Mapping class groups of del Pezzo manifolds

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August 6, 2022

1 Topology of del Pezzo surfaces

The main object of study in these notes are *del Pezzo manifolds* and their topological mapping class groups.

Definition 1.1. A del Pezzo manifold is any of:

$$M = \mathbb{CP}^1 \times \mathbb{CP}^1 \text{ or } \operatorname{Bl}_P \mathbb{CP}^2;$$

where P is a set of $0 \leq n \leq 8$ points in $\mathbb{CP}^2$.

Remark 1.2. 1. One reason these manifolds are grouped together is that as projective algebraic varieties, these are the surfaces whose anticanonical bundles are ample (when $P \subseteq \mathbb{CP}^2$ consists of points in general position).

2. Topologically, the behavior changes dramatically once $n \geq 9$; this will be addressed in Benson's subsequent lectures.

1.1 Underlying smooth manifold

Given the discussion of blowups of complex surfaces from the previous lecture, we can describe the topology of del Pezzo manifolds using connected sums.

Lemma 1.3. There's a diffeomorphism $\operatorname{Bl}_P \mathbb{CP}^2 \to \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$ where $|P| = n$.

Remark 1.4. This diffeomorphism is independent of the set of points P. The exceptional divisors of $\operatorname{Bl}_P \mathbb{CP}^2$ correspond to the lines in each copy of $\overline{\mathbb{CP}^2}$.

Proof Sketch. The exceptional divisor E has self-intersection number -1, and so the normal neighborhood of of an exceptional divisor E is isomorphic as a complex vector bundle to $\mathcal{O}(-1)$ whose total space is diffeomorphic to $\overline{\mathbb{CP}^2} - p$ for some point $p \in \overline{\mathbb{CP}^2}$. The blowup can be constructed by replacing a neighborhood of each point $q \in P$ by $\overline{\mathbb{CP}^2} - p$; this construction can be used to give a diffeomorphism between $\operatorname{Bl}_P \mathbb{CP}^2$ and $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$. $\square$

Definition 1.5. Let M_n denote the smooth manifold $\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$.

Remark 1.6. There are other descriptions of M_n as a smooth manifold that may be useful in thinking about its topology. One such way is via the diffeomorphism $M_n \cong (\mathbb{CP}^1 \times \mathbb{CP}^1) \#(n-1)\overline{\mathbb{CP}^2}$. Another way is via a conic bundle $\pi : M_n \to \mathbb{CP}^1$, which will be discussed in the second lecture.

1.2 Algebraic topology of M_n

Let's elaborate on the algebraic topology of M_n.

- $\pi_1(M_n) = 1$ by Van Kampen's theorem, with $M_n = (M_{n-1} - *) \cup (\overline{\mathbb{CP}^2} - *)$.
- By Mayer–Vietoris, there is an isomorphism

$$\mathbb{Z}^{n+1} \cong H_2(\mathbb{CP}^2;\mathbb{Z}) \oplus \bigoplus_{i=1}^{n} H_2(\overline{\mathbb{CP}^2};\mathbb{Z}) \xrightarrow{\sim} H_2(M_n;\mathbb{Z}).$$

  Let $\{H, E_1, \dots, E_n\}$ be the $\mathbb{Z}$-basis of $H_2(M_n;\mathbb{Z})$ coming from the direct sum decomposition. Then H is the class of a hyperplane in $\mathrm{Bl}_n\,\mathbb{CP}^2$ that does not pass through any point in P and $E_1, \dots, E_n$ are the classes of the exceptional divisors.

To compute the intersection form, note that

$$Q_{\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}} \cong Q_{\mathbb{CP}^2} \oplus nQ_{\overline{\mathbb{CP}^2}} \cong (1) \oplus n(-1).$$

In matrix notation, the intersection form is given by

$$Q_{M_n} = \mathrm{diag}(1, -1, \dots, -1).$$

More explicitly, we can see geometrically that the off-diagonal terms of the matrix above must be zero since:

1. Exceptional divisors E_j and E_k over distinct points p_j and p_k in $P \subseteq \mathbb{CP}^2$ are disjoint, so $E_j \cdot E_k = 0$ if $j \neq k$.
2. The class H is represented by a line in $\mathbb{CP}^2$ that does not pass through any points of P. Therefore, this line also does not intersect any exceptional divisor in $\mathrm{Bl}_P\,\mathbb{CP}^2$, so $H \cdot E_k = 0$ for all k.

