Finite-order mapping classes of del Pezzo surfaces

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math.uchicago.edu/~seraphina/gatech.pdf

Finite group actions on M (closed, oriented).

$$q : \mathrm{Homeo}^+(M) \to \mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}^+(M))$$

Question: For $g \in \mathrm{Mod}(M)$ of order $n < \infty$, does there exist $f \in \mathrm{Homeo}^+(M)$ of order n such that $q(f) = [f] = g$?

$\dim_{\mathbb{R}} 2$: Yes! Nielsen realization thm (Nielsen '43)
$\dim_{\mathbb{C}} 1$

What about for higher dimensional complex manifolds?

$\dim_{\mathbb{C}} = 2$

$\dim_{\mathbb{R}} = 4.$

Easy Examples:

$M = \mathbb{CP}^2$

$$\left(\sigma : [X : Y : Z] \mapsto [-X : Y : Z]\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in PGL_3\mathbb{C}.$$

σ isotopic to Id b/c $PGL_3\mathbb{C}$ connected!

$$\tau : [X : Y : Z] \mapsto [\bar{X} : \bar{Y} : \bar{Z}]$$

complex conjugation of coordinates

τ not isotopic to Id $\curvearrowright H_2$.

$\tau_* = -\mathrm{Id}$ on H_2.

$\sigma^2 = \tau^2 = \mathrm{Id} \in \mathrm{Diff}^+(M)$.

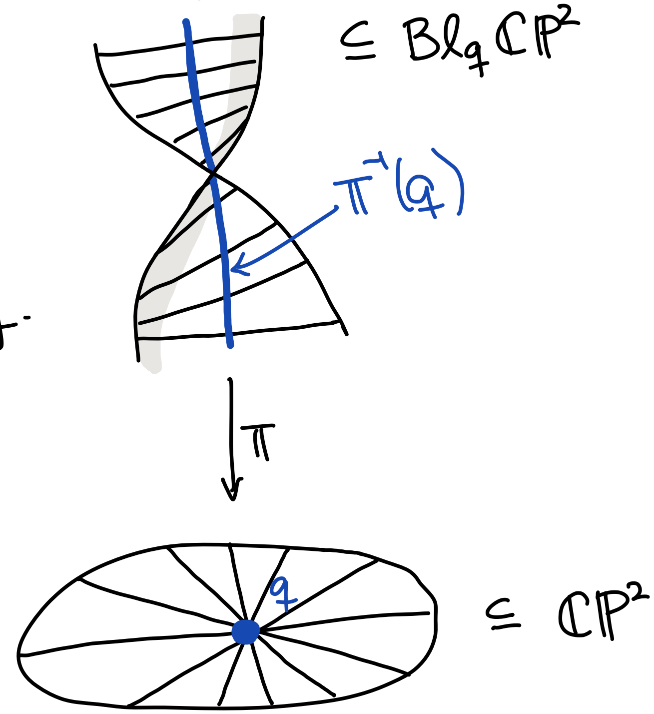
Blowups of $\mathbb{CP}^2$:

A quick reminder: $\quad q \in \mathbb{CP}^2$.

$Bl_q\mathbb{CP}^2$ = "Blowup of $\mathbb{CP}^2$ at a point q"

= $\mathbb{CP}^2$, but replace q with the $\underbrace{\text{set}}_{\mathbb{CP}^1}$ of $\underbrace{\text{complex lines}}_{\mathbb{CP}^1}$ through q.

$\cong \mathbb{CP}^1$, exceptional divisor E.



$Q_M(E,E) = -1$.

del Pezzo Surfaces:

Defn: A del Pezzo surface is a complex surface

$$M = \mathbb{CP}^1 \times \mathbb{CP}^1 \text{ or } Bl_P \mathbb{CP}^2$$

where $0 \leq |P| \leq 8$ and the points of P lie in general position.

no 3 on a line
no 6 on a conic
no 8 on a singular cubic s.t. P contains a singularity

Lemma: ① $M_n := Bl_P \mathbb{CP}^2 \cong \mathbb{CP}^2 \# n \overline{\mathbb{CP}^2}$ with $|P| = n$.

diffeo

② M_n simply connected,

gen of $H_2(\mathbb{CP}^2)$ gen $H_2(\overline{\mathbb{CP}^2})$.

