$$

Note $(\mathbb{Z}\{H\}, R_n) \cong (H_2(M_0; \mathbb{Z}), -Q_{M_0})$.

(d) If $n = 2$ and $v = H - E_1 - E_2$ then G_v preserves the summands in the decomposition

$$\mathbb{Z}^3 = \mathbb{Z}\{H\} \oplus \mathbb{Z}\{E_1, E_2\}.$$

Note $(\mathbb{Z}\{H\}, R_2) \cong (H_2(M_0; \mathbb{Z}), -Q_{M_0})$.

(e) If $v = E_k - E_{k+1}$ with $k \geq 4$ then G_v preserves the summands in the decomposition

$$\mathbb{Z}^{n+1} = \mathbb{Z}\{H, E_1, \ldots, E_k\} \oplus \mathbb{Z}\{E_{k+1}, \ldots, E_n\}.$$

Note $(\mathbb{Z}\{H, E_1, \ldots, E_k\}, R_n) \cong (H_2(M_k; \mathbb{Z}), -Q_{M_k})$.

All subgroups G_v with $v \neq E_n$ preserve some orthogonal decomposition of $(\mathbb{Z}^{n+1}, R_n)$ specified in the statement of the lemma. Therefore g must be contained in G_{E_n}. $\square$

In the rest of the paper, we often consider the image of the subgroup $\mathrm{O}^+(1,n)(\mathbb{Z}) \leq \mathrm{O}(1,n)(\mathbb{Z})$ in $\mathrm{Mod}(M_n)$ under the isomorphism $\Phi : \mathrm{O}(1,n)(\mathbb{Z}) \to \mathrm{Mod}(M_n)$.

Definition 2.7. For any $0 \leq n \leq 9$, let $\mathrm{Mod}^+(M_n)$ denote the index 2 subgroup $\mathrm{O}^+(1,n)(\mathbb{Z})$ of $\mathrm{Mod}(M_n)$ under the isomorphism $\Phi : \mathrm{O}(1,n)(\mathbb{Z}) \to \mathrm{Mod}(M_n)$. Let $\mathrm{Mod}^+(M_*)$ denote the index 2 subgroup $\langle c \rangle \cong \mathbb{Z}/2\mathbb{Z}$ of $\mathrm{Mod}(M_*)$ under the isomorphism $\mathrm{Aut}(H_2(M_*; \mathbb{Z}), Q_{M_*}) \to \mathrm{Mod}(M_*)$, where c is the map swapping the isotropic generators S_1 and S_2 of $H_2(M_*; \mathbb{Z})$.

With this definition in hand, we reformulate Lemma 2.6 as a statement about irreducibility of mapping classes.

Corollary 2.8. *Let $2 \leq n \leq 9$ and let Φ denote the isomorphism $\mathrm{O}(1,n)(\mathbb{Z}) \to \mathrm{Mod}(M_n)$. Suppose $g \in \mathrm{Mod}^+(M_n)$ has finite order. If g is irreducible then g is in $W_n := \Phi(G_{E_n}) \leq \mathrm{Mod}^+(M_n)$ up to conjugacy in $\mathrm{Mod}^+(M_n)$.*

Proof. There is an equality of subgroups $\mathrm{O}(1,n)(\mathbb{Z}) = \mathrm{O}(n,1)(\mathbb{Z}) \leq \mathrm{GL}(n+1, \mathbb{Z})$ and an isomorphism $\Phi : \mathrm{O}(1,n)(\mathbb{Z}) = \mathrm{O}(n,1)(\mathbb{Z}) \to \mathrm{Mod}(M_n)$. If $g \in \mathrm{Mod}^+(M_n)$ is irreducible then there does not exist any isometry

$$\iota : (H_2(N; \mathbb{Z}), -Q_N) \oplus (H_2(\#k\overline{\mathbb{CP}^2}; \mathbb{Z}), -Q_{\#k\overline{\mathbb{CP}^2}}) \to (H_2(M_n; \mathbb{Z}), -Q_{M_n}) \cong (\mathbb{Z}^{n+1}, R_n)$$

such that $\Phi^{-1}(g)$ preserves the image of each summand $(H_2(N; \mathbb{Z}), -Q_N)$ and $(H_2(\#k\overline{\mathbb{CP}^2}; \mathbb{Z}), -Q_{\#k\overline{\mathbb{CP}^2}})$ under ι. Lemma 2.6 implies that $\Phi^{-1}(g) \in G_{E_n}$ up to conjugacy in $\mathrm{Mod}^+(M_n)$. $\square$

For any reflection $\mathrm{Ref}_v \in \mathrm{O}(1,n)(\mathbb{Z})$, we also denote the corresponding mapping class $\Phi(\mathrm{Ref}_v)$ by Ref_v in the rest of the paper.

3 Order 2 elements of $\mathrm{Mod}(M_n)$ with $1 \leq n \leq 8$

The goal of this section is to prove Theorems 1.3 and 1.2 and Corollary 1.4.

3.1 The Weyl group $W(\mathbb{E}_n)$

Let X be a del Pezzo surface diffeomorphic to M_n. By [Dol12, p. 378], the action of any complex automorphism $f \in \mathrm{Aut}(X)$ on $H_2(M_n;\mathbb{Z})$, denoted by $f_* \in \mathrm{Aut}(H_2(M_n;\mathbb{Z}), Q_{M_n})$, leaves the canonical class $K_X \in H_2(M_n;\mathbb{Z})$ invariant. The canonical class is given by $K_X = -3H + \sum_{i=1}^n E_i$.

The restriction of Q_{M_n} to $\mathbb{E}_n := (\mathbb{Z}\{K_X\})^\perp$ turns $\mathbb{E}_n$ into an even, negative-definite lattice if $n \le 8$ by [Dol12, p. 361]. For $n \ge 3$, there is a $\mathbb{Z}$-basis of $\mathbb{E}_n$

$$\{H - E_1 - E_2 - E_3,\, E_1 - E_2,\, \ldots,\, E_{n-1} - E_n\}$$

([Dol12, Lemma 8.2.6]). Define the *Weyl group* $W(\mathbb{E}_n)$ to be the subgroup of $\mathrm{Mod}(M_n)$ generated by the reflections Ref_v for v in this basis. Observe that $W(\mathbb{E}_n)$ coincides with the subgroup W_n containing all irreducible involutions of $\mathrm{Mod}^+(M_n)$, up to conjugacy in $\mathrm{Mod}^+(M_n)$, as considered in Corollary 2.8. Moreover, W_n is the stabilizer of K_X in $\mathrm{O}(1,n)(\mathbb{Z})$ by [Dol12, Corollary 8.2.15] and

$$\begin{aligned} W_3 &\cong W(A_2) \times W(A_1), \\ W_4 &\cong W(A_4), \\ W_5 &\cong W(D_5), \\ W_n &\cong W(E_n) \qquad \text{for } 6 \le n \le 8. \end{aligned}$$

Remark 3.1. The subgroup of W_n generated by the reflections $\mathrm{Ref}_{E_k - E_{k+1}}$ for $1 \le k \le n-1$ is isomorphic to S_n via its action on the set $\{E_1, \ldots, E_n\}$.

3.2 Involutions in $\mathrm{Mod}^+(M_n)$ for $n = *$ and $0 \le n \le 4$

In this section we examine the order 2 elements of $\mathrm{Mod}^+(M)$ for $M = M_n$ with $0 \le n \le 4$ and $M = M_*$. We account for the only irreducible mapping classes of order 2 in $\mathrm{Mod}^+(M)$ for $M = M_*$ or M_n with $0 \le n \le 4$ in the following lemma.

Lemma 3.2 ($n = *$ and 0). *Let $M = M_0$ or M_*. Any $g \in \mathrm{Mod}(M)$ is irreducible.*

Proof. There does not exist $c \in H_2(M;\mathbb{Z})$ such that $Q_M(c,c) = -1$. Therefore, there is no isometric embedding

$$(H_2(\#k\overline{\mathbb{CP}^2};\mathbb{Z}), Q_{\#k\overline{\mathbb{CP}^2}}) \hookrightarrow (H_2(M;\mathbb{Z}), Q_M)$$

for any $k > 0$. □

The rest of the mapping classes of order 2 considered in this section are reducible.

Lemma 3.3 ($1 \le n \le 4$). *Let $1 \le n \le 4$. If $g \in \mathrm{Mod}^+(M_n)$ has order 2 then g is reducible.*

Proof. The group $\mathrm{Mod}^+(M_1)$ is generated by Ref_{E_1}. The group $\mathrm{Mod}^+(\mathbb{CP}^2)$ is trivial and the image of the standard inclusion

$$\iota_* : \mathrm{Mod}^+(\mathbb{CP}^2) \times \mathrm{Mod}(\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}^+(M_1)$$

is precisely $\mathrm{Mod}^+(M_1)$. Therefore, any $g \in \mathrm{Mod}^+(M_1)$ is reducible.

The group $W_2 \le \mathrm{Mod}^+(M_2)$ is generated by $\mathrm{Ref}_{E_1 - E_2}$ and $\mathrm{Ref}_{H - E_1 - E_2}$ which commute in $\mathrm{Mod}^+(M_2)$. The image of the standard inclusion

$$\iota_* : \mathrm{Mod}^+(M_*) \times \mathrm{Mod}(\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}^+(M_2)$$

is precisely W_2. Then because any $g \in W_2$ is reducible, Corollary 2.8 implies that any $g \in \mathrm{Mod}^+(M_2)$ of finite order is reducible.

The group $W_3 \leq \mathrm{Mod}^+(M_3)$ is given by $\langle \mathrm{Ref}_{H-E_1-E_2-E_3}\rangle \times \langle \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2-E_3}\rangle$. The group $\langle \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2-E_3}\rangle$ is isomorphic to S_3 so the elements of order 2 are conjugate in S_3 to $\mathrm{Ref}_{E_1-E_2}$. Therefore, any $g \in W_3$ of order 2 is conjugate to $\mathrm{Ref}_{H-E_1-E_2-E_3} \circ \mathrm{Ref}_{E_1-E_2}$ or $\mathrm{Ref}_{E_1-E_2}$ in W_3. Replace g with its conjugate $\mathrm{Ref}_{H-E_1-E_2-E_3} \circ \mathrm{Ref}_{E_1-E_2}$ or $\mathrm{Ref}_{E_1-E_2}$ and observe in both cases that g preserves $\mathbb{Z}\{H-E_1,\, H-E_2\}$. Therefore, g is reducible because it is contained in the image of the standard inclusion

$$\iota_* : \mathrm{Mod}^+(M_*) \times \mathrm{Mod}(\#2\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}^+(M_3).$$

Because any $g \in W_3$ of order 2 is reducible, Corollary 2.8 implies that any $g \in \mathrm{Mod}^+(M_3)$ of order 2 is reducible.

By the proof of [Dol12, Theorem 8.5.8], the group $W_4 \leq \mathrm{Mod}^+(M_4)$ is isomorphic to S_5 generated by the subgroup $S_4 = \langle \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2-E_3}, \mathrm{Ref}_{E_3-E_4}\rangle$ and an element of order 5. This means that any $g \in W_4$ of order 2 is conjugate in W_4 to an element in S_4. The image of the standard inclusion

$$\iota_* : \mathrm{Mod}^+(\mathbb{CP}^2) \times \mathrm{Mod}(\#4\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}^+(M_4)$$

contains $S_4 \leq W_4$ meaning that g is reducible. Then because any $g \in W_4$ of order 2 is reducible, Corollary 2.8 implies that any $g \in \mathrm{Mod}^+(M_4)$ of order 2 is reducible. □

3.3 Irreducible mapping classes and involutions in the Cremona group

The smallest integer $n \geq 1$ such that there exist irreducible mapping classes of order 2 in $\mathrm{Mod}^+(M_n)$ is $n = 5$. In order to discuss these irreducible classes, we first need to consider some classical involutions in the plane Cremona group $\mathrm{Cr}(2)$, i.e. the group of birational automorphisms of $\mathbb{CP}^2$. Conjugacy classes of involutions in the plane Cremona group are classified by the following theorem.

Theorem 3.4 (Bayle–Beauville, [BB00, Theorem 2.6])**.** *Every birational involution of $\mathbb{CP}^2$ is conjugate in $\mathrm{Cr}(2)$ to one and only one of the following:*

1. *a de Jonquiéres involution of degree $d \geq 2$,*
2. *a Geiser involution, or*
3. *a Bertini involution.*

We now briefly recall the definitions of these involutions.