2 Intersection form, mapping class group, and some diffeomorphisms

Applying theorems of Freedman ([Fre82]) and Quinn ([Qui86]) to M_n gives an isomorphism of groups

$$\mathrm{Mod}(M_n) \cong \mathrm{Aut}(H_2(M_n;\mathbb{Z}), Q_{M_n}) = \mathrm{O}(1,n)(\mathbb{Z})$$

where $\mathrm{O}(1,n)(\mathbb{Z})$ is the group consisting of all $(n+1)\times(n+1)$-matrices A with integral coefficients such that for all $v, w \in H_2(M_n;\mathbb{Z})$,

$$Q_{M_n}(A \cdot v, A \cdot w) = Q_{M_n}(v, w).$$

Equivalently, A is an element of $\mathrm{O}(1,n)(\mathbb{Z})$ if and only if

$$A \cdot \mathrm{diag}(1, -1, \dots, -1) \cdot A^T = \mathrm{diag}(1, -1, \dots, -1).$$

Example 2.1. For any $v \in H_2(M_n;\mathbb{Z})$ with $Q_{M_n}(v,v) = \pm 1$ or ± 2, define a *reflection* Ref_v to be the map

$$\mathrm{Ref}_v(w) = w - 2\frac{w \cdot v}{v \cdot v} v.$$

One can check that any such reflection Ref_v is a linear map preserving the intersection form Q_{M_n}, i.e. $\mathrm{Ref}_v \in \mathrm{O}(1,n)(\mathbb{Z})$. It has order 2.

Moreover, it is the unique element of $\mathrm{O}(1,n)(\mathbb{Z})$ with the property that

$$\mathrm{Ref}_v(v) = -v, \qquad \text{and} \qquad \mathrm{Ref}_v(w) = w \text{ if } w \perp v, \text{ i.e. } w \cdot v = 0.$$

In the special case if $v \cdot v = -2$,

$$\mathrm{Ref}_v(w) = w + (w \cdot v)v.$$

Example 2.2. Consider $E_1 - E_2 \in H_2(M_n; \mathbb{Z})$ which has self-intersection number -2. Then $\mathrm{Ref}_{E_1-E_2}$ acts by

$$H_2(M_n;\mathbb{Z}) = \underbrace{\mathbb{Z}\{H, E_3, \ldots, E_n\}}_{\perp E_1 - E_2 \implies \circlearrowleft \mathrm{Id}} \oplus \underbrace{\mathbb{Z}\{E_1, E_2\}}_{\circlearrowleft \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}.$$

To compute the action of $\mathrm{Ref}_{E_1-E_2}$ on the second summand, note that $\mathrm{Ref}_{E_1-E_2}$ has order 2 and that

$$\mathrm{Ref}_{E_1-E_2}(E_1) = E_1 + (E_1 \cdot (E_1 - E_2))(E_1 - E_2) = E_2.$$

There is an embedded 2-sphere S in M_n representing the class $E_1 - E_2$. One way to see this is take two spheres, S_1 and S_2, representing E_1 and $-E_2$ and tube them together. So $\mathrm{Ref}_{E_1-E_2}$ is the class of a *Dehn twist* about the sphere S.

2.1 Some examples of diffeomorphisms of connected sums

Recall the following two examples from previous lectures.

Example 2.3. Any $g \in \mathrm{Aut}(\mathbb{CP}^2) = \mathrm{PGL}_3(\mathbb{C})$ induces the identity map on $H_2(\mathbb{CP}^2; \mathbb{Z})$.

Example 2.4. The diffeomorphism $c : \mathbb{CP}^2 \to \mathbb{CP}^2$ given by complex conjugation on the coordinates,

$$c : [X : Y : Z] \mapsto [\bar{X} : \bar{Y} : \bar{Z}]$$

induces the negation map on $H_2(\mathbb{CP}^2; \mathbb{Z})$.

We can "glue" diffeomorphisms of the pieces in connected sums together to realize certain actions on homology.

Example 2.5. In this example, we will take two order-2 diffeomorphisms

$$f : \mathbb{CP}^2 \to \mathbb{CP}^2, \qquad g : \overline{\mathbb{CP}^2} \to \overline{\mathbb{CP}^2}$$

and glue them together to form a new order-2 diffeomorphism of $M_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. More specifically, take

$$f([X : Y : Z]) = [-X : Y : Z] \quad \text{and} \quad g([X : Y : Z]) = [\bar{X} : \bar{Y} : \bar{Z}].$$

Then $f_* = \mathrm{Id}$ on $H_2(\mathbb{CP}^2)$ and $g_* = -\mathrm{Id}$ on $H_2(\overline{\mathbb{CP}^2})$. These two diffeomorphisms generate a subgroup $G \cong \mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Diff}^+(\mathbb{CP}^2)$ and $\mathrm{Diff}^+(\overline{\mathbb{CP}^2})$ respectively.