③ $H_2(M_n) \underset{MV}{=} \mathbb{Z}^{n+1} = \mathbb{Z}\{H, E_1, E_2, \ldots, E_n\} \cong H_2(\mathbb{CP}^2) \oplus H_2(\overline{\mathbb{CP}^2})^{\oplus n}$.

④ $Q_{M_n} = \langle 1 \rangle \oplus n\langle -1 \rangle$

intersection form $H_2(M_n) \times H_2(M_n) \to \mathbb{Z}$.

$Q_{M_n}(H, H) = 1$ $\quad Q_{M_n}(E_k, E_k) = -1.$ $\quad Q_{M_n}(H, E_k) = Q_{M_n}(E_j, E_k) = 0$

Some classical examples:

3 classical examples of involutions arising from the classification of birational automorphisms of order 2 of $\mathbb{CP}^2$:

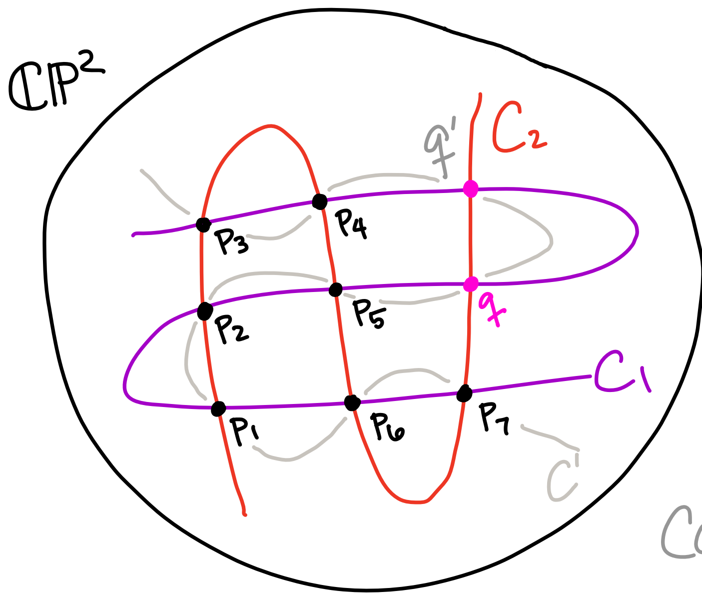
1. Geiser involution
2. Bertini involution
3. de Jonquiéres involution.

More examples:

Ex ①: The Geiser involution $\gamma \in \mathrm{Aut}(\mathrm{Bl}_P \mathbb{CP}^2)$

Let $P = \{7 \text{ points in general position}\} \subseteq \mathbb{CP}^2$.

$q \in \mathbb{CP}^2 - P \qquad \gamma(q) = q'$.



$\mathrm{Pencil}(P \cup \{q\})$

= cubic curves passing through $P \cup \{q\}$.

$= \{\lambda C_1 + \mu C_2 : [\lambda : \mu] \in \mathbb{CP}^1\}$

$= \mathbb{CP}^1$.

Cayley-Bacharach $\Rightarrow$ All cubic curves $C \in \mathrm{Pencil}(P \cup \{q\})$ pass through q'.

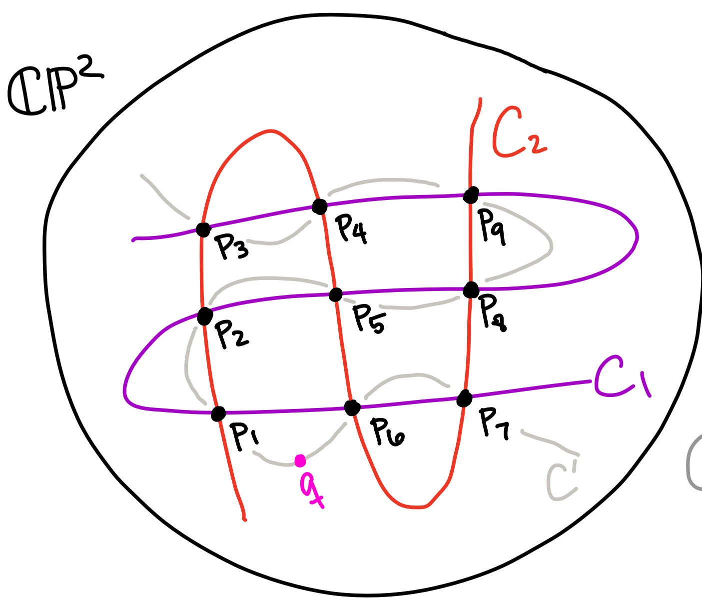
γ lifts to an automorphism of $\mathrm{Bl}_P \mathbb{CP}^2$.