3.3.1 de Jonquiéres involutions

This description of de Jonquiéres involutions follows the exposition of [Bla07, Example 3.1]. Fix $g \geq 1$ and $a_1, \ldots, a_{2m} \in \mathbb{C}$ distinct with $m = g+1$. Consider the map $\varphi_0 : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ defined

$$\varphi_0 : ([X_1 : X_2], [Y_1 : Y_2]) \mapsto \left([X_1 : X_2], \left[Y_2 \prod_{i=m+1}^{2m} (X_1 - a_i X_2) : Y_1 \prod_{i=1}^{m} (X_1 - a_i X_2) \right] \right).$$

The map φ_0 is rational and defined on the open set U, which is the complement of the set of $2m$ points

$$P = \{p_i = ([a_i : 1], [1 : 0]) : 1 \leq i \leq m\} \cup \{p_i = ([a_i : 1], [0 : 1]) : m+1 \leq i \leq 2m\}.$$

Then φ_0 lifts to an automorphism φ of $X := \mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ of order 2. To see this, consider the rational map $\psi_0 : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1$ given by projecting φ_0 to the second coordinate, i.e.

$$\psi_0 : ([X_1 : X_2], [Y_1 : Y_2]) \mapsto \left[Y_2 \prod_{i=m+1}^{2m} (X_1 - a_i X_2) : Y_1 \prod_{i=1}^{m} (X_1 - a_i X_2) \right].$$

The set of basepoints of ψ_0 is equal to P and the construction in the proof of [Bea96, Theorem II.7] shows that ψ_0 extends to a rational morphism $X \to \mathbb{CP}^1$. Hence φ_0 extends to a birational morphism $\psi : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$. Finally, the universal property of blowups ([Bea96, Proposition II.8]) shows that $\psi : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$ extends to an automorphism φ of X. See the following diagram; here, $b : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is the blowup of the points in P.

$$\begin{array}{ccc} \mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1) & \xrightarrow{\ \varphi\ } & \mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1) \\ \downarrow b & \searrow \psi & \downarrow b \\ \mathbb{CP}^1 \times \mathbb{CP}^1 & \xrightarrow{\ \psi_0 \times \mathrm{Id} = \varphi_0\ } & \mathbb{CP}^1 \times \mathbb{CP}^1 \end{array}$$

One way to think about this extension $\varphi \in \mathrm{Aut}(X)$ is by restricting to the open and dense subset $V \subseteq X$ defined by

$$V := (\mathbb{CP}^1 - \{[a_i : 1] : 1 \le i \le 2m\}) \times \mathbb{CP}^1.$$

Because φ_0 restricts to an automorphism of V, the automorphism φ is the unique continuous extension of φ_0 to X.

Let e_i denote the homology classes of the exceptional fibers over $p_i \in P$ for all $1 \le i \le 2m$ and let S_1, S_2 denote the homology classes of X coming from the first and second factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$ respectively. Then $H_2(X;\mathbb{Z}) = \mathbb{Z}\{S_1, S_2, e_1, \ldots, e_{2m}\}$ with $Q_X(S_k, e_i) = 0$ and $Q_X(S_k, S_\ell) = 1 - \delta_{k\ell}$ for all $k, \ell = 1, 2$ and $1 \le i \le 2m$.

Consider the projection map $\mathrm{pr}_0 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ onto the first coordinate which extends to a map $\mathrm{pr} : X \to \mathbb{CP}^1$. Then $\mathrm{pr} \circ \varphi = \mathrm{pr}$ because $\mathrm{pr}_0 \circ \varphi_0 = \mathrm{pr}_0$; see Figure 3 for an illustration. The fiber of pr over any $q \in \mathbb{CP}^1$ with $q \neq [a_i : 1]$ for all i is $\mathrm{pr}^{-1}(q) = \{q\} \times \mathbb{CP}^1$ in X, which represents the homology class S_2. The map φ restricts to a complex automorphism of each fiber $\mathrm{pr}^{-1}(q) = \{q\} \times \mathbb{CP}^1$ and so $\varphi_*(S_2) = S_2$. Over any $[a_i : 1] \in \mathbb{CP}^1$, the fiber $\mathrm{pr}^{-1}([a_i : 1])$ is a bouquet of two $\mathbb{CP}^1$, i.e. two copies of $\mathbb{CP}^1$ intersecting transversely at one point. One component is the exceptional fiber e_i and the other component is the strict transform of the line $\mathrm{pr}_0^{-1}([a_i : 1])$ in X. To determine the action of φ on the exceptional fiber e_i, compute that $\varphi_0([a_i : 1], [Y_1 : Y_2]) = p_i \in P$ for any point of the form $([a_i : 1], [Y_1 : Y_2]) \in U$. Because φ is an automorphism of X of order 2, this means that the strict transform of $\mathrm{pr}_0^{-1}([a_i : 1])$ in X must be sent to the exceptional fiber e_i by φ and vice versa. Hence φ swaps the two components of $\mathrm{pr}^{-1}([a_i : 1])$. More explicitly, this means that $\varphi_*(e_i) = S_2 - e_i$, where $S_2 - e_i$ is the homology class of this strict transform of $\mathrm{pr}_0^{-1}([a_i : 1])$.

The homological data described above determines the action of φ_* on $H_2(M_n;\mathbb{Z})$.

Lemma 3.5. *Let $n \ge 5$ be odd and let $g_1, g_2 \in \mathrm{Mod}(M_n)$. Consider some primitive $C_i \in H_2(M_n;\mathbb{Z})$ and some $\mathbb{Z}$-submodule $N_i := \mathbb{Z}\{C_i, v_1^i, \ldots, v_{n-1}^i\}$ of $H_2(M_n;\mathbb{Z})$ for $i = 1, 2$ such that the restriction of Q_{M_n} to N_i with respect to the given basis is $\langle 0 \rangle \oplus (n-1)\langle -1 \rangle$. If*

$$g_i(C_i) = C_i, \qquad g_i(v_k^i) = C_i - v_k^i$$

for all $1 \le k \le n-1$ and $i = 1, 2$ then g_1 and g_2 are conjugate in $\mathrm{Mod}(M_n)$. In particular, any such g_1 is conjugate to $[\varphi]$ where φ is the de Jonquiéres involution on $X = \mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ defined above where $|P| = n - 1$.

Proof. Suppose the restriction of Q_{M_n} to $\mathbb{Z}\{v_1^i, \ldots, v_{n-1}^i\}^\perp \subseteq H_2(M_n;\mathbb{Z})$ is odd for $i = 1$ or 2. Then one can check that the restriction of Q_{M_n} to $\mathbb{Z}\{C_i - v_1^i, v_2^i, \ldots, v_{n-1}^i\}^\perp$ is even, and

$$g_i(C_i - v_1^i) = g_i(C_i) - g_i(v_1^i) = C_i - (C_i - v_1^i).$$

Furthermore, the restriction of Q_{M_n} to $\mathbb{Z}\{C_i, C_i - v_1^i, v_2^i, \ldots, v_{n-1}^i\}$ with respect to the given basis is $\langle 0 \rangle \oplus (n-1)\langle -1 \rangle$. Hence after possibly replacing v_1^i with $C_i - v_1^i$, we may assume that the restriction of Q_{M_n} to $\mathbb{Z}\{v_1^i, \ldots, v_{n-1}^i\}^\perp$ is even for each i.

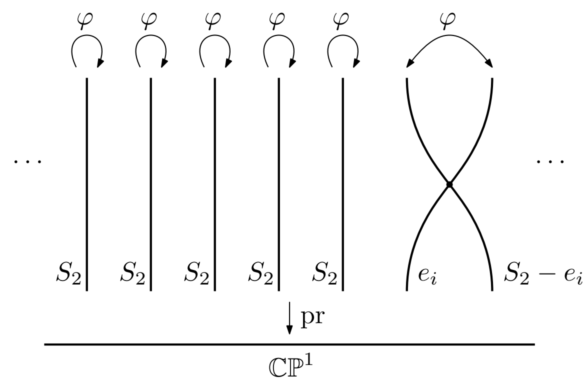

Figure 3: The action of a de Jonquiéres involution φ of $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and the conic bundle pr : $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1) \to \mathbb{CP}^1$ preserved by φ. Each (vertical) line in the figure above represents a complex submanifold of $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ isomorphic to $\mathbb{CP}^1$; they are labelled with their respective homology classes. Each vertical connected component represents a fiber of the map pr. The singular fibers are two copies of $\mathbb{CP}^1$ intersecting transversely at one point; there are $|P|$-many singular fibers.

For each $i = 1, 2$, there is an orthogonal decomposition

$$H_2(M_n; \mathbb{Z}) = \mathbb{Z}\{v_2^i, \ldots, v_{n-1}^i\} \oplus \mathbb{Z}\{C_i, c_i', v_1^i\}$$

where $c_i' \in \mathbb{Z}\{v_2^i, \ldots, v_{n-1}^i\}^\perp$ is such that $Q_{M_n}(C_i, c_i') = 1$ which exists by unimodularity of Q_{M_n} restricted to $\mathbb{Z}\{v_2^i, \ldots, v_{n-1}^i\}^\perp$. Denote $Q_{M_n}(c_i', v_1^i)$ by A and let

$$c_i := (c_i' + Av_1^i) - \left(\frac{Q_{M_n}(c_i' + Av_1^i, c_i' + Av_1^i)}{2}\right) C_i.$$

Note that $Q_{M_n}(c_i' + Av_1^i, c_i' + Av_1^i)$ is even because $c_i' + Av_1^i$ is in $\mathbb{Z}\{v_1^i, \ldots, v_{n-1}^i\}^\perp$. Compute that that with respect to the $\mathbb{Z}$-basis (C_i, c_i, v_1^i),

$$Q_{M_n}|_{\mathbb{Z}\{C_i,\, c_i,\, v_1^i\}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

There is another orthogonal decomposition

$$H_2(M_n; \mathbb{Z}) = \mathbb{Z}\{C_i - v_2^i, \ldots, C_i - v_{n-1}^i\} \oplus \mathbb{Z}\{C_i,\, c_i - v_1^i - \cdots - v_{n-1}^i,\, C_i - v_1^i\}.$$

The only automorphism of $\mathbb{Z}\{C_i,\, c_i,\, v_1^i\}$ preserving Q_{M_n} and fixing C_i and v_1^i is the identity. This uniquely determines g_i since g_i restricts to an isometry

$$g_i : \mathbb{Z}\{C_i,\, c_i,\, v_1^i\} \to \mathbb{Z}\{C_i,\, c_i - v_1^i - \cdots - v_{n-1}^i,\, C_i - v_1^i\}$$

with respect to the restrictions of Q_{M_n} satisfying

$$g_i(C_i) = C_i, \qquad g_i(v_1^i) = C_i - v_1^i.$$

Finally let $\Phi \in \mathrm{Mod}(M_n)$ such that for all $1 \le k \le n-1$,

$$\Phi(v_k^1) = v_k^2, \qquad \Phi(C_1) = C_2, \qquad \Phi(c_1) = c_2.$$

Then $g_1 = \Phi^{-1} \circ g_2 \circ \Phi$. □

The birational involution φ_0 has (algebraic) degree $m+1$. Because $m = g+1 \geq 2$ in all constructions in this paper, any de Jonquiéres involution that we consider has degree $d > 2$. Moreover, φ_0 is birationally equivalent to the de Jonquiéres involutions of [BB00, Example 2.4(c)]. In the following lemma, we consider an explicit birational equivalence with an automorphism f of a surface $\mathrm{Bl}_{P_0}\mathbb{CP}^2$.

Lemma 3.6. *For any odd $n \geq 5$, there exist $P_0 \subseteq \mathbb{RP}^2 \subseteq \mathbb{CP}^2$ with $|P_0| = n$ and an involution $f \in \mathrm{Aut}(\mathrm{Bl}_{P_0}\mathbb{CP}^2)$ conjugate to a de Jonquierés involution φ_0 described above in $\mathrm{Cr}(2)$ such that*

1. *$H_2(\mathrm{Bl}_{P_0}\mathbb{CP}^2;\mathbb{Z}) \cong H_2(\mathrm{Bl}_P(\mathbb{CP}^1\times\mathbb{CP}^1);\mathbb{Z})$ as $\mathbb{Z}[G]$-modules with $G = \mathbb{Z}/2\mathbb{Z}$ acting by $\langle [f]\rangle$ and $\langle[\varphi]\rangle$ respectively,*
2. *f commutes with the anti-biholomorphism $\tau : \mathrm{Bl}_{P_0}\mathbb{CP}^2 \to \mathrm{Bl}_{P_0}\mathbb{CP}^2$ induced by complex conjugation on $\mathbb{CP}^2$, and*
3. *$[f]$ is conjugate to $\prod_{k=1}^{\frac{n-1}{2}}\left(\mathrm{Ref}_{H-E_1-E_{2k}-E_{2k+1}}\circ\mathrm{Ref}_{E_{2k}-E_{2k+1}}\right)$ in $\mathrm{Mod}(M_n)$ after identifying $M_n \cong \mathrm{Bl}_{P_0}\mathbb{CP}^2$.*

Proof. Let $a_1,\ldots,a_{n-1}\in\mathbb{R}$ be distinct and let φ_0 and $P\subseteq\mathbb{CP}^1\times\mathbb{CP}^1$ be defined as above. Fix some $p = ([a:1],[b:1])$ with $a,b\in\mathbb{R}_{\neq 0}$ and $a\neq a_i$ for all i such that $\varphi_0(p) = p$ and let $q_1 = [0:0:1]$, $q_2 = [0:1:0]$. Consider the Hirzebruch surfaces $\mathbb{F}_0 = \mathbb{CP}^1\times\mathbb{CP}^1$ and $\mathbb{F}_1 = \mathrm{Bl}_{q_1}\mathbb{CP}^2$. There is an isomorphism $\mathrm{Bl}_p\mathbb{F}_0 \cong \mathrm{Bl}_{q_1,q_2}\mathbb{CP}^2$ which can be seen by explicitly writing

$$\mathrm{Bl}_{q_1,q_2}\mathbb{CP}^2 = \{([A:B],[C:D],[X:Y:Z]) : A(X+Y)-aBY = C(X+Z)-bDZ = 0\}\subseteq\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^2$$

and noting that the projection map onto the first two factors defines a blowup $\psi_1 : \mathrm{Bl}_{q_1,q_2}\mathbb{CP}^2\to\mathbb{F}_0$ which is an isomorphism onto $\mathbb{F}_0-\{p\}$. Let $\psi_2 : \mathrm{Bl}_{q_1,q_2}\mathbb{CP}^2\to\mathbb{F}_1$ be a blowup given by projecting onto the first and third factors. The exceptional divisor of ψ_2 is the strict transform of $\mathrm{pr}_0^{-1}([a:1])$ in $\mathrm{Bl}_p\mathbb{F}_0$.