The pointwise fixed sets of f and g each contain[1] 2-dimensional submanifolds in $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$; one can check that

$$\mathrm{Fix}(f) \cong \mathbb{CP}^1 \sqcup \{[1 : 0 : 0]\} \subseteq \mathbb{CP}^2, \qquad \mathrm{Fix}(g) \cong \mathbb{RP}^2 \subseteq \overline{\mathbb{CP}^2}.$$

Now choose some points $p \in \mathbb{CP}^1 \subseteq \mathrm{Fix}(f)$ and $q \in \mathrm{Fix}(g)$. There exist 4-dimensional disks $D_1 \subseteq \mathbb{CP}^2$ and $D_2 \subseteq \overline{\mathbb{CP}^2}$ centered at p and q which are preserved by f and g respectively. Now form a particular instance of the connected sum $M_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ by letting

$$M_1 = (\mathbb{CP}^2 - D_1) \cup_{\partial D_1 \sim \partial D_2} (\overline{\mathbb{CP}^2} - D_2)$$

where the gluing $\partial D_1 \sim \partial D_2$ is by an orientation-reversing map that respects the $\mathbb{Z}/2\mathbb{Z}$-action by $\langle f \rangle$ and $\langle g \rangle$ on ∂D_1 and ∂D_2 respectively. (See Figure 1.) Then define $F \in \mathrm{Diff}^+(M_1)$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{CP}^2 - D_1, \\ g(x) & \text{if } x \in \overline{\mathbb{CP}^2} - D_2. \end{cases}$$

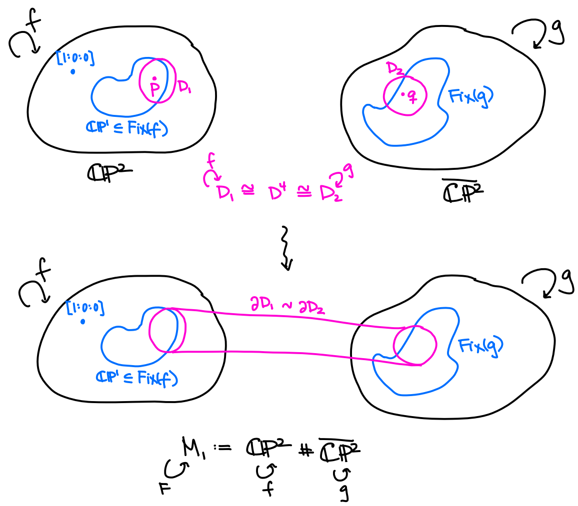

Figure 1: Construction of the equivariant connected sum in Example 2.5

This diffeomorphism is well-defined by the construction of M_1. Moreover, F_* acts by

$$H_2(M_1) \cong \underbrace{H_2(\mathbb{CP}^2)}_{\circlearrowleft f_*} \oplus \underbrace{H_2(\overline{\mathbb{CP}^2})}_{\circlearrowleft g_*}.$$

This corresponds to the action of Ref_{E_1} on $H_2(M_1)$.

3 Hyperbolic space and reflection groups

With these examples in hand, we examine the group $\mathrm{O}(1,n)(\mathbb{Z})$ in earnest by studying the action of an index 2-subgroup of $\mathrm{O}(1,n)(\mathbb{Z})$ on hyperbolic space. The simplest way to see such an action is via the hyperboloid model of $\mathbb{H}^n$.

3.1 Hyperboloid model

Consider the diagonal, bilinear symmetric form of signature $(1-n)$ and type $(1,n)$ on $\mathbb{R}^{n+1}$,

$$Q_n = \mathrm{diag}(1, -1, \ldots, -1).$$

The hyperboloid model of $\mathbb{H}^n$ sits in $\mathbb{R}^{n+1}$ via

$$\mathbb{H}^n := \{p = (x,\, y_1,\, \ldots,\, y_n) \in \mathbb{R}^{n+1} : x > 0,\, Q_n(p,p) = 1\}.$$

The restriction of Q_n to each $T_p\mathbb{H}^n$ is negative definite and $-Q_n$ defines the hyperbolic metric on $\mathbb{H}^n$. Note that $\{Q_n(p) = 1\}$ has two connected components, and we restrict to one of these components. See Figure 2.