More examples:

Ex ②: The Bertini involution $\beta \in \mathrm{Aut}(Bl_P\mathbb{CP}^2)$

Let $P = \{8 \text{ points in general position}\} \subseteq \mathbb{CP}^2$.

$q \in \mathbb{CP}^2 - P \qquad \beta(q) = q'$.



$V(P) = \{\text{Sextic curves singular at each point in } P\}$

E.g. $C_1^2,\ C_2^2,\ C_1 \cdot C_2 \in V(P)$.

$V(P) \cong \mathbb{CP}^2$.

$S(q) = \{\text{Sextic curves passing through } q\}$

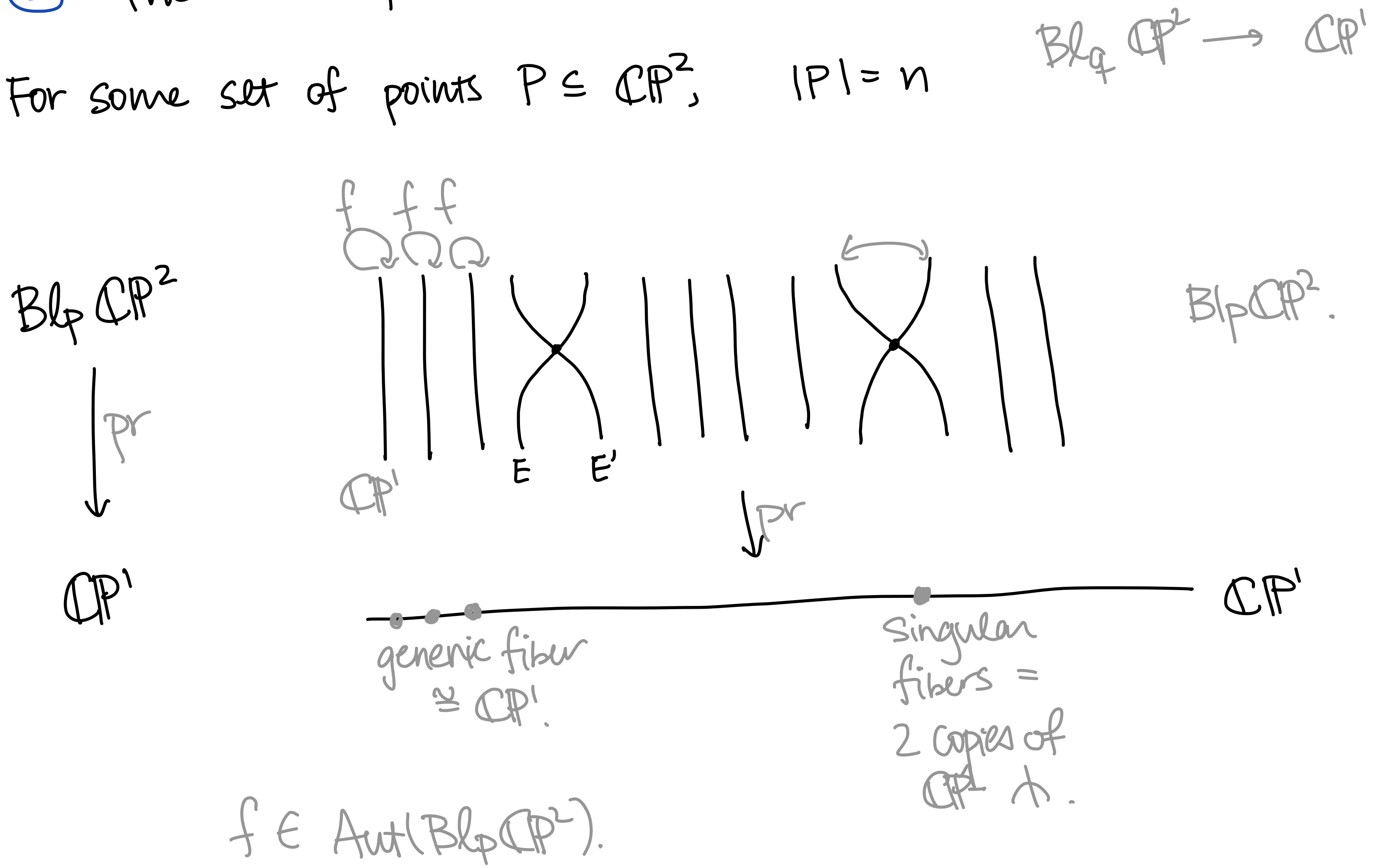
All $C \in V(P) \cap S(q)$. also pass through common 10th pt q'.