The rational map $F:\mathbb{F}_0 \dashrightarrow \mathbb{F}_1$ given by $F = \psi_2\circ\psi_1^{-1}$ is the elementary transformation centered at p (cf. [Dol12, Section 7.4.2] or [BB00, (2.5)]) and is a morphism restricted to $\mathbb{F}_0-\{p\}$. Note that $\psi_1^{-1}(P)$ is not contained in the exceptional divisor of ψ_2. Hence F extends to $\mathrm{Bl}_P\mathbb{F}_0 \dashrightarrow \mathrm{Bl}_{F(P)}\mathbb{F}_1$; also denote this map by F. The maps ψ_1 and ψ_2 similarly extend and fit into the following commutative diagram:

$$\begin{array}{ccc} & S := \mathrm{Bl}_{\{p\}\cup P}\mathbb{F}_0 & \\ {}^{\psi_1}\swarrow & & \searrow^{\psi_2} \\ \mathrm{Bl}_P\mathbb{F}_0 & \overset{F}{\dashrightarrow} & \mathrm{Bl}_{F(P)}\mathbb{F}_1 \end{array}$$

Let $e\in H_2(S;\mathbb{Z})$ denote the exceptional divisor over p. Let φ be the automorphism of $\mathrm{Bl}_P\mathbb{F}_0$ induced by φ_0. Because $\varphi(p) = p$, the map φ extends to an involution $\tilde\varphi$ of S. Because φ preserves the fibers of pr, the map $\tilde\varphi$ descends to an involution f of $\mathrm{Bl}_{F(P)}\mathbb{F}_1$. Note that f and φ are conjugate in $\mathrm{Cr}(2)$ since $f = F\circ\varphi\circ F^{-1}$ as birational automorphisms of $\mathbb{CP}^2$. There are isometries

$$\begin{aligned}(H_2(S;\mathbb{Z}),Q_S) &\cong (H_2(\mathrm{Bl}_P\mathbb{F}_0;\mathbb{Z}),Q_{\mathrm{Bl}_P\mathbb{F}_0})\oplus(\mathbb{Z}\{e\},Q_S|_{\mathbb{Z}\{e\}})\\ &\cong (H_2(\mathrm{Bl}_{F(P)}\mathbb{F}_1;\mathbb{Z}),Q_{\mathrm{Bl}_{F(P)}\mathbb{F}_1})\oplus(\mathbb{Z}\{S_2-e\},Q_S|_{\mathbb{Z}\{S_2-e\}})\end{aligned}$$

where the action of $[\tilde\varphi]$ on $H_2(S;\mathbb{Z})$ restricts to the actions of $[\varphi]$ and $[f]$ on $H_2(\mathrm{Bl}_P\mathbb{F}_0;\mathbb{Z})$ and $H_2(\mathrm{Bl}_{F(P)}\mathbb{F}_1;\mathbb{Z})$ respectively. The $\mathbb{Z}[\langle[\tilde\varphi]\rangle]$-submodule $N := \mathbb{Z}\{S_2, e_1,\ldots,e_{n-1}\}$ is contained in both $H_2(\mathrm{Bl}_P\mathbb{F}_0;\mathbb{Z})$ and $H_2(\mathrm{Bl}_{F(P)}\mathbb{F}_1;\mathbb{Z})$. Because the actions of $[f]$ and $[\varphi]$ agree on N, Lemma 3.5 shows there is an isometry

$$\iota : (H_2(\mathrm{Bl}_P\mathbb{F}_0;\mathbb{Z}),Q_{\mathrm{Bl}_P\mathbb{F}_0})\to(H_2(\mathrm{Bl}_{F(P)}\mathbb{F}_1;\mathbb{Z}),Q_{\mathrm{Bl}_{F(P)}\mathbb{F}_1})$$

which is also a $\mathbb{Z}[G]$-module isomorphism with $G = \mathbb{Z}/2\mathbb{Z}$ acting by $\langle[f]\rangle$ and $\langle[\varphi]\rangle$ respectively.

Note that $F \circ \tau_0 = \tau \circ F$ where $\tau_0 : \mathbb{F}_0 \to \mathbb{F}_0$ and $\tau : \mathbb{F}_1 \to \mathbb{F}_1$ are diffeomorphisms induced by complex conjugation of the coordinates of $\mathbb{CP}^1$ and $\mathbb{CP}^2$ respectively. Then τ_0 commutes with φ and $F(P) \subseteq \mathbb{RP}^2 \subseteq \mathbb{CP}^2$ because $P \subseteq \mathbb{F}_0$ is pointwise fixed by τ_0. Moreover, τ commutes with $F \circ \varphi \circ F^{-1}$, meaning that f must commute with τ as a diffeomorphism of $\mathrm{Bl}_{F(P)} \mathbb{CP}^2 \to \mathrm{Bl}_{F(P)} \mathbb{CP}^2$.

Consider

$$g = \prod_{k=1}^{\frac{n-1}{2}} \left(\mathrm{Ref}_{H-E_1-E_{2k}-E_{2k+1}} \circ \mathrm{Ref}_{E_{2k}-E_{2k+1}}\right) \in \mathrm{Mod}(M_n).$$

Note that each reflection in g fixes $H - E_1$. Also,

$$g(E_{2k+1}) = \mathrm{Ref}_{H-E_1-E_{2k}-E_{2k+1}} \circ \mathrm{Ref}_{E_{2k}-E_{2k+1}}(E_{2k+1}) = \mathrm{Ref}_{H-E_1-E_{2k}-E_{2k+1}}(E_{2k}) = H - E_1 - E_{2k+1},$$
$$g(E_{2k}) = \mathrm{Ref}_{H-E_1-E_{2k}-E_{2k+1}} \circ \mathrm{Ref}_{E_{2k}-E_{2k+1}}(E_{2k}) = \mathrm{Ref}_{H-E_1-E_{2k}-E_{2k+1}}(E_{2k+1}) = H - E_1 - E_{2k}.$$

Let $g_1 = g$ with $C_1 = H - E_1 \in H_2(M_n;\mathbb{Z})$ and $v_k^1 = E_{k+1} \in H_2(M_n;\mathbb{Z})$ for all $1 \le k \le n-1$. Lemma 3.5 implies that g, $[f]$, and $[\varphi]$ are conjugate in $\mathrm{Mod}(M_n) \cong \mathrm{Mod}(\mathrm{Bl}_{F(P)} \mathbb{F}_1) \cong \mathrm{Mod}(\mathrm{Bl}_P \mathbb{F}_0)$. ☐

In the next two lemmas, we consider the action of φ_* on $H_2(M_n;\mathbb{Z})$. In this lemma and the rest of the paper, let $H_2(M_n;\mathbb{Z})^G$ denote the subgroup fixed by G, i.e.

$$H_2(M_n;\mathbb{Z})^G := \{c \in H_2(M_n;\mathbb{Z}) : g(c) = c \text{ for all } g \in G\}.$$

Lemma 3.7. *Let $n \ge 5$ be odd and let $\varphi \in \mathrm{Aut}(\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1))$ be the de Jonquiéres involution. Identify $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1) \cong M_n$ and let $G = \langle [\varphi] \rangle \cong \mathbb{Z}/2\mathbb{Z} \le \mathrm{Mod}^+(M_n)$. As a $\mathbb{Z}[G]$-module, $H_2(M_n;\mathbb{Z}) \cong \mathbb{Z}[G]^{\oplus 2} \oplus C^{\oplus(n-3)}$ where $C \cong \mathbb{Z}$ as a $\mathbb{Z}$-module and G acts by negation on C, and*

$$H_2(M_n;\mathbb{Z})^G = \mathbb{Z}\{S_2,\, 2S_1 - e_1 - \cdots - e_{n-1}\}.$$

Proof. The fixed set of φ is a smooth curve Γ with a surjective morphism $\Gamma \to \mathbb{CP}^1$ of degree 2 ramified over $(n-1)$-points (see [Bla07, Example 3.1]); Γ is a curve of genus $\frac{n-3}{2}$. There is an isomorphism

$$H_2(M_n;\mathbb{Z}) \cong \mathbb{Z}^{\oplus t} \oplus C^{\oplus c} \oplus \mathbb{Z}[G]^{\oplus r}$$

as $\mathbb{Z}[G]$-modules for some $t, r, c \in \mathbb{Z}$ by [Edm89, Proposition 1.1] where $C \cong \mathbb{Z}$ as $\mathbb{Z}$-modules and φ_* acts by negation in C. By [Edm89, Proposition 2.4], $\beta_0(\Gamma) + \beta_2(\Gamma) = t + 2$ and $\beta_1(\Gamma) = c$ where $\beta_k(\Gamma)$ is the kth mod 2 Betti number of Γ. Therefore $t = 0$ and $c = n - 3$ so that

$$H_2(M_n;\mathbb{Z}) \cong \mathbb{Z}[G]^{\oplus 2} \oplus C^{\oplus(n-3)}$$

as $\mathbb{Z}[G]$-modules. As $\mathbb{Q}[G]$-modules,

$$H_2(M_n);\mathbb{Q}) \cong (C \otimes \mathbb{Q})^{\oplus(n-1)} \oplus \mathbb{Q}^{\oplus 2}.$$

A calculation shows that S_2 and $2S_1 - e_1 - \cdots - e_{n-1}$ are fixed by G. Therefore,

$$H_2(M_n;\mathbb{Z})^G = \mathbb{Q}\{S_2,\, 2S_1 - e_1 - \cdots - e_{n-1}\} \cap H_2(M_n;\mathbb{Z}) = \mathbb{Z}\{S_2,\, 2S_1 - e_1 - \cdots - e_{n-1}\}. \quad ☐$$

Lemma 3.8. *Let $n \ge 5$ be odd. If φ and f are de Jonquiéres involutions on some $\mathrm{Bl}_{P_0}(\mathbb{CP}^1 \times \mathbb{CP}^1) \cong M_n$ and $\mathrm{Bl}_P \mathbb{CP}^2 \cong M_n$ respectively then $[\varphi], [f] \in \mathrm{Mod}^+(M_n)$ are irreducible.*

Proof. Suppose for some $1 \le k \le n$, there exists an isometric embedding

$$\iota : (H_2(\#k\overline{\mathbb{CP}^2};\mathbb{Z}), Q_{\#k\overline{\mathbb{CP}^2}}) \hookrightarrow (H_2(M_n;\mathbb{Z}), Q_{M_n})$$

such that φ_* restricts to an automorphism of the image. Let $v_1, \ldots, v_k$ denote the orthogonal $\mathbb{Z}$-basis of $H_2(\#k\overline{\mathbb{CP}^2};\mathbb{Z})$; note that $Q_{M_n}(\iota(v_i), \iota(v_j)) = -\delta_{ij}$ for all $1 \le i, j \le k$. Because φ_* acts as an element of $\mathrm{O}(k)(\mathbb{Z})$ on the image of ι,

$$\varphi_*(\iota(v_1)) = \iota(v_1),\ -\iota(v_1),\ \text{or}\ \pm\,\iota(v_i) \text{ for some } i \ne 1.$$

We address the three cases separately.

1. Suppose there exists some $c \in H_2(M_n;\mathbb{Z})$ such that $Q_{M_n}(c,c) = -1$ and $\varphi_*(c) = c$. If $\varphi_*(c) = c$ then $c \in H_2(M_n;\mathbb{Z})^G$. Compute that the restriction of Q_{M_n} to $H_2(M_n;\mathbb{Z})^G$ is

$$Q_{M_n}|_{H_2(M_n;\mathbb{Z})^G} = \begin{pmatrix} 0 & 2 \\ 2 & -(n-1) \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 1 & -\frac{n-1}{2} \end{pmatrix}$$

with respect to the $\mathbb{Z}$-basis of $H_2(M_n;\mathbb{Z})^G$ given in Lemma 3.7. Therefore, $Q_{M_n}(x,x) \equiv 0 \pmod 2$ for all $x \in H_2(M_n;\mathbb{Z})^G$. This is a contradiction because $Q_{M_n}(c,c) = -1$.