Also, the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ is isomorphic to $\mathrm{O}^+(1,n)(\mathbb{R})$, the group of matrices preserving Q_n and each sheet of the hyperboloid. It has index 2 as a subgroup of $\mathrm{O}(1,n)(\mathbb{R})$. Then since $\mathrm{O}(1,n)(\mathbb{Z})$ is a subgroup of $\mathrm{O}(1,n)(\mathbb{R})$, we consider the index-2 subgroup of $\mathrm{O}(1,n)(\mathbb{Z})$

$$\mathrm{O}^+(1,n)(\mathbb{Z}) := \mathrm{O}(1,n)(\mathbb{Z}) \cap \underbrace{\mathrm{O}^+(1,n)(\mathbb{R})}_{\mathrm{Isom}(\mathbb{H}^n)} \leq \mathrm{O}(1,n)(\mathbb{R})$$

[1] In lecture, I had omitted the unique isolated fixed point of $\mathrm{Fix}(f)$. However, this construction only depends on the existence of 2-dimensional components in each $\mathrm{Fix}(f)$ and $\mathrm{Fix}(g)$.

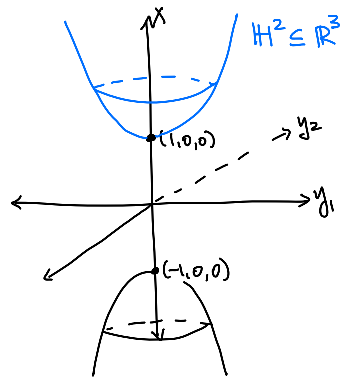

Figure 2: The hyperboloid model of $\mathbb{H}^2 \subseteq \mathbb{R}^3$.

that acts by isometries on $\mathbb{H}^n$.

Reflections Ref_v where $v \in \mathbb{R}^{n+1}$ with $Q_n(v,v) = \pm 1, \pm 2$ act by an actual reflection of $\mathbb{H}^n$ (and $\mathbb{R}^{n+1}$) across the hyperplane $v^\perp$. For small n, the group $\mathrm{O}^+(1,n)(\mathbb{Z})$ is generated by such reflections.

Theorem 3.1 (Vinberg [Vin72, Section 4, Table 4])**.** For all $n \leq 17$, the group $\mathrm{O}^+(1,n)(\mathbb{Z})$ contains a finite index, hyperbolic reflection subgroup acting by isometries on $\mathbb{H}^n$.

Finally, recall the classification of isometries of $\mathbb{H}^n$. All isometries of $\mathbb{H}^n$ fall under one of three types:

1. elliptic – fixes a point in $\mathbb{H}^n$; e.g. isometries of finite order,
2. hyperbolic – acts by translation along a unique axis; it fixes two points on the boundary sphere $\partial\mathbb{H}^n$,
3. parabolic – neither elliptic nor hyperbolic; it fixes a unique point on the boundary sphere $\partial\mathbb{H}^n$.

We will consider cases (1) and (3) in these lectures.

4 Representing mapping classes by diffeomorphisms

4.1 The subgroup of $\mathrm{Mod}(M_n)$ represented by diffeomorphisms

For any $n \geq 0$, consider the quotient map

$$q_n : \mathrm{Homeo}^+(M_n) \to \mathrm{Mod}(M_n) \cong \mathrm{O}(1,n)(\mathbb{Z}).$$

Theorem 4.1 (Wall [Wal64a, Special case of Theorem 2])**.** Let $0 \leq n \leq 9$. The restriction of q_n to $\mathrm{Diff}^+(M_n)$ is surjective onto $\mathrm{O}(1,n)(\mathbb{Z})$.

Remark 4.2. This theorem is a special case of a more general theorem by Wall. The proof idea given below only applies to this special case and is not the proof given by Wall.

Proof idea. Take all the generators of $\mathrm{O}(1,n)(\mathbb{Z})$ and exhibit them by diffeomorphisms. The generators are determined given in Wall ([Wal64b, (1.4), (1.6)]):

1. For $n = 2$,
$$\mathrm{O}(1,2)(\mathbb{Z}) = \langle \mathrm{Ref}_{H-E_1-E_2}, \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2}, \mathrm{Ref}_H \rangle.$$

2. For $3 \le n \le 9$,

$$\mathrm{O}(1,n)(\mathbb{Z}) = \langle \mathrm{Ref}_{H-E_1-E_2-E_3}, \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{E_2-E_3}, \ldots, \mathrm{Ref}_{E_{n-1}-E_n}, \mathrm{Ref}_{E_n}, \mathrm{Ref}_H \rangle.$$

In the case $3 \le n \le 9$, note that all the generators $\mathrm{Ref}_{E_k-E_{k+1}}$ and $\mathrm{Ref}_{H-E_1-E_2-E_3}$ are represented by Dehn twists about (-2)-spheres. For the last two generators, adapt the construction of a diffeomorphism representative of Ref_{E_1} in M_1 to M_n for all $3 \le n \le 9$. □

Here is one way in which the behavior changes as n grows on the mapping class group side.