β lifts to aut of $Bl_8\mathbb{CP}^2$.

More examples:

Ex ③: The de Jonquiéres involution: $n \geq 5$ odd.

For some set of points $P \subseteq \mathbb{CP}^2$, $|P| = n$



Mod(M) for M^4 $\left(\mathrm{Mod}(M^4) = \pi_0(\mathrm{Homeo}^+(M)) \neq \pi_0(\mathrm{Diff}^+(M))\right)$.

Let M^4 be smooth, oriented, closed, simply connected.

Thm (Freedman '82, Quinn '86): The map

$$\Phi : \mathrm{Mod}(M) \to \mathrm{Aut}(H_2(M;\mathbb{Z}), Q_M), \quad [f] \mapsto f_*$$

is an isomorphism of groups. [no analog of Torelli].

Applied to M_n: $\quad H_2(M_n;\mathbb{Z}) = \mathbb{Z}\{H, E_1, \dots, E_n\}$

$$\Rightarrow \mathrm{Mod}(M_n) \cong \mathrm{Aut}(H_2(M_n), Q_{M_n}) \cong O(1,n)(\mathbb{Z})$$

Rmk: $\exists$ index 2 subgp $\quad O^+(1,n)(\mathbb{Z}) \leq \mathrm{Mod}(M_n)$.

$$\begin{array}{c} O^+(1,n)(\mathbb{Z}) \\ \wedge| \\ O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^n). \end{array}$$

$2 \leq n \leq 9 \Rightarrow O^+(1,n)(\mathbb{Z})$ hyperbolic reflection gp! (Wall, Vinberg).

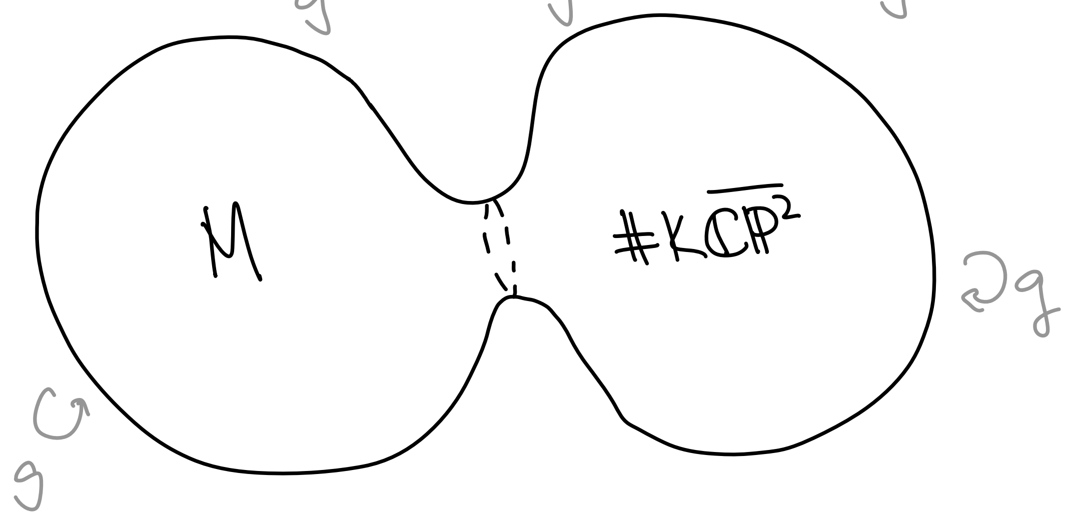
"Q": $M_n \overset{?}{\longleftrightarrow} \mathbb{H}^n$?