2. Suppose there exists some $c \in H_2(M_n;\mathbb{Z})$ such that $Q_{M_n}(c,c) = -1$ and $\varphi_*(c) = -c$. Then $c \in \mathbb{Z}\{S_2\}^\perp$ because $\varphi_*(S_2) = S_2$. The only elements of $x \in \mathbb{Z}\{S_2\}^\perp$ with $Q_{M_n}(x,x) = -1$ are of the form $x = aS_2 \pm e_k$ for some $a \in \mathbb{Z}$ and $1 \le k \le 2m$ because $\mathbb{Z}\{S_2\}^\perp = \mathbb{Z}\{S_2, e_1, \ldots, e_{2m}\}$. On the other hand, if

$$-aS_2 \mp e_k = \varphi_*(aS_2 \pm e_k) = aS_2 \pm (S_2 - e_k)$$

then $a \pm 1 = -a$. This is a contradiction since $a \in \mathbb{Z}$.

3. Suppose there exist some $c_1, c_2 \in H_2(M_n;\mathbb{Z})$ such that $Q_{M_n}(c_k, c_\ell) = -\delta_{k\ell}$ and $\varphi_*(c_1) = c_2$. Then $Q_{M_n}(c_1 - c_2, c_1 - c_2) = -2$ and $\varphi_*(c_1 - c_2) = -(c_1 - c_2)$. Since $\varphi_*(S_2) = S_2$,

$$c_1 - c_2 \in \mathbb{Z}\{S_2\}^\perp = \mathbb{Z}\{S_2, e_1, \ldots, e_{n-1}\}.$$

Then

$$c_1 - c_2 = aS_2 + (-1)^{a_k} e_k + (-1)^{a_j} e_j$$

some $a, a_k, a_j \in \mathbb{Z}$ and $1 \le k, j \le n-1$ because $Q_{M_n}(c_1 - c_2, c_1 - c_2) = -2$. Moreover,

$$c_1 + c_2 \in H_2(M_n;\mathbb{Z})^G = \mathbb{Z}\{S_2,\ 2S_1 - e_1 - \cdots - e_{n-1}\}$$

where the second equality holds by Lemma 3.7. However, for any $A, B \in \mathbb{Z}$,

$$(c_1 - c_2) + (AS_2 + B(2S_1 - e_1 - \cdots - e_{n-1})) \notin 2H_2(M_n;\mathbb{Z}).$$

This is a contradiction because $(c_1 - c_2) + (c_1 + c_2) = 2c_1 \in 2H_2(M_n;\mathbb{Z})$.

Therefore, $[\varphi]$ is irreducible in $\mathrm{Mod}^+(M_n)$. Because $H_2(M_n;\mathbb{Z})$ as a $\langle f_* \rangle$-module is isomorphic to $H_2(M_n;\mathbb{Z})$ as a $\langle \varphi_* \rangle$-module by Lemma 3.6, $[f] \in \mathrm{Mod}^+(M_n)$ is irreducible as well. □

3.3.2 Geiser and Bertini involutions

In this section we describe the Geiser involution $\gamma : X_7 \to X_7$ and the Bertini involution $\beta : X_8 \to X_8$ for any del Pezzo surface X_n diffeomorphic to M_n for $n = 7$ and 8; we follow the exposition of [BB00].

Let $X_7 = \mathrm{Bl}_P\, \mathbb{CP}^2$ with P a set of 7 points in general position in $\mathbb{CP}^2$. For any $p \in \mathbb{CP}^2 - P$, the pencil of cubic curves passing through the points $P \cup \{p\}$ has a ninth base point q. The map $\gamma : p \mapsto q$ defines a birational map $\gamma : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ and induces an order 2 automorphism of X_7, which we also denote by γ. Another way to construct this map is to consider the linear system $|-K_{X_7}|$ which defines a double covering $f : X_7 \to \mathbb{CP}^2$ branched along a smooth curve C of genus 3. Then γ is the nontrivial deck transformation of this branched cover, and the fixed set $\mathrm{Fix}(\gamma)$ in X_7 is C.

Let $X_8 = \mathrm{Bl}_P\, \mathbb{CP}^2$ with P a set of 8 points in general position in $\mathbb{CP}^2$. Consider the linear system $|-2K_{X_8}|$ which defines a double covering $f : X_8 \to Q$ onto a quadric cone $Q \subseteq \mathbb{CP}^3$ branched along the vertex v of Q and a smooth curve C of genus 4. Then β is the nontrivial deck transformation of this branched cover, and the fixed set $\mathrm{Fix}(\gamma)$ in X_8 is $C \sqcup \{q\}$, where q is the ninth base point of the pencil of cubics defined by P.

By [Dol12, p. 410], the Geiser involution γ acts on the subgroup $\mathbb{E}_7 = \mathbb{Z}\{K_{X_7}\}^\perp$ of $H_2(X_7;\mathbb{Z}) \cong H^2(X_7;\mathbb{Z}) \cong \mathrm{Pic}(X_7)$ by negation and is the product of seven, pairwise-commuting involutions in W_7. By [Dol12, p. 414], the Bertini involution β acts on the subgroup $\mathbb{E}_8 = \mathbb{Z}\{K_{X_8}\}^\perp$ of $H_2(X_8;\mathbb{Z}) \cong H^2(X_8;\mathbb{Z}) \cong \mathrm{Pic}(X_8)$ by negation and is the product of eight, pairwise-commuting involutions in W_8. In particular, γ and β fix K_{X_7} and K_{X_8} respectively.

We conclude this section by noting that $[\gamma]$ and $[\beta]$ are irreducible elements of $\mathrm{Mod}^+(M_n)$ for $n = 7, 8$ respectively.

Lemma 3.9. *The mapping classes $[\gamma] \in \mathrm{Mod}^+(M_7)$ and $[\beta] \in \mathrm{Mod}^+(M_8)$ are irreducible.*

Proof. Let $G_7 = \langle[\gamma]\rangle$ and $G_8 = \langle[\beta]\rangle$.

1. If $v \in H_2(M_7;\mathbb{Z})$ is fixed by G_7 then consider $H_2(M_7;\mathbb{Q})^{G_7}$, the subspace of $H_2(M_7;\mathbb{Q})$ that is pointwise fixed by G_7. There is a decomposition of $H_2(M_7;\mathbb{Q})$ as a $\mathbb{Q}[G_7]$-module

$$H_2(M_7;\mathbb{Q}) = \mathbb{Q}\{K_{X_7}\} \oplus \mathbb{E}_7 \otimes \mathbb{Q}$$

and $H_2(M_7;\mathbb{Q})^{G_7} = \mathbb{Q}\{K_{X_7}\}$. Taking intersections with $H_2(M_7;\mathbb{Z})$ on both sides shows that $H_2(M_7;\mathbb{Z})^{G_7} = \mathbb{Z}\{K_{X_7}\}$.

2. The restriction of Q_{M_8} to $\mathbb{Z}\{K_{X_8}\}$ is unimodular so there is an orthogonal decomposition of $H_2(M_8;\mathbb{Z})$ as a $\mathbb{Z}[G_8]$-module as

$$H_2(M_8;\mathbb{Z}) \cong \mathbb{E}_8 \oplus \mathbb{Z}\{K_{X_8}\} \cong C^{\oplus 8} \oplus \mathbb{Z}.$$

In both cases, $H_2(M_n;\mathbb{Z})^{G_n} = \mathbb{Z}\{K_{X_n}\}$. If $g = [\gamma]$ or $g = [\beta]$ is reducible, there are two possibilities:

1. There exists some $c \in H_2(M_n;\mathbb{Z})$ such that $Q_{M_n}(c,c) = -1$ and $g(c) = \pm c$. If $g(c) = c$ then $c = aK_{X_n}$ for some $a \in \mathbb{Z}$. However,

$$Q_{M_n}(aK_{X_n}, aK_{X_n}) = a^2(9-n) \geq 0.$$

Because $Q_{M_n}(c,c) = -1$, this is a contradiction. If $g(c) = -c$ then $c \in \mathbb{E}_n$. This is a contradiction because $\mathbb{E}_n$ is an even lattice.

2. There exist some $c_1, c_2 \in H_2(M_n;\mathbb{Z})$ such that $Q_{M_n}(c_k, c_\ell) = -\delta_{k\ell}$ and $g(c_1) = c_2$. Then $c_1 + c_2 \in H_2(M_n;\mathbb{Z})^{G_n}$, meaning that $c_1 + c_2 = aK_{X_n}$ for some $a \in \mathbb{Z}$ and

$$Q_{M_n}(aK_{X_n}, aK_{X_n}) = a^2(9-n) \geq 0.$$

Because $Q_{M_n}(c_1 + c_2, c_1 + c_2) = -2$, this is a contradiction. □

3.4 Involutions in $\mathrm{Mod}^+(M_n)$ for $5 \leq n \leq 8$

With the discussion of conjugacy classes of involutions in the plane Cremona group above, we are ready to continue analyzing the cases $5 \leq n \leq 8$. The next lemma gives a criterion for reducibility of mapping classes $g \in \mathrm{Mod}(M_n)$.

Lemma 3.10. *Let $g \in \mathrm{Mod}(M_n)$. Then g is reducible in the following cases:*

1. *if $n \leq 7$ and there exists $c \in H_2(M_n;\mathbb{Z})$ with $Q_{M_n}(c,c) = 1$ such that $g(c) = c$;*

2. *if $n \leq 8$ and there exists $c_1, c_2 \in H_2(M_n;\mathbb{Z})$ with $Q_{M_n}(c_k, c_\ell) = 1 - \delta_{k\ell}$ such that $g(c_1) = c_2$ or $g(c_k) = c_k$ for $k = 1, 2$;*

3. *if $n \leq 9$ and there exists $c \in H_2(M_n;\mathbb{Z})$ with $Q_{M_n}(c,c) = -1$ such that $g(c) = c$.*

Proof. 1. The restriction of the intersection form Q_{M_n} to $\mathbb{Z}\{c\}^\perp$ is unimodular and negative-definite by [GS99, Lemma 1.2.12]. For $n \le 7$, there is only one unimodular and negative-definite symmetric form of rank n, and so there is an isometry

$$\iota : H_2(\mathbb{CP}^2;\mathbb{Z}) \oplus H_2(\#n\overline{\mathbb{CP}^2};\mathbb{Z}) \to H_2(M_n;\mathbb{Z})$$

such that $\iota(H) = c$ and the image of $H_2(\#n\overline{\mathbb{CP}^2};\mathbb{Z})$ under ι is $\mathbb{Z}\{c\}^\perp$. Then g is contained in the image of ι_* because g preserves $\mathbb{Z}\{c\}^\perp$ and $\mathbb{Z}\{c\}$. Therefore, g is reducible.

2. The restriction of the intersection form Q_{M_n} to $\mathbb{Z}\{c_1, c_2\}^\perp$ is unimodular and negative-definite with rank $n-1$ by [GS99, Lemma 1.2.12]. Because $n-1 \le 7$, there is only one unimodular and negative-definite symmetric form of rank $n-1$. Therefore, g is reducible by the same reasoning as the proof of (1).

3. The restriction of the intersection for Q_{M_n} to $\mathbb{Z}\{c\}^\perp$ is unimodular and indefinite with signature $(1, n-1)$ by [GS99, Lemma 1.2.12]. If $n \ne 2$ then the signature $\sigma(Q_{M_n}|_{\mathbb{Z}\{c\}^\perp}) = 2 - n$ is not divisible by 8, so the lattice $(\mathbb{Z}\{c\}^\perp, Q_{M_n}|_{\mathbb{Z}\{c\}^\perp})$ is odd by [GS99, Lemma 1.2.20]. There is an isometry

$$\iota : H_2(M_{n-1};\mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z}) \to H_2(M_n;\mathbb{Z})$$

such that the image of $H_2(M_{n-1};\mathbb{Z})$ under ι is $\mathbb{Z}\{c\}^\perp$ and the image of $H_2(\overline{\mathbb{CP}^2};\mathbb{Z})$ under ι is $\mathbb{Z}\{c\}$ by [GS99, Theorem 1.2.21]. If $n = 2$ then the signature $\sigma(Q_{M_n}|_{\mathbb{Z}\{c\}^\perp}) = 0$. So $(\mathbb{Z}\{c\}^\perp, Q_{M_n}|_{\mathbb{Z}\{c\}^\perp}) \cong (H_2(M;\mathbb{Z}), Q_M)$ for $M = M_*$ or M_1 because these are the only two indefinite lattices of rank 2. There is an isometry

$$\iota : H_2(M;\mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z}) \to H_2(M_2;\mathbb{Z})$$

such that the image of $H_2(M;\mathbb{Z})$ under ι is $\mathbb{Z}\{c\}^\perp$ and the image of $H_2(\overline{\mathbb{CP}^2};\mathbb{Z})$ under ι is $\mathbb{Z}\{c\}$ by [GS99, Theorem 1.2.21].