Theorem 4.3 (Friedman–Morgan [FM88, Theorem 10])**.** *Let $n \ge 10$. The image of $\mathrm{Diff}^+(M_n)$ under q_n in $\mathrm{Mod}(M_n) \cong \mathrm{O}(1,n)(\mathbb{Z})$ has infinite index.*

Compare with Vinberg's result, which here says that there are some reflections that cannot be represented by diffeomorphisms.

4.2 Elliptic diffeomorphisms

The Nielsen realization problem asks: For any finite subgroup $G \le \mathrm{Mod}(M)$, does there exist a section $s : G \to \mathrm{Homeo}^+(M)$ of the map $\pi : \mathrm{Homeo}^+(M) \to \mathrm{Mod}(M)$? For today, we will insist that the section has image contained in $\mathrm{Diff}^+(M) \le \mathrm{Homeo}^+(M)$.

In the case of $G = \mathbb{Z}/2\mathbb{Z}$, there is a positive answer:

Theorem 4.4 (Lee [Lee22, Theorem 1.3, Corollary 1.5])**.** *Let $n \le 8$. Any mapping class $g \in \mathrm{Mod}(M_n)$ of order 2 is realized by a diffeomorphism of order 2. Moreover, g is the mapping class of a Geiser, Bertini, or de Jonquiéres (of algebraic degree $d > 2$) involution if:*

1. *$g \in \mathrm{Mod}(M_n) \cap \mathrm{O}^+(1,n)(\mathbb{Z})$, and*

2. *there's no isomorphism of lattices*

$$H_2(M_n;\mathbb{Z}) \cong H_2(M) \oplus H_2(\#k\overline{\mathbb{CP}^2})$$

for any $k \ge 1$ and any del Pezzo M that g preserves.

Remark 4.5. The three involutions above (Geiser, Bertini, and de Jonquiéres) arise together naturally in the classification of conjugacy classes of order 2 in the group $\mathrm{Cr}(2)$ of birational automorphisms of $\mathbb{CP}^2$; see Bayle–Beauville [BB00, Theorem 2.6].

The second condition should be thought of as an obstruction to the mapping class being realized by a diffeomorphism constructed by gluing as in Example 2.5.

Since the mapping class of any Dehn twist has order 2 in $\mathrm{Mod}(M_n)$, we obtain the following corollary:

Corollary 4.6. *Any Dehn twist of a del Pezzo manifold is realizable by a diffeomorphism of order 2.*

In contrast, Dehn twists are not isotopic to any finite-order diffeomorphism

1. in a K3 manifold (Farb–Looijenga [FL21, Corollary 1.10]), or

2. more generally, in any spin manifold of nonzero signature (Konno [Kon22, Theorem 1.1]).

4.3 Proof methods to show nonrealizability

On the other hand, there do exist finite subgroups $G \le \mathrm{Mod}(M_n)$ that have no lift to $\mathrm{Diff}^+(M_n)$. Here, we consider an example of such a subgroup and give a proof outline of its nonrealizability by diffeomorphisms. For more examples and details of proofs of nonrealizability, see [Lee21].

The example of a nonrealizable subgroup is:

$$G := \langle \mathrm{Ref}_{E_1-E_2}, \mathrm{Ref}_{H-E_1-E_2} \rangle \le \mathrm{Mod}(M_2).$$

We will use the following two tools from the theory of finite group actions on 4-manifolds. Here, $p \in \mathbb{Z}$ denotes any prime number.

Theorem 4.7 (Edmonds [Edm89, Proposition 2.4])**.** Let $G = \mathbb{Z}/p\mathbb{Z}$ act on a closed, oriented, simply-connected 4-manifold M. For the sake of simplicity, let $p < 23$; see [Edm89] for the theorem for all primes p. There is a decomposition

$$H_2(M;\mathbb{Z}) \cong \mathbb{Z}^t \oplus \mathbb{Z}[\zeta_p]^c \oplus \mathbb{Z}[G]^r$$

as a G-representation, where G acts trivially on Z, acts by multiplication-by-ζ_p on $\mathbb{Z}[\zeta_p]$, and by left-multiplication on $\mathbb{Z}[G]$. If $\mathrm{Fix}(G)$ is the fixed set of G and is nonempty, then

$$\beta_1(\mathrm{Fix}(G)) = c, \qquad \beta_0(\mathrm{Fix}(G)) + \beta_2(\mathrm{Fix}(G)) = t + 2$$

where $\beta_m(\mathrm{Fix}(G))$ denotes the mod p Betti number of $\mathrm{Fix}(G)$.