Irreducibility in Mod(M)

Defn: Let $g \in \mathrm{Mod}(M_n)$ $\circlearrowleft H_2(M_n;\mathbb{Z})$. Then g is reducible if for some del Pezzo surface M and some $k > 0$, $M_n \underset{\text{diff}}{\cong} M \# k\overline{\mathbb{CP}^2}$.

$$\underset{\circlearrowleft g}{H_2(M_n;\mathbb{Z})} = \underset{\circlearrowleft g}{H_2(M)} \oplus \underset{\circlearrowleft g}{H_2(k\overline{\mathbb{CP}^2})}.$$

On level of H_2



Otherwise, g is irreducible.

A structure thm

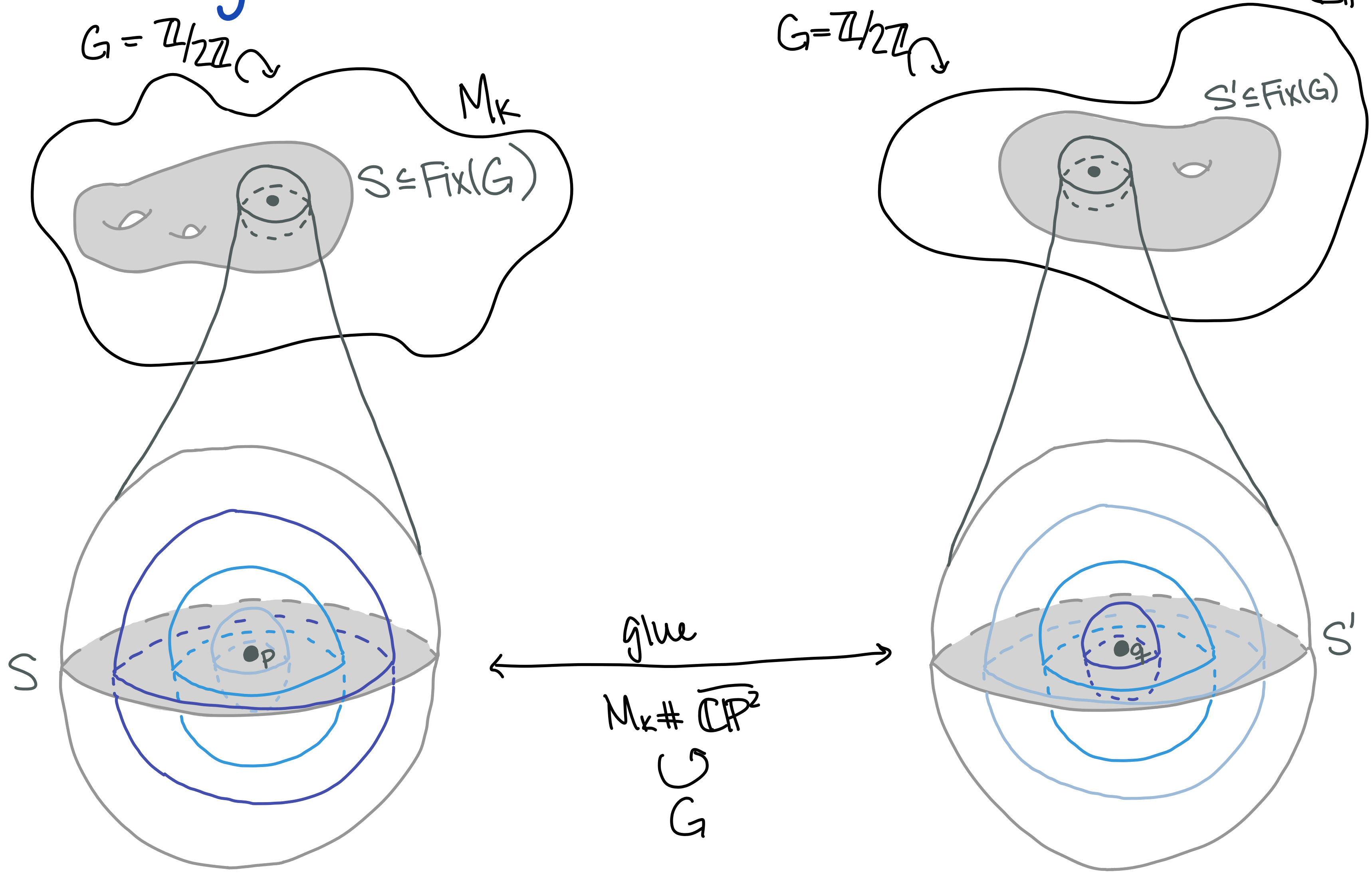
$O^+(1,n)(\mathbb{Z})$.