Therefore, g is contained in the image of ι_*, so g is reducible. □

One way to determine all conjugacy classes of $W_5 = W(D_5)$ of order 2 is to consult [Car72, Table 3], but we apply [Car72, Lemma 5] instead. To do so, we consider the *roots* of $\mathbb{E}_n$. For each $5 \le n \le 8$, let

$$\mathcal{R}_n := \{e \in \mathbb{E}_n : Q_{M_n}(e, e) = -2\}.$$

Then $\mathcal{R}_n$ is finite by [Dol12, Proposition 8.2.7]. One can check that $\mathcal{R}_n$ is a root system for the Euclidean space $\mathbb{E}_n \otimes \mathbb{R}$ with the bilinear form $-Q_{M_n}$ extended $\mathbb{R}$-linearly. Any element of $\mathcal{R}_n$ is called a *root* of $\mathbb{E}_n$. For $n = 5$, we first determine the maximal set of mutually orthogonal roots of $\mathbb{E}_5$, up to W_5-action.

Lemma 3.11. *Up to W_5-action and up to sign, the unique maximal set of mutually orthogonal roots of $\mathbb{E}_5$ is*

$$S = \{H - E_1 - E_2 - E_3,\ H - E_1 - E_4 - E_5,\ E_2 - E_3,\ E_4 - E_5\}. \tag{1}$$

Proof. Let S be a maximal set of mutually orthogonal roots of $\mathbb{E}_5$. By [Dol12, Proposition 8.2.7], the roots of $\mathbb{E}_5$ are of the form $E_i - E_j$ and $\pm(H - E_i - E_j - E_k)$ for i, j, k distinct. The group W_5 acts transitively on the roots by [Dol12, Proposition 8.2.17], so we may assume that $\alpha_1 = H - E_1 - E_2 - E_3 \in S$.

1. Suppose $\alpha_2 = H - E_i - E_j - E_k \in S$ with $\alpha_1 \ne \alpha_2$. Because $Q_{M_5}(\alpha_1, \alpha_2) = 0$, up to relabeling the vectors E_i, we may assume that $\alpha_2 = H - E_1 - E_4 - E_5$. No other roots of the form $H - E_i - E_j - E_k$ are orthogonal to both α_1 and α_2.

   If $\alpha_3 = E_i - E_j \in S$, then $\{i, j\} = \{2, 3\}$ or $\{4, 5\}$. Since $E_2 - E_3$ and $E_4 - E_5$ are orthogonal, we see that the set S as given in (1) is the unique maximal set containing multiple roots of the form $H - E_1 - E_2 - E_3$.

2. Suppose there are no other roots of the form $H - E_i - E_j - E_k \in S$. If $E_i - E_j \in S$, either $\{i,j\} = \{4,5\}$ of $\{i,j\} \subseteq \{1,2,3\}$. Without loss of generality, we may assume that $E_2 - E_3, E_4 - E_5 \in S$. No other roots of the form $E_i - E_j$ are orthogonal to all elements of S. This set S is then contained in the maximal set given in (1). □

Proposition 3.12. *There is exactly one conjugacy class of irreducible involutions in* $\mathrm{Mod}^+(M_5)$*, and the elements of this conjugacy class are realized by de Jonquiéres involutions of (algebraic) degree* 3.

Proof. The group W_5 is the Weyl group $W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$. Consider various subsets I of the maximal mutually orthogonal set of S as in (1), up to W_5-orbits. For each $I \subseteq S$, consider $g = \prod_{\alpha \in I} \mathrm{Ref}_\alpha \in W_5$.

1. If $I = S$ then

$$g = (\mathrm{Ref}_{H-E_1-E_2-E_3} \circ \mathrm{Ref}_{E_2-E_3}) \circ (\mathrm{Ref}_{H-E_1-E_4-E_5} \circ \mathrm{Ref}_{E_4-E_5}).$$

By Lemmas 3.6 and 3.8, g is irreducible and realized by de Jonquiéres involutions of degree 3.

2. If $I = \{H - E_1 - E_2 - E_3,\ H - E_1 - E_4 - E_5\}$ then $g(c) = c$ with $c = 2H - E_1 - E_3 - E_5$ because $c \in \mathbb{Z}\{I\}^\perp$. Because $Q_{M_5}(c,c) = 1$, Lemma 3.10(1) implies that g is reducible.

3. If $I \subseteq \{H - E_1 - E_2 - E_3,\ E_2 - E_3,\ E_4 - E_5\}$, then g is in the image of the standard inclusion

$$\iota_* : \mathrm{Mod}(M_3) \times \mathrm{Mod}(\#2\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}(M_5).$$

Therefore, g is reducible.

4. If $I = \{H - E_1 - E_2 - E_3,\ H - E_1 - E_4 - E_5,\ E_2 - E_3\}$ then consider

$$\alpha_1 := H - E_5 \in \mathbb{Z}\{H - E_1 - E_4 - E_5,\ E_2 - E_3\}^\perp.$$

Let $\alpha_2 := 2H - E_1 - E_2 - E_3 - E_5$ and compute that

$$g(\alpha_1) = \mathrm{Ref}_{H-E_1-E_2-E_3}(\alpha_1) = \alpha_2.$$

By Lemma 3.10(2) , g is reducible because $g(\alpha_1) = \alpha_2$.

Any element of order 2 in W_5 can be written as a product of reflections about mutually orthogonal roots by [Car72, Lemma 5]. Hence, we have shown that there is a unique irreducible conjugacy class of order 2 in W_5 and this class is realized by a de Jonquiéres involution of degree 3. Corollary 2.8 then implies that this is the only irreducible conjugacy class of order 2 in $\mathrm{Mod}^+(M_5)$. □

For the rest of this paper, we use the list of conjugacy classes of each W_n given in [Car72]. The classification of [Car72] is stated in terms of a graph Γ (called *Carter graph*) that one can associate to each conjugacy class C of W_n (cf. [Car72, p. 6]). We briefly describe the Carter graph of a conjugacy class C of order 2 here:

First, let $g \in W_n$ be any element and let C be its conjugacy class. The graph Γ of C depends a priori on a factorization $g = w_1 w_2$ where $w_1, w_2 \in W_n$ each have order 2. To construct Γ, consider a set of roots $\{v_i : 1 \le i \le k+h\} \subseteq \mathbb{E}_n$ such that

$$w_1 = \prod_{i=1}^{k} \mathrm{Ref}_{v_i}, \qquad w_2 = \prod_{i=k+1}^{k+h} \mathrm{Ref}_{v_i}$$

and

1. $V_1 \cap V_2 = 0$, where $V_i \subseteq \mathbb{E}_n \otimes \mathbb{R}$ denotes the (-1)-eigenspace of w_i for $i = 1, 2$,

2. the roots in $R_1 := \{v_1, \ldots, v_k\}$, which span V_1, are mutually orthogonal (with respect to $\mathbb{R}$-bilinear extension of Q_{M_n} restricted to $\mathbb{E}_n$), and
3. the roots in $R_2 := \{v_{k+1}, \ldots, v_{k+h}\}$, which span V_2, are mutually orthogonal.

The existence of such a factorization of g is guaranteed by [Car72, Proposition 38, Corollary (ii)]. Finally, let Γ be the graph with vertex set $\{v_1, \ldots, v_{k+h}\}$ and with $\left(\frac{4Q_{M_n}(v_i,v_j)^2}{Q_{M_n}(v_i,v_i)Q_{M_n}(v_j,v_j)}\right)$-many edges between the vertices v_i and v_j.

Now assume that g has order 2 and consider any factorization $g = w_1w_2$ as above. Because w_1 and w_2 each have order 2 and commute, they are simultaneously diagonalizable in $\mathrm{GL}(\mathbb{E}_n \otimes \mathbb{R})$. Hence there is an orthogonal decomposition $V_2 \cong (V_2 \cap V_1) \oplus (V_2 \cap V_1^{\perp})$ since $V_1^{\perp}$ is precisely the (1)-eigenspace of w_1 in $\mathbb{E}_n \otimes \mathbb{R}$. Because $V_2 \cap V_1 = 0$, there is an inclusion $V_2 \subseteq V_1^{\perp}$. Finally this shows that the roots in $R_1 \cup R_2$ are mutually orthogonal. Moreover, the number of roots in $R_1 \cup R_2$ is equal to the dimension of the (-1)-eigenspace of g acting on $\mathbb{E}_n \otimes \mathbb{R}$.

Therefore in the case of conjugacy classes C of order 2, Carter's construction is independent of the choices of w_i and the factorization of each w_i into reflections about mutually orthogonal roots. So if $g \in C$ is a product of reflections Ref_{v_k} about mutually orthogonal roots v_k with $1 \leq k \leq m$ then the unique Carter graph of C is $\Gamma = (A_1)^m$ which has m vertices and no edges. This is the Dynkin diagram of the Weyl subgroup $\langle \mathrm{Ref}_{v_1}, \ldots, \mathrm{Ref}_{v_m} \rangle \cong W(A_1)^m$ of W_n. Moreover, m is the dimension of the (-1)-eigenspace of g acting on $\mathbb{E}_n \otimes \mathbb{R}$.

Throughout the rest of this section, we use the notation

$$\alpha_{ijk} := H - E_i - E_j - E_k \in H_2(M_n; \mathbb{Z})$$

for $1 \leq i, j, k \leq n$ distinct.

Lemma 3.13. *Any element $g \in \mathrm{Mod}^+(M_6)$ of order 2 is reducible.*

Proof. Consider the set

$$S = \{\alpha_{123},\, \alpha_{145},\, E_2 - E_3,\, E_4 - E_5\}$$

of four mutually orthogonal roots of $\mathbb{E}_6$. According to [Car72, Table 9], the conjugacy classes of order 2 are in bijection with the graphs $\Gamma = (A_1)^m$ with $1 \leq m \leq 4$. Therefore, such conjugacy classes are represented by elements of the form $\prod_{\alpha \in I} \mathrm{Ref}_\alpha$ for some $I \subseteq S$. All such involutions g satisfy $g(E_6) = E_6$, making them reducible by Lemma 3.10(3) . ☐

Proposition 3.14. *There are two conjugacy classes of irreducible involutions in $\mathrm{Mod}^+(M_7)$ and the elements of these conjugacy classes are realized by de Jonquiéres involutions of (algebraic) degree 4 and Geiser involutions.*

Proof. Consider the set of mutually orthogonal roots

$$S = \{\alpha_{127},\, \alpha_{347},\, \alpha_{567},\, E_1 - E_2,\, E_3 - E_4,\, E_5 - E_6,\, 2H - E_1 - \cdots - E_6\}.$$

According to [Car72, Table 10], the Carter graphs of the conjugacy classes of order 2 are of the form $\Gamma = (A_1)^k$ for some $1 \leq k \leq 7$. Each graph

$$\Gamma = A_1,\, (A_1)^2,\, (A_1)^5,\, (A_1)^6,\, (A_1)^7$$

has a unique associated conjugacy class of order 2 in W_7. Each graph

$$\Gamma = (A_1)^3,\, (A_1)^4$$

has two associated conjugacy classes of order 2 in W_7.

1. The conjugacy classes of $\Gamma = A_1$ and $\Gamma = (A_1)^2$ are represented by $g = \mathrm{Ref}_{E_1-E_2}$ and $g = \mathrm{Ref}_{E_1-E_2} \circ \mathrm{Ref}_{E_3-E_4}$ respectively. In both cases, g is reducible by Lemma 3.10(1) because $g(H) = H$.

2. The conjugacy class of $\Gamma = (A_1)^5$ is represented by $g = \prod_{\alpha \in I} \mathrm{Ref}_\alpha$ with

$$I = \{\alpha_{123},\, \alpha_{145},\, E_2 - E_3,\, E_4 - E_5,\, E_6 - E_7\}.$$

   Then g is in the image of the standard inclusion

$$\iota_* : \mathrm{Mod}^+(M_5) \times \mathrm{Mod}(\#2\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}^+(M_7).$$

   Therefore, g is reducible.

3. The conjugacy class of $\Gamma = (A_1)^6$ is represented by $g = \prod_{\alpha \in I} \mathrm{Ref}_\alpha$ with

$$I = \{\alpha_{127},\, \alpha_{347},\, \alpha_{567},\, E_1 - E_2,\, E_3 - E_4,\, E_5 - E_6\}.$$

   By Lemmas 3.6 and 3.8, g is irreducible and realized by a de Jonquiéres involution of degree 4.

4. The conjugacy class of $\Gamma = (A_1)^7$ is represented by $g = \prod_{\alpha \in S} \mathrm{Ref}_\alpha$. Then g acts by negation on $\mathbb{E}_7$ and is realized by the Geiser involution as described in Section 3.3.2. By Lemma 3.9, g is irreducible.

The remaining two cases are $\Gamma = (A_1)^3$ and $(A_1)^4$.