This is useful because the fixed set of a finite, *cyclic* group is a union of 2- and 0-dimensional submanifolds.[2]

Theorem 4.8 (Hirzebruch G-signature theorem [HZ74, Section 9.2, (12)])**.** Let $G = \mathbb{Z}/p\mathbb{Z}$ act on a closed, oriented M^4 by orientation-preserving diffeomorphisms. Then

$$p\sigma(M/G) = \sigma(M) + \sum_{\substack{C^2 \subseteq \mathrm{Fix}(G) \\ \text{2-dim'l components}}} \mathrm{defect}(C) + \sum_{\substack{z \in \mathrm{Fix}(G) \\ \text{isolated fixed points}}} \mathrm{defect}(z)$$

where

1. $\sigma(M/G)$ is the signature of the restriction of Q_M to the fixed subspace $H_2(M;\mathbb{R})^G$,
2. $\mathrm{defect}(z)$ is a quantity that is determined by the action of G on T_zM which happens to vanish for $p = 2$, and
3. $\mathrm{defect}(C)$ is $\left(\frac{p^2-1}{2}\right) Q_M([C],[C])$ if C is orientable.

We now show that G does not have a lift to $\mathrm{Diff}^+(M_2)$. For the sake of contradiction, suppose there exist diffeomorphisms $f, g \in \mathrm{Diff}^+(M_2)$ such that $[f] = \mathrm{Ref}_{E_1-E_2}$ and $[g] = \mathrm{Ref}_{H-E_1-E_2}$ and $\langle f, g\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Step 1. Apply Edmonds (Theorem 4.7) to see that

$$\mathrm{Fix}(f) \cong \mathbb{S}^2 \sqcup \{p\} \text{ or } \{p_1, p_2, p_3\}.$$

Apply G-signature theorem (Theorem 4.8) to see that

$$\mathrm{Fix}(f) = \mathbb{S}^2 \sqcup \{p\}, \qquad [\mathbb{S}^2] \cdot [\mathbb{S}^2] = 1.$$

Let $S := \mathbb{S}^2 \subseteq \mathrm{Fix}(f)$.

[2] In lecture, I had mistakenly claimed this also holds for any finite group but this is not the case.

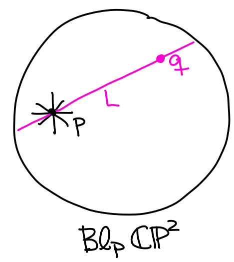

Figure 3: A fiber L of π in $\mathrm{Bl}_p\,\mathbb{CP}^2$ containing the point q.

Step 2. Because f and g commute, g must act by a diffeomorphism on S. Therefore, $g_*([S]) = \pm[S]$. Moreover, $H_2(M_2;\mathbb{Z})$ decomposes into a direct sum of eigenspaces of g_*:

$$H_2(M_2) = \underbrace{\mathbb{Z}\{H - E_1 - E_2\}}_{\circlearrowleft - \mathrm{Id}} \oplus \underbrace{\mathbb{Z}\{H - E_1,\, H - E_2\}}_{\circlearrowleft \mathrm{Id}}.$$

Step 3. The class $[S]$ must be contained in the (1)- or (-1)-eigenspace of g_* computed above. To reach a contradiction, compute that there is no class in $\mathbb{Z}\{H - E_1 - E_2\}$ or $\mathbb{Z}\{H - E_1, H - E_2\}$ with self-intersection number 1.

5 Examples of parabolic diffeomorphisms of del Pezzo manifolds

Fix $n \leq 8$. Parabolic isometries of $\mathbb{H}^n$ fix a unique point on the boundary sphere $\partial\mathbb{H}^n$. In the hyperboloid model of $\mathbb{H}^n$, points of the boundary sphere corresponds to the vectors in $\mathbb{R}^{n+1}$ with zero self-intersection (up to scaling by $\mathbb{R}^+$). The parabolic elements of $\mathrm{O}^+(1,n)(\mathbb{R})$ fixes a vector $v \in \mathbb{R}^{n+1}$ with $Q_n(v,v) = 0$ (rather than merely preserving a line in $\mathbb{R}^{n+1}$ spanned by v; see [Thu97, Problem 2.5.24]). Moreover, the parabolic elements of $\mathrm{O}^+(1,n)(\mathbb{Z})$ fix an integral vector $v \in H_2(M_n;\mathbb{Z})$ with $v \cdot v = 0$.

There is a surface representing $v \in H_2(M_n)$ that has algebraic self-intersection 0. It turns out that that there exist disjoint, distinct spheres representing v, and they come as the smooth fibers of a *conic bundle* which we describe below. We would like to represent the mapping classes that fix v by diffeomorphisms that not only preserve the homology class v but a surface representing v.