Thm (L. '22): Let M_n be a del Pezzo surface with $n \geq 2$. A mapping class $g \in \mathrm{Mod}^+(M_n)$ of order 2 is irreducible if and only if g is realized by a de Jonquiéres, Geiser, or Bertini involution on $Bl_P \mathbb{CP}^2$ for some $P \subseteq \mathbb{CP}^2$ finite.

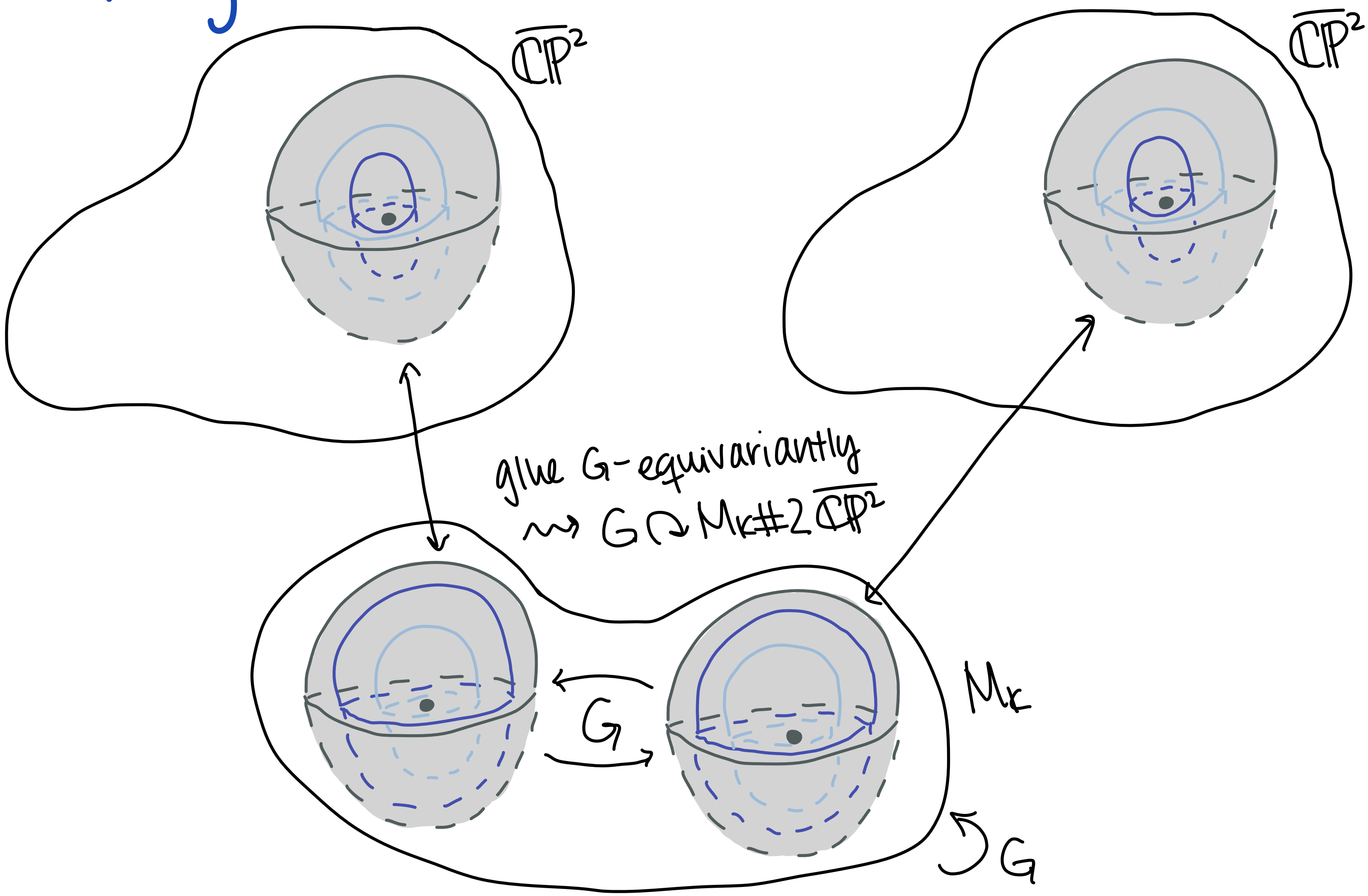
* generalizes to $\mathrm{Mod}(M_n)$.
* Nielsen realization for irred order 2.
* Next goal: Nielsen realization for all involutions

  Plan: glue irreducible pieces together to realize reducible cases!