1. There are two conjugacy classes of order 2 associated to $\Gamma = (A_1)^3$. Consider the two elements

$$h_1 = \mathrm{Ref}_{\alpha_{127}} \circ \mathrm{Ref}_{\alpha_{347}} \circ \mathrm{Ref}_{\alpha_{567}},$$
$$h_2 = \mathrm{Ref}_{E_1-E_2} \circ \mathrm{Ref}_{E_3-E_4} \circ \mathrm{Ref}_{E_5-E_6}.$$

   Let $\alpha_1 = 2H - E_1 - E_3 - E_5 - E_7$ and $\alpha_2 = H - E_7$ and note that $h_1(\alpha_i) = \alpha_i$ for each $i = 1, 2$ because

$$\alpha_1,\, \alpha_2 \in \mathbb{Z}\{\alpha_{127},\, \alpha_{347},\, \alpha_{567}\}^\perp.$$

   Because $Q_{M_7}(\alpha_k, \alpha_\ell) = 1 - \delta_{k\ell}$, Lemma 3.10(2) shows that h_1 is reducible. Moreover, h_2 is reducible by Lemma 3.10(1) because $h_2(H) = H$; the subspace fixed by h_2 is

$$H_2(M_7; \mathbb{Z})^{\langle h_2 \rangle} = \mathbb{Z}\{H,\, E_7\} \oplus \mathbb{Z}\{E_1 + E_2,\, E_3 + E_4,\, E_5 + E_6\}.$$

   Suppose h_1 and h_2 are conjugate in $\mathrm{Mod}(M_7)$, so that there exist some $c_1, c_2 \in H_2(M_7; \mathbb{Z})^{\langle h_2 \rangle}$ such that $Q_{M_7}(c_i, c_j) = Q_{M_7}(\alpha_i, \alpha_j)$ for all i, j. Then

$$c_i = A_i E_7 + \left(\sum_{k=1}^{3} B_{i,k}(E_{2k-1} + E_{2k}) \right) + C_i H$$

   for some $A_i, B_{i,k}, C_i \in \mathbb{Z}$ for $i = 1, 2$ and $k = 1, 2, 3$ with $C_i^2 = A_i^2 + 2\sum_{k=1}^{3} B_{i,k}^2$. Taking both sides mod 2, we see that $C_i \equiv A_i \pmod 2$ for $i = 1, 2$ so that $C_1C_2 - A_1A_2 \equiv 0 \pmod 2$. However, $Q_{M_7}(c_1, c_2) = 1$ and

$$Q_{M_7}(c_1, c_2) = -A_1A_2 + \left(\sum_{k=1}^{3} -2B_{1,k}B_{2,k} \right) + C_1C_2 \equiv -A_1A_2 + C_1C_2 \pmod 2.$$

   This is a contradiction. Therefore, both h_1 and h_2 are reducible and are not conjugate to each other in $\mathrm{Mod}(M_7)$.

2. There are two conjugacy classes of order 2 associated to $\Gamma = (A_1)^4$. Consider the two elements

$$h_1 = \mathrm{Ref}_{E_1-E_2} \circ \mathrm{Ref}_{E_3-E_4} \circ \mathrm{Ref}_{E_6-E_7} \circ \mathrm{Ref}_{H-E_1-E_2-E_5}$$
$$h_2 = (\mathrm{Ref}_{H-E_1-E_2-E_3} \circ \mathrm{Ref}_{E_2-E_3}) \circ (\mathrm{Ref}_{H-E_1-E_4-E_5} \circ \mathrm{Ref}_{E_4-E_5}).$$

Then h_1 and h_2 are in the image of the standard inclusion

$$\iota_* : \mathrm{Mod}^+(M_5) \times \mathrm{Mod}(\#2\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}^+(M_7)$$

because h_1 and h_2 both preserve $\mathbb{Z}\{E_6, E_7\}$. Therefore, h_1 and h_2 are both reducible.

By Lemmas 3.6 and 3.8, the restriction of h_2 to $\iota(H_2(M_5;\mathbb{Z}))$ is irreducible and realizable by a de Jonquiéres involution. Moreover, h_2 restricts to a trivial action on $\iota(H_2(\#2\overline{\mathbb{CP}^2};\mathbb{Z}))$. By Lemma 3.7, there is a decomposition as a $\mathbb{Z}[\langle h_2\rangle]$-module

$$H_2(M_7;\mathbb{Z}) \cong \iota(H_2(M_5;\mathbb{Z})) \circ \iota(H_2(\#2\overline{\mathbb{CP}^2};\mathbb{Z})) \cong \mathbb{Z}[\langle h_2\rangle]^{\oplus 2} \oplus C^{\oplus 2} \oplus \mathbb{Z}^{\oplus 2}$$

where $C \cong \mathbb{Z}$ as a $\mathbb{Z}$-module and h_2 acts by negation on C. On the other hand, the $\mathbb{Z}[\langle h_1\rangle]$-module structure of $H_2(M_7;\mathbb{Z})$ is

$$H_2(M_7;\mathbb{Z}) = \mathbb{Z}\{H-E_1,\, H-E_2\} \oplus \mathbb{Z}\{H-E_1-E_2,\, E_5\} \oplus \mathbb{Z}\{E_3,\, E_4\} \oplus \mathbb{Z}\{E_6,\, E_7\} \cong \mathbb{Z}[\langle h_1\rangle]^{\oplus 4}.$$

If h_1 and h_2 are conjugate in $\mathrm{Mod}(M_7)$ then the $\mathbb{Z}[\langle h_1\rangle]$- and $\mathbb{Z}[\langle h_2\rangle]$-module structures of $H_2(M_7;\mathbb{Z})$ agree. Therefore, h_1 and h_2 are not conjugate in W_7 and the two conjugacy classes of order 2 associated to $\Gamma = (A_1)^4$ are represented by h_1 and h_2.

Therefore, the conjugacy classes of the Carter graphs $\Gamma = (A_1)^6$ and $(A_1)^7$ are the only two irreducible conjugacy classes of order 2 in W_7 and they are realized by a de Jonquiéres involution f of degree 4 and a Geiser involution γ respectively. By Lemmas 3.7 and 3.9,

$$H_2(M_7;\mathbb{Z})^{\langle f\rangle} \cong \mathbb{Z}^2, \qquad H_2(M_7;\mathbb{Z})^{\langle \gamma\rangle} \cong \mathbb{Z}$$

so $[f]$ and $[\gamma]$ are not conjugate in $\mathrm{Mod}(M_7)$. Corollary 2.8 then implies that these are the only two irreducible conjugacy classes of order 2 in $\mathrm{Mod}^+(M_7)$. □

Proposition 3.15. *There is exactly one conjugacy class of irreducible involutions in* $\mathrm{Mod}^+(M_8)$ *and the elements of this conjugacy class are realized by Bertini involutions.*

Proof. According to [Car72, Table 11], the Carter graphs of the conjugacy classes of W_8 of order 2 are of the form $\Gamma = \Gamma(A_1)^k$ for some $1 \le k \le 8$. Each graph

$$\Gamma = A_1,\, (A_1)^2,\, (A_1)^3,\, (A_1)^5,\, (A_1)^6,\, (A_1)^7,\, (A_1)^8$$

has a unique associated conjugacy class of order 2 in W_8. The graph

$$\Gamma = (A_1)^4$$

has two associated conjugacy classes of order 2 in W_8.

1. The conjugacy classes of $\Gamma = (A_1)^k$ for $1 \le k \le 7$ and $k \ne 4$ are represented by $g = \prod_{\alpha\in I} \mathrm{Ref}_\alpha$ for some

$$I \subseteq \{\alpha_{127},\, \alpha_{347},\, \alpha_{567},\, E_1-E_2,\, E_3-E_4,\, E_5-E_6,\, 2H-E_1-\cdots-E_6\}.$$

Then $g(E_8) = E_8$ and therefore g is reducible by Lemma 3.10(3).

2. There are two conjugacy classes of order 2 associated to $\Gamma = (A_1)^4$. Consider

$$h_1 = \mathrm{Ref}_{E_1-E_2} \circ \mathrm{Ref}_{E_3-E_4} \circ \mathrm{Ref}_{E_6-E_7} \circ \mathrm{Ref}_{H-E_1-E_2-E_5}$$
$$h_2 = (\mathrm{Ref}_{H-E_1-E_2-E_3} \circ \mathrm{Ref}_{E_2-E_3}) \circ (\mathrm{Ref}_{H-E_1-E_4-E_5} \circ \mathrm{Ref}_{E_4-E_5}).$$

Both h_1 and h_2 are reducible by Lemma 3.10(3) because $h_i(E_8) = E_8$ for $i = 1, 2$. By the same proof as in the analogous case in Proposition 3.14, h_1 and h_2 are not conjugate in $\mathrm{Mod}(M_8)$. Therefore, the two conjugacy classes of order 2 associated to $\Gamma = (A_1)^4$ are represented by h_1 and h_2.

3. The conjugacy class of $\Gamma = (A_1)^8$ is represented by g which acts by negation on $\mathbb{E}_8$ and fixes $3H - E_1 - \cdots - E_8$. The involution g is realized by the Bertini involution as described in Section 3.3.2. By Lemma 3.9, g is irreducible.

Therefore, there is a unique irreducible conjugacy class of order 2 in W_8 and this class is realized by a Bertini involution. Corollary 2.8 then implies that this class is the only irreducible conjugacy class of order 2 in $\mathrm{Mod}^+(M_8)$. □

We conclude by combining all of the lemmas above to prove Theorem 1.3.

Proof of Theorem 1.3. Lemma 3.3 shows that each involution in $\mathrm{Mod}^+(M_n)$ of $1 \leq n \leq 4$ is reducible. Lemma 3.13 and Propositions 3.12, 3.14, and 3.15 show that the only irreducible involutions $g \in \mathrm{Mod}^+(M_n)$ for $5 \leq n \leq 8$ are those conjugate to the mapping classes of involutions on some $X = \mathrm{Bl}_P\, \mathbb{CP}^2$ induced by de Jonquiéres (of degree $d > 2$), Geiser, and Bertini involutions where P is the set of its base points. Suppose $g \in \mathrm{Mod}^+(M_n)$ is realized by such an automorphism $\tilde{g}$ of X via the diffeomorphism $\varphi : M_n \to X$. For any $f \in \mathrm{Mod}^+(M_n)$, there exists a diffeomorphism $F \in \mathrm{Diff}^+(M_n)$ with $[F] = f$ by Theorem 2.2. Hence $f^{-1}gf \in \mathrm{Mod}^+(M_n)$ is realized by $\tilde{g}$ via the diffeomorphism $\varphi \circ F : M_n \to X$. □

Before considering the extension of Theorem 1.3 to Theorem 1.2, we consider the notion of *minimal pairs* considered by Bayle–Beauville ([BB00]) in their classification of conjugacy classes of involutions in $\mathrm{Cr}(2)$. A pair (S, σ) where S is a rational surface and σ is an involution of S is called *minimal* if any birational morphism $F : S \to S_0$ such that there exists an involution σ_0 of S_0 with $F \circ \sigma = \sigma_0 \circ F$ is an isomorphism. Corollary 1.4 is a reformulation of Theorem 1.3 using this language.

Proof of Corollary 1.4. Let M be a del Pezzo manifold and let $g \in \mathrm{Mod}^+(M)$ be an irreducible mapping class of order 2. By Theorem 1.3, g is realized by an involution σ of a rational surface S diffeomorphic to M. If (S, σ) is not minimal then there exists some smooth rational curve $E \subseteq S$ such that $Q_M([E], [E]) = -1$ satisfying $\sigma(E) = E$ or $E \cap \sigma(E) = \emptyset$ by [BB00, Lemma 1.1]. In both cases, $M = M_n$ for some $1 \leq n \leq 8$. In the first case, $[\sigma]$ is reducible by Lemma 3.10(3). In the second case, $M = M_n$ for some $5 \leq n \leq 8$ by Theorem 1.3. Note that $\mathbb{Z}\{[E], \sigma_*([E])\}$ is a $\mathbb{Z}$-submodule of $H_2(M; \mathbb{Z})$ preserved by $[\sigma]$ to which the restriction of Q_M is unimodular of signature $(0, 2)$. Then $(\mathbb{Z}\{[E], \sigma_*([E])\}^\perp, Q_M)$ is preserved by $[\sigma]$ and is a unimodular lattice of signature $(1, n-2)$; it is isometric to $(H_2(M_{n-2}; \mathbb{Z}), Q_{M_{n-2}})$. Hence $[\sigma]$ is reducible. Therefore, (S, σ) must be minimal if $[\sigma]$ is irreducible.

Now suppose (S, σ) is a minimal pair where S is a rational surface diffeomorphic to some del Pezzo manifold M and σ is an involution of S. All possible pairs (S, σ) are listed in [BB00, Theorem 1.4]; we consider each case (i)-(vi) separately.

(i) There exists a smooth $\mathbb{CP}^1$-fibration $f : S \to \mathbb{CP}^1$ and an involution τ of $\mathbb{CP}^1$ such that $f \circ \sigma = \tau \circ f$. Because S is a geometrically ruled surface, S is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ by Noether–Enriques ([Bea96, Theorem III.4]). Hence S must be isomorphic to a Hirzebruch $\mathbb{F}_m$ for some $m \geq 0$ by [GS99, Theorem 3.4.8]. If $m > 0$ then any complex automorphism σ of S must preserve the unique irreducible curve C of S with self-intersection number $-m$ (given by a section of f) and σ must also fix the homology class $[F]$ of the fiber F of f. Because $[F]$ and $[C]$ span $H_2(S;\mathbb{Z})$, this implies that $[\sigma] = \mathrm{Id} \in \mathrm{Mod}(M)$ so $[\sigma]$ does not have order 2. If $m = 0$ then $S = \mathbb{CP}^1 \times \mathbb{CP}^1$ is diffeomorphic to M_*. Any element of $\mathrm{Mod}(M_*)$ is irreducible by Lemma 3.2.