5.1 Topology of conic bundles

One way to view the topology of M_n is via a map $\pi : M_n \to \mathbb{CP}^1$ which we now describe.

First, consider the case $n = 1$; fix a point $p \in \mathbb{CP}^2$ and let $M_1 = \mathrm{Bl}_p\,\mathbb{CP}^2$. Identify $\mathbb{CP}^1$ with the space of lines in $\mathbb{CP}^2$ through p. For any point $q \in \mathbb{CP}^2 - p \subseteq \mathrm{Bl}_p\,\mathbb{CP}^2$, let

$$\pi : q \mapsto \ell \in \mathbb{CP}^1$$

where ℓ is the point corresponding to the unique line L in $\mathbb{CP}^2$ through p and q. See Figure 3. One can check that π extends to a well-defined map on $\mathrm{Bl}_p\,\mathbb{CP}^2$. The fiber of π over any point $\ell \in \mathbb{CP}^1$ corresponding to a line L in $\mathbb{CP}^2$ is the strict transform of L in $\mathrm{Bl}_p\,\mathbb{CP}^2$.

One can also check that $\pi : \mathrm{Bl}_p\,\mathbb{CP}^2 \to \mathbb{CP}^1$ is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. Topologically, this is the unique, nontrivial $\mathbb{S}^2$-bundle over $\mathbb{S}^2$. Let $v = [F] \in H_2(M_1)$ be the class of a fiber F of π. Since any two fibers of π are homologous and disjoint, we see that v is independent of the choice of F and satisfies $v^2 = 0$.

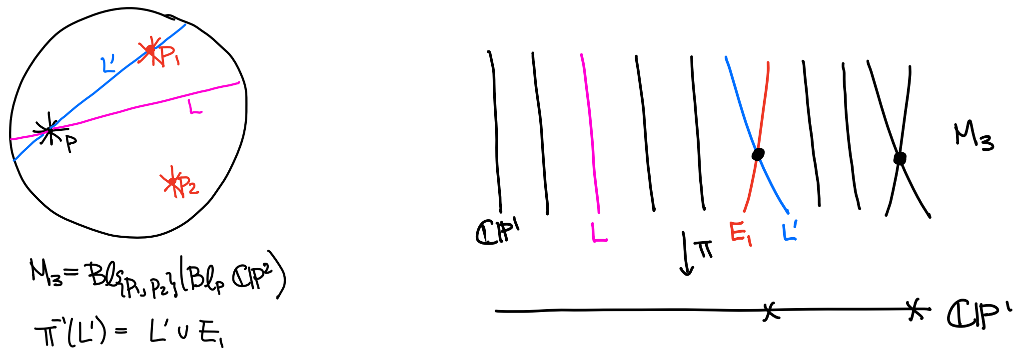

Figure 4: Two ways to view a conic bundle $\pi : M_n \to \mathbb{CP}^1$. Left: The pink line L is a smooth fiber of π in M_3. The union of blue line L' and the red exceptional divisor E_1 over p_1 is a singular fiber of π over L'. Right: Each smooth fiber is depicted by a vertical line. Over the specified finitely many points of $\mathbb{CP}^1$, the fiber of π is a union of two copies of $\mathbb{CP}^1$.

Now, consider the case $n \geq 2$. Identify M_n with $\mathrm{Bl}_P(\mathrm{Bl}_p \mathbb{CP}^2)$ where $P \subseteq \mathbb{CP}^2 - p$ is a set of $(n-1)$-points where no two points of P lie on a line containing p.[3] There is a map $M_n \to \mathrm{Bl}_p \mathbb{CP}^2$ that blows down the exceptional divisors above the points of P. Let π be the composition of this blowdown map and the map $M_1 \to \mathbb{CP}^1$ constructed above:

$$\pi : M_n \to \mathrm{Bl}_p \mathbb{CP}^2 \to \mathbb{CP}^1.$$

This is an example of a *conic bundle*.

Let $\ell \in \mathbb{CP}^1$ be a point corresponding to a line L through p in $\mathbb{CP}^2$. If L does not pass through any point of P, then the preimage $\pi^{-1}(\ell)$ is still the line L. If L does pass through a point $q \in P$, then the preimage $\pi^{-1}(\ell)$ is the union of L and the exceptional divisor E_k over q, intersecting positively once.