Making new actions out of old:



Making new actions out of old:



Complex Equivariant Connected Sums

$$
\begin{array}{ccccccc}
M_K & \overline{\mathbb{CP}^2} & \cdots & \overline{\mathbb{CP}^2} & \#n_1\overline{\mathbb{CP}^2} & \cdots & \#n_m\overline{\mathbb{CP}^2} \\
\circlearrowleft & \circlearrowleft & & \circlearrowleft & \circlearrowleft & & \circlearrowleft \\
G & G & & G & G & & G
\end{array}
$$

G acts (anti-)biholomorphically

$$
\begin{array}{l}
M_n = M_K \# n\overline{\mathbb{CP}^2} \\
\circlearrowleft \\
G
\end{array}
$$

"Complex equivariant connected sum"

Nielsen realization for involutions:

Let M be any del Pezzo surface.

Cor: If $g \in \mathrm{Mod}(M)$ has order 2 then there exists some $f \in \mathrm{Diff}^+(M)$ of order 2 such that $[f] = g$.
In particular, $\langle g \rangle$ is realized by a complex equivariant connected sum.

* complex equiv connected sum necessary!

Nielsen Realization for $M_2 = \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$.

Thm (L. '21): A finite subgroup $G \leq \mathrm{Mod}(M_2)$ has a lift to $\mathrm{Diff}^+(M_2) \leq \mathrm{Homeo}^+(M_2)$ under $q: \mathrm{Homeo}^+(M_2) \to \mathrm{Mod}(M_2)$ if and only if G is realized by a complex equivariant connected sum.

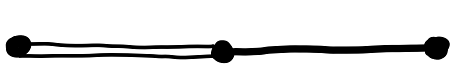
Cor: If $g \in \mathrm{Mod}(M_2)$ has finite order n then there exists $f \in \mathrm{Diff}^+(M_2)$ with order n such that $[f] = g$.

Cor: There exist some finite subgroups $G \leq \mathrm{Mod}(M_2)$ that do not have any lift to $\mathrm{Diff}^+(M_2)$ such that all elements $g \in G$ of order n admit representatives $f \in \mathrm{Diff}^+(M_2)$.

Proof Sketch:

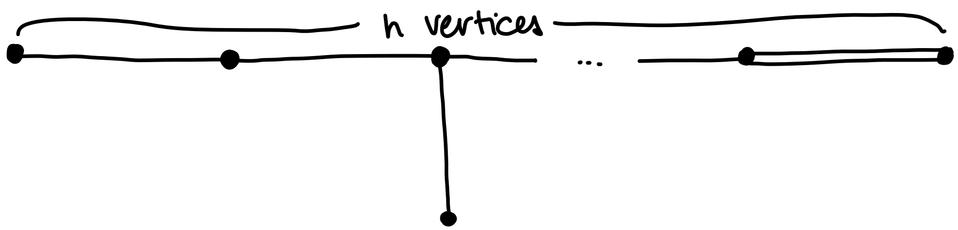
① $\mathrm{Mod}(M_n) \underset{\text{Freedman-Quinn}}{\cong} O(1,n)(\mathbb{Z})$

$$\begin{array}{ccc} O(1,n)(\mathbb{Z}) & & \\ \rotatebox{90}{$\leq$} & & \\ O^+(1,n)(\mathbb{Z}) & \leq & \mathrm{Isom}(\mathbb{H}^n) \end{array}$$



$O^+(1,2)(\mathbb{Z})$

$O^+(1,3)(\mathbb{Z})$



$O^+(1,n)(\mathbb{Z})$, $4 \leq n \leq 9$

② Enumerate the stabilizers of points in $\mathbb{H}^n$ up to conjugacy.

③ Construct smooth G-actions using complex equiv. conn. sums

OR

Obstruct using homological data

Further questions...

Q1: What do irreducible mapping classes of order p "look like"?

Q2: Nielsen realization for arbitrary finite subgroups of $\mathrm{Mod}(M_n)$, $n \geq 3$?

Q3: A more explanatory proof for all results here?

Thank you!