(ii) There exists a fibration $f : S \to \mathbb{CP}^1$ such that $f \circ \sigma = f$; the smooth fibers of f are diffeomorphic to $\mathbb{CP}^1$ on which σ induces a nontrivial involution and any singular fiber is the union of submanifolds diffeomorphic to $\mathbb{CP}^1$ exchanged by σ meeting at one point.

Suppose f has s-many singular fibers with $s > 0$. By the proof of [BB00, Theorem 1.4], any singular fiber contains an exceptional divisor. Blowing down one of the components (call it e_i for $1 \leq i \leq s$) in each singular fiber yields a geometrically ruled surface $f : S' \to \mathbb{CP}^1$, which is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ by Noether–Enriques ([Bea96, Theorem III.4]). This means that if $S_2 \in H_2(S;\mathbb{Z})$ is the class coming from a fiber of $f : S' \to \mathbb{CP}^1$ then

$$\sigma_*(S_2) = S_2, \qquad \sigma_*(e_i) = S_2 - e_i$$

for all $1 \leq i \leq s$. Because $H_2(S;\mathbb{Z}) = \mathbb{Z}\{S_1, S_2, e_1, \ldots, e_s\}$ where S_1 is the class of a section of f, Lemmas 3.5 and 3.8 imply that $[\sigma] \in \mathrm{Mod}(M_{s+1})$ is irreducible.

If $s = 0$ then the same argument as in case (i) holds.

(iii), (iv) The surface S is isomorphic to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. Lemma 3.2 shows that $[\sigma] \in \mathrm{Mod}(M)$ is irreducible.

(v), (vi) The surface S is a del Pezzo surface of degree 2 or 1 and f is the Geiser or Bertini involution respectively. Lemma 3.9 shows that $[\sigma] \in \mathrm{Mod}(M)$ is irreducible. □

3.5 Extension to $\mathrm{Mod}(M_n)$ for $1 \leq n \leq 8$

In this section we compare the involutions $g \in \mathrm{Mod}(M_n)$ to involutions of $\mathrm{Mod}^+(M_n)$ to prove Theorem 1.2. The following lemma will be used to construct involutions realizing irreducible order 2 elements $g \in \mathrm{Mod}(M_n) - \mathrm{Mod}^+(M_n)$.

Lemma 3.16. *Let P be a finite subset of $4 \leq n \leq 8$ points in general position contained in $\mathbb{RP}^2 \subseteq \mathbb{CP}^2$ and let $\tau_0 : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the map given by complex conjugation of the coordinates. Let $f : \mathrm{Bl}_P \mathbb{CP}^2 \to \mathrm{Bl}_P \mathbb{CP}^2$ be a complex automorphism of order 2 induced by a birational map $f_0 : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ with base points given by P and let $\tau : \mathrm{Bl}_P \mathbb{CP}^2 \to \mathrm{Bl}_P \mathbb{CP}^2$ be the map induced by τ_0. Then τ and f commute.*

Proof. For any polynomial $F \in \mathbb{C}[X,Y,Z]$, write $\overline{F} \in \mathbb{C}[X,Y,Z]$ to denote the polynomial obtained by conjugating the coefficients of F. There exist homogeneous polynomials $F, G, H \in \mathbb{C}[X,Y,Z]$ of degree m such that

$$f_0(q) = [F(q) : G(q) : H(q)]$$

for all $q \notin P$. Then $g_0 := \tau_0 f_0 \tau_0$ is given by

$$g_0(q) = [\overline{F}(q) : \overline{G}(q) : \overline{H}(q)].$$

If $q \in \mathbb{CP}^2$ such that g_0 is not defined at q then

$$\tau_0(0) = \tau_0(\overline{F}(q)) = F(\tau_0(q)).$$

Similarly, $G(\tau_0(q)) = H(\tau_0(q)) = 0$ and so $\tau_0(q) \in P$. Therefore, $q \in P$ because the points of P are fixed by τ_0. This shows that g_0 is birational and lifts to an automorphism g of $\mathrm{Bl}_P \mathbb{CP}^2$.

By construction, $g = \tau f \tau$ as a diffeomorphism of $\mathrm{Bl}_P \mathbb{CP}^2$. The action of g_* on $H_2(\mathrm{Bl}_P \mathbb{CP}^2;\mathbb{Z})$ coincides with the action of f_* because τ_* acts by negation on $H_2(\mathrm{Bl}_P \mathbb{CP}^2;\mathbb{Z})$. Therefore, $f = \tau f \tau$ because the homomorphism $\mathrm{Aut}(\mathrm{Bl}_P \mathbb{CP}^2) \to \mathrm{Aut}(H_2(\mathrm{Bl}_P \mathbb{CP}^2;\mathbb{Z}), Q_{\mathrm{Bl}_P \mathbb{CP}^2})$ is injective ([Dol12, Proposition 8.2.39]). □

We finally extend Theorem 1.3 to prove Theorem 1.2.

Proof of Theorem 1.2. Let $-I \in \operatorname{Mod}(M_n)$ denote the mapping class which acts by negation on $H_2(M_n;\mathbb{Z})$, and let $-g = (-I) \circ g$ for any $g \in \operatorname{Mod}(M_n)$. If g preserves some $\mathbb{Z}$-submodule $N \leq H_2(M_n;\mathbb{Z})$ then $-g$ preserves N as well. Therefore, g is reducible if and only if $-g$ is reducible.

Let $g \in \operatorname{Mod}(M_n)$ be an irreducible element of order 2. If $g \in \operatorname{Mod}^+(M_n)$ then Theorem 1.3 shows that g is realized by a de Jonquiéres (of degree $d > 2$), Geiser, or Bertini involution. If $g \notin \operatorname{Mod}^+(M_n)$ then $-g \in \operatorname{Mod}^+(M_n)$. Theorem 1.3 shows that $-g$ is realized by de Jonquiéres (of degree $d > 2$), Geiser, or Bertini involutions.

1. If $-g$ is realized by Geiser or Bertini involutions then let $X = \operatorname{Bl}_P \mathbb{CP}^2$ where P is a set of n points in general position contained in $\mathbb{RP}^2 \subseteq \mathbb{CP}^2$. Let f be the Geiser or Bertini involution of X and let τ be the diffeomorphism of X induced by complex conjugation on $\mathbb{CP}^2$. By Lemma 3.16, $f \circ \tau$ has order 2 in $\operatorname{Diff}^+(M_n)$. Then $[f \circ \tau] = g$ because $[\tau] = -I$ and $[f] = -g$.
2. If $-g$ is realized by de Jonquiéres involutions then Lemma 3.6 shows that there exist $X = \operatorname{Bl}_P \mathbb{CP}^2$ where P is a set of n points in $\mathbb{RP}^2 \subseteq \mathbb{CP}^2$ and an automorphism $f \in \operatorname{Aut}(X)$ induced by a de Jonquiéres involution that commutes with the anti-biholomorphism τ induced by complex conjugation on $\mathbb{CP}^2$. Therefore, $f \circ \tau$ has order 2 and $[f \circ \tau] = g$. □

4 The smooth Nielsen realization problem for involutions

In this section we describe a construction that we call *complex equivariant connected sums* and use it to prove the smooth Nielsen realization problem for involutions (Corollary 1.5).

4.1 Complex equivariant connected sums

Finding representative diffeomorphisms of a mapping class g of order two has distinct flavors depending on the irreducibility of g. We define *complex equivariant connected sums* in order to realize order 2 reducible mapping classes of del Pezzo manifolds. The definition here is specialized to $G = \mathbb{Z}/2\mathbb{Z}$ and is a special case of *equivariant connected sums* which appear in [HT04, (1.C)]. For a more general description, also see [Lee21, Section 2.2].

Let N_1, N_2 be smooth manifolds and let $G = \mathbb{Z}/2\mathbb{Z}$. Fix a G-invariant Riemannian metric on both N_1 and N_2. Consider diffeomorphisms $g_i \in \operatorname{Diff}^+(N_i)$ of order two for $i = 1, 2$. Suppose there are points $p_i \in N_i$ for $i = 1, 2$ such that p_i is fixed by g_i and the tangent representations $G_i \to \operatorname{SO}(T_{p_i} N_i)$ are equivalent by an orientation-reversing isomorphism $\rho : T_{p_1} N_1 \to T_{p_2} N_2$. By the equivariant tubular neighborhood theorem ([Bre73, Theorem VI.2.2]), there exist G-invariant neighborhoods of $p_i \in N_i$ for each $i = 1, 2$ which are G-equivariantly diffeomorphic to $T_{p_i} N_i$. We can now form as usual a connected sum $N_1 \# N_2$ by taking the G-equivariant neighborhoods of p_1 and p_2 in $N_1 - p_1$ and $N_2 - p_2$ respectively and equivariantly identifying concentric annuli around p_1 and p_2 via the orientation-reversing map ρ. Then the connected sum $N_1 \# N_2$ has a natural smooth action of G. The G-manifold $(N_1 \# N_2, G)$ is called an *equivariant connected sum*. See Figure 4 for an illustration.

Consider $G \times N_2$. Suppose there exist points $p_1 \in N_1$ which is not fixed by g_1 and any $p_2 \in N_2$. Similarly as in the first case, the G-equivariant identification of the neighborhoods of the points in the G-orbit $\{p_1, g_1(p_1)\}$ of $p_1 \in N_1$ and the neighborhoods of the points $G \times \{p_2\}$ in $G \times N_2$ is denoted $(N_1 \#(G \times N_2), G)$ and is also called an *equivariant connected sum*. See Figure 1 for an illustration.

With these definitions in mind, we define a *complex equivariant connected sum*.

Definition 4.1. Let M be a smooth, oriented manifold and let $G \cong \mathbb{Z}/2\mathbb{Z} \leq \operatorname{Diff}^+(M)$. The pair (M, G) is called a *complex equivariant connected sum* if one of the following holds:

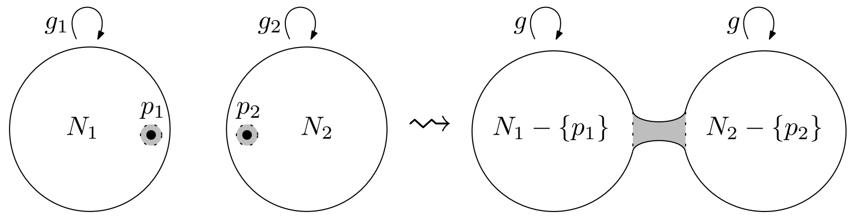

Figure 4: The equivariant connected sum $(N_1 \# N_2, G)$ where $G = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

1. (M, G) is G-equivariantly diffeomorphic to (N, G) or $(\overline{N}, G)$ where N is a complex manifold and $\overline{N}$ is the same manifold with the opposite orientation; each $g \in G \leq \mathrm{Diff}^+(N)$ is biholomorphic or anti-biholomorphic,
2. (M, G) is G-equivariantly diffeomorphic to an equivariant connected sum $(N_1 \# N_2, G)$ where (N_1, G) and (N_2, G) are complex equivariant connected sums, or
3. (M, G) is G-equivariantly diffeomorphic to an equivariant connected sum $(N_1 \# (\mathbb{Z}/2\mathbb{Z}) \times N_2, G)$ where (N_1, G) is a complex equivariant connected sum.

If $G_0 \leq \mathrm{Mod}(M)$ is a finite group such that there exists a complex equivariant connected sum (M, G) and $G \leq \mathrm{Diff}^+(M)$ is a lift of G_0 under the quotient $\pi : \mathrm{Homeo}^+(M) \to \mathrm{Mod}(M)$ then we say that G_0 is *realizable by a complex equivariant connected sum*.

The following lemma is used in realizing reducible mapping classes of order 2.