Then $\pi : M_n \to \mathbb{CP}^1$ defines a trivial fiber bundle over $\mathbb{CP}^1$ minus finitely many points with every fiber isomorphic to $\mathbb{CP}^1$. The smooth fibers $F \cong \mathbb{CP}^1$ of $q : M_n \to \mathbb{CP}^1$ are homologous in M_n and their homology class $v := [F]$ satisfies $v^2 = 0$. Each singular fiber is diffeomorphic to the wedge of two copies of $\mathbb{CP}^1$, the exceptional divisor E over q and the line L passing through p and q. See Figure 4.

5.2 A parabolic diffeomorphism

Here we give a quick description of an example of a parabolic diffeomorphism f on M_n for n odd. See Figure 5.

Step 1 There exists an order-2 complex automorphism Φ (of certain complex structures on M_n) which

1. acts by a diffeomorphism of order 2 on each smooth fiber F, and
2. swaps the two spheres $\mathbb{CP}^1$ of each singular fiber.

The existence of a homeomorphism of M_n satisfying these properties requires that n is odd.

Step 2 Let f be a diffeomorphism that has support $\mathrm{supp}(f)$ contained in the union of normal neighborhoods of all exceptional divisors E lying over the points of P such that $f_*([E]) = -[E]$ for all such E. Let $F = f \circ \Phi$.

Step 3 Compute that $[F]$ is an element of $\mathrm{O}^+(1, n)(\mathbb{Z})$ that induces a parabolic isometry of $\mathbb{H}^n$.

Remark 5.1. The diffeomorphism F above preserves smooth fibers of π away from a neighborhood of each of the singular fibers.

[3] In lecture, I forgot to specify the last condition but it is necessary.

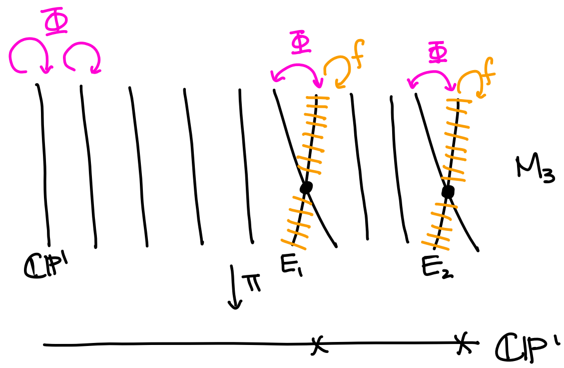

Figure 5: An example construction of a parabolic diffeomorphism $F = f \circ \Phi$ of M_n when n is odd. Here, E_1 and E_2 denote the exceptional divisors over the two points of P in $M_3 = \mathrm{Bl}_P(\mathrm{Bl}_p\,\mathbb{CP}^2)$.

References

[BB00] Lionel Bayle and Arnaud Beauville. Birational involutions of $\mathbb{P}^2$. *Asian J. Math*, 4:11–18, 2000.

[Edm89] Allen L. Edmonds. Aspects of group actions on four-manifolds. *Topol. Its Appl.*, 31:109–124, 1989.

[FL21] Benson Farb and Eduard Looijenga. The Nielsen realization problem for K3 surfaces. *Jour. Diff. Geom.*, 2021. to appear.

[FM88] Robert Friedman and John W. Morgan. On the diffeomorphism types of certain algebraic surfaces. I. *J. Differ. Geom.*, 27:297–369, 1988.

[Fre82] Michael Hartley Freedman. The topology of four-dimensional manifolds. *J. Differ. Geom.*, 17:357–453, 1982.

[HZ74] F. Hirzebruch and D. Zagier. *The Atiyah–Singer Theorem and Elementary Number Theory*. Publish or Perish, Inc., 1974.

[Kon22] Hokuto Konno. Dehn twists and the Nielsen realization problem for spin 4-manifolds. 2022. arXiv:2203.11631.

[Lee21] Seraphina Eun Bi Lee. The Nielsen realization problem for high degree del Pezzo surfaces. 2021. arXiv:2112.13500.

[Lee22] Seraphina Eun Bi Lee. Isotopy classes of involutions of del pezzo surfaces. 2022. arXiv:2202.10616.

[Qui86] Frank Quinn. Isotopy of 4-manifolds. *J. Differ. Geom.*, 24:343–372, 1986.

[Thu97] William P. Thurston. *Three-Dimensional Geometry and Topology*, volume 1. Princeton University Press, 1997.

[Vin72] É. B. Vinberg. On groups of unit elements of certain quadratic forms. *Math. USSR Sbornik*, 16(1):17–35, 1972.

[Wal64a] C.T.C. Wall. Diffeomorphisms of 4-manifolds. *J. London Math. Soc.*, s1-39(1):131–140, 1964.

[Wal64b] C.T.C. Wall. On the orthogonal groups of unimodular quadratic forms. II. *J. Reine Angew. Math.*, 1964(213):122–136, 1964.

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