Lemma 4.2. *Let M and N be smooth 4-manifolds and let $f_M \in \mathrm{Diff}^+(M)$ and $f_N \in \mathrm{Diff}^+(N)$ be diffeomorphisms of order 2 fixing real surfaces $S_M \subseteq M$ and $S_N \subseteq N$ respectively. There is an equivariant connected sum $(M \# N, \langle f \rangle)$ where $\langle f \rangle \cong \mathbb{Z}/2\mathbb{Z}$ such that $f|_{M-p\subseteq M\#N} = f_M$ and $f|_{N-q\subseteq M\#N} = f_N$ with $p \in S_M$ and $q \in S_N$. Moreover, f fixes a real surface in $M \# N$.*

Proof. Fix f_M- and f_N-invariant metrics on M and N. For any $p \in S_M$, the action of $d(f_M)_p$ on T_pM fixes $T_pS_M \subseteq T_pM$ and acts by negation on $T_pS_M^{\perp} \subseteq T_pM$. Similarly, the action of $d(f_N)_q$ on T_qN fixes $T_qS_N \subseteq T_qN$ and acts by negation on $T_qS_N^{\perp} \subseteq T_qN$ for all $q \in S_N$. There is an orientation-reversing isomorphism $\varphi : T_qN \to T_pM$ taking T_qS_N to T_pS_M in an orientation-reversing way and taking $T_qS_N^{\perp}$ to $T_pS_M^{\perp}$ in an orientation-preserving way. By construction, $\langle f_M \rangle \to \mathrm{SO}(T_pM)$ and $\langle f_N \rangle \to \mathrm{SO}(T_qN)$ are equivalent by the orientation-reversing isomorphism φ which forms the equivariant connected sum $(M \# N, \mathbb{Z}/2\mathbb{Z})$. Moreover, $S_M \# S_N \subseteq M \# N$ is a real surface fixed by the resulting smooth $\mathbb{Z}/2\mathbb{Z}$-action. □

4.2 The proof of Corollary 1.5

Let M be a del Pezzo manifold. Throughout this section, we say that $\mathrm{Id} \in \mathrm{Mod}(M)$ is *realizable by an order 2 complex equivariant connected sum* if there exists a complex equivariant connected sum (M, G) and $G \cong \mathbb{Z}/2\mathbb{Z} \leq \mathrm{Diff}^+(M)$ such that $G \leq \ker(\pi)$, where π denotes the usual quotient $\mathrm{Homeo}^+(M) \to \mathrm{Mod}(M)$. Note that in this case, G is not a lift of its image $\pi(G) = \mathrm{Id} \leq \mathrm{Mod}(M)$.

The following lemma forms the base case of the inductive proof of Corollary 1.5.

Lemma 4.3. *Let $M = M_0$ or M_*. Any element $g \in \mathrm{Mod}(M)$ is realizable by order 2 complex equivariant connected sum fixing a real surface.*

Proof. Note that $\mathrm{Mod}(M_0) = \mathrm{Mod}(\mathbb{CP}^2) \cong \{\pm \mathrm{Id}\}$, where the nontrivial element acts on $H_2(\mathbb{CP}^2; \mathbb{Z})$ by negation. Then $[f_-] = -\mathrm{Id} \in \mathrm{Mod}(\mathbb{CP}^2)$ where f_- is the involution $[X : Y : Z] \mapsto [\overline{X} : \overline{Y} : \overline{Z}]$ given by complex conjugation and fixes a real surface in M_0. Moreover, $[f_+] = \mathrm{Id} \in \mathrm{Mod}(\mathbb{CP}^2)$ where f_+ is the involution $[X : Y : Z] \mapsto [-X : Y : Z]$ which also fixes a real surface in M_0.

Note that $\mathrm{Mod}(M_*) = \langle c_1, c_2 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ where

$$c_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the $\mathbb{Z}$-basis (S_1, S_2) of $H_2(M_*; \mathbb{Z})$. Define

$$f_{c_1}([X:Y],[W:Z]) = ([\overline{X}:\overline{Y}],[\overline{W}:\overline{Z}]), \qquad f_{c_2}([X:Y],[W:Z]) = ([W:Z],[X:Y]).$$

The group $\langle f_{c_1}, f_{c_2} \rangle \leq \mathrm{Diff}^+(M_*)$ is a lift of $\mathrm{Mod}(M_*)$ under the quotient map $\pi : \mathrm{Homeo}^+(M_*) \to \mathrm{Mod}(M_*)$ with $\pi(f_{c_i}) = c_i$ for each $i = 1, 2$. It is straightforward to check that all nontrivial elements of $\langle f_{c_1}, f_{c_2} \rangle$ fix a real surface in M_*. The identity element is realized by $f_0 : M_* \to M_*$ where $f_0([X:Y],[W:Z]) = ([-X:Y],[W:Z])$ which fixes a real surface in M_*. □

The inductive step is handled by the lemma below. The proof is straightforward but included for the sake of completeness.

Lemma 4.4. *Fix $n \geq 1$. Suppose any $h \in \mathrm{Mod}(M_k)$ of order dividing 2 is realizable by an order 2 complex equivariant connected sum fixing a real surface for all $0 \leq k < n$ and $k = *$. If $g \in \mathrm{Mod}(M_n)$ is a reducible element of order dividing 2, then g is realizable by an order 2 complex equivariant connected sum fixing a real surface.*

Proof. Suppose g is contained in the image of a standard inclusion

$$\iota_* : \mathrm{Mod}(M_k) \times \mathrm{Mod}(\#\ell\overline{\mathbb{CP}^2}) \hookrightarrow \mathrm{Mod}(M_n)$$

for some $k \leq n-1$ and $\ell = n-k$ or $k = *$ and $\ell = n-1$. Suppose $g = \iota_*(h_1, h_2)$ with $(h_1, h_2) \in \mathrm{Mod}(M_k) \times \mathrm{Mod}(\#\ell\overline{\mathbb{CP}^2})$. Up to conjugacy in $\mathrm{Mod}(\#\ell\overline{\mathbb{CP}^2})$, any $h_2 \in \mathrm{Mod}(\#\ell\overline{\mathbb{CP}^2})$ of order dividing 2 satisfies:

$$h_2 : E_i \mapsto E_{j_i} \text{ or } \pm E_i$$

for all $1 \leq i \leq \ell$ and some $j_i \neq i$. Because h_2 preserves $\mathbb{Z}\{E_i, E_{j_i}\}$ or $\mathbb{Z}\{E_i\}$, we may assume that $\ell = 2$ or 1 respectively. Without loss of generality, suppose $h_2(E_1) = E_2$ or $h_2(E_1) = \pm E_1$.

Let $(M_k, \langle h \rangle)$ be an order 2 complex equivariant connected sum (where h is a diffeomorphism of M_k fixing a real surface $S \subseteq M_k$) such that $[h] = h_1$.

1. Suppose $\ell = 1$. If $h_2(E_1) = E_1$ then let $f : \overline{\mathbb{CP}^2} \to \overline{\mathbb{CP}^2}$ with $f : [X:Y:Z] \mapsto [-X:Y:Z]$. If $h_2(E_1) = -E_1$ then let $f : \overline{\mathbb{CP}^2} \to \overline{\mathbb{CP}^2}$ with $f : [X:Y:Z] \mapsto [\overline{X}:\overline{Y}:\overline{Z}]$. In either case, f fixes a real surface in $\overline{\mathbb{CP}^2}$. There is a complex equivariant connected sum $(M_k\#\overline{\mathbb{CP}^2}, \mathbb{Z}/2\mathbb{Z})$ fixing a real surface realizing (h_1, h_2) by Lemma 4.2.

2. Suppose $\ell = 2$ and $h_2(E_1) = E_2$. Then $(M_k\#((\mathbb{Z}/2\mathbb{Z}) \times \overline{\mathbb{CP}^2}), \mathbb{Z}/2\mathbb{Z})$ gives the desired complex equivariant connected sum.

Any reducible $g \in \mathrm{Mod}(M_n)$ of order dividing 2 is conjugate to some $g_0 \in \mathrm{Mod}(M_n)$ contained in the image of a standard inclusion ι_*; let $g = f^{-1}g_0 f$ for some $f \in \mathrm{Mod}(M_n)$. By Theorem 2.2, there exists a diffeomorphism $F \in \mathrm{Diff}^+(M_n)$ with $[F] = f$. If (M_n, G) is a complex equivariant connected sum realizing g_0 then $(M_n, F^{-1}GF)$ is a complex equivariant connected sum realizing g. □

With the inductive step in hand, we prove the smooth Nielsen realization problem for involutions on del Pezzo manifolds.

Proof of Corollary 1.5. We will show that for any del Pezzo manifold M, any $g \in \mathrm{Mod}(M)$ of order dividing 2 is realized by a complex equivariant connected sum of order 2 fixing a real surface.

The claim holds for $M = M_*$ and M_0 by Lemma 4.3. Fix $1 \leq n \leq 8$ and suppose that the claim holds for $M = M_k$ for all $0 \leq k < n$. We will prove the claim for $M = M_n$.

Let $g \in \mathrm{Mod}(M_n)$ be an element of order dividing 2. If g is reducible then g is realized by a complex equivariant connected sum of order 2 fixing a real surface by Lemma 4.4.

Suppose g is irreducible. If $g \in \mathrm{Mod}^+(M)$ then Theorem 1.2 shows that g is realized by a complex automorphism of some $\mathrm{Bl}_P\,\mathbb{CP}^2 \cong M$ induced by de Jonquiéres, Geiser, or Bertini involutions. All such automorphisms fix a complex curve in $\mathrm{Bl}_P\,\mathbb{CP}^2$. If $g \notin \mathrm{Mod}^+(M)$ then Theorem 1.2 shows that g is realized by some anti-biholomorphism f of order 2 of a complex surface $\mathrm{Bl}_P\,\mathbb{CP}^2 \cong M$ and $-g$ is represented by an automorphism of $\mathrm{Bl}_P\,\mathbb{CP}^2$ induced by a de Jonquiéres, Geiser, or Bertini involution.

To show that f fixes a real surface in M, we apply the Hirzebruch G-signature theorem ([HZ74, Section 9.2, (12)]) which says that if f_0 is a smooth involution of M then

$$2\sigma(M/\langle f_0\rangle) = \sigma(M) + \sum_C \mathrm{def}_C$$

where

1. $\sigma(M/\langle f_0\rangle)$ is the signature of the restriction of Q_M to the fixed subspace of $H_2(M;\mathbb{R})$ under $(f_0)_*$ (cf. [HZ74, Section 2.1, (22)]),
2. $\sigma(M)$ is the signature of the 4-manifold M, and
3. the sum $\sum_C \mathrm{def}_C$ is taken over the 2-dimensional components of the fixed set of f_0 and def_C denotes the quantity called the *defect* of C. To be precise, the statement of the Hirzebruch G-signature theorem also involves defects def_p associated to isolated fixed points p. However, $\mathrm{def}_p = 0$ for all isolated fixed points p when f_0 has order 2. See [HZ74, Section 9.2] or [Lee21, Remark 4.4] for more details.

We compute $2\sigma(M/\langle f\rangle)$ and $\sigma(M)$ in each of the three cases.

1. Suppose $-g \in \mathrm{Mod}(M_n)$ is represented by a de Jonquiéres involution and $n = 5$ or 7. By Lemma 3.7, the $\mathbb{Z}[\langle g\rangle]$-module structure of $H_2(M_n;\mathbb{R})$ is isomorphic to

$$H_2(M_n;\mathbb{R}) \cong C^{\oplus 2} \oplus \mathbb{R}^{\oplus n-1}$$

where $C \cong \mathbb{R}$ as an $\mathbb{R}$-vector space and g acts by negation. Moreover, this decomposition must be orthogonal and

$$C^{\oplus 2} = \mathbb{R}\{S_2,\, 2S_1 - e_1 - \cdots - e_{n-1}\}.$$

With respect to this basis, the restriction of Q_{M_n} to $C^{\oplus 2}$ is

$$Q_{M_n}|_{C^{\oplus 2}} = \begin{pmatrix} 0 & 2 \\ 2 & -(n-1) \end{pmatrix}$$

which has signature 0. The signature of a direct sum of orthogonal subspaces is the sum of the respective signatures, meaning that

$$1 - n = \sigma(M_n) = \sigma(C^{\oplus 2}) + \sigma(H_2(M_n;\mathbb{R})^{\langle g\rangle}) = \sigma(M_n/\langle f\rangle)$$

The G-signature theorem implies that

$$\sum_C \mathrm{def}_C = 2\sigma(M_n/\langle f\rangle) - \sigma(M_n) = 1 - n \neq 0.$$

Therefore, there exist real surfaces $C \subseteq M_n$ fixed by f.

2. Suppose $-g \in \mathrm{Mod}(M_n)$ is represented by a Geiser or Bertini involution and $n = 7$ or 8 respectively. There is an orthogonal decomposition

$$H_2(M_n; \mathbb{R}) = \mathbb{R}\{K_{X_n}\} \oplus \mathbb{E}_n \otimes \mathbb{R};$$

here, $-g$ acts by negation on $\mathbb{E}_n$ and fixes $\mathbb{R}\{K_{X_n}\}$. Therefore,

$$H_2(M_n; \mathbb{R})^{\langle g \rangle} = \mathbb{E}_n \otimes \mathbb{R}.$$

The restriction of Q_{M_n} to $\mathbb{E}_n$ is negative-definite so $\sigma(M_n/\langle f \rangle) = -n$. The G-signature theorem implies that

$$\sum_C \mathrm{def}_C = 2\sigma(M_n/\langle f \rangle) - \sigma(M_n) = -n - 1 \neq 0.$$

Therefore, there exist real surfaces $C \subseteq M_n$ fixed by f.

Therefore, any $g \in \mathrm{Mod}(M_n)$ of order dividing 2 is realized by a complex equivariant connected sum of order 2 fixing a real surface. The corollary now follows by induction on n. □

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