Mapping class groups of del Pezzo surfaces

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del Pezzo manifolds

Definition

A del Pezzo manifold M is one of

$$M = \mathbb{CP}^1 \times \mathbb{CP}^1 \qquad \textit{or} \qquad \mathrm{Bl}_n\, \mathbb{CP}^2$$

where $0 \leq n \leq 8$.

Lemma

There is a diffeomorphism

$$M_n := \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2} \cong \mathrm{Bl}_n\, \mathbb{CP}^2.$$

Goal

Examples of $G \leq \mathrm{Mod}(M_n) := \pi_0(\mathrm{Homeo}^+(M_n))$ *and their diffeomorphism representatives.*

Examples of diffeomorphisms

Example

$$\sigma : [X : Y : Z] \mapsto [-X : Y : Z] \curvearrowright \mathbb{CP}^2$$

Then $\sigma \in \mathrm{PGL}_3(\mathbb{C})$ and $[\sigma] = \mathrm{Id}$.

Example

$$\tau : [X : Y : Z] \mapsto [\overline{X} : \overline{Y} : \overline{Z}] \curvearrowright \mathbb{CP}^2$$

Then $\tau_* = -\mathrm{Id} : H_2(M) \to H_2(M)$ so $[\tau] \neq \mathrm{Id}$.

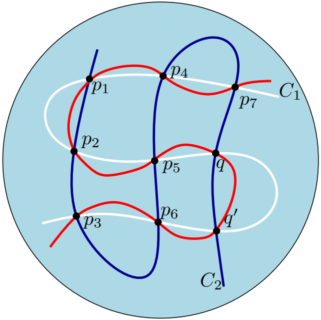
Example

Three classical involutions on (some) $\mathrm{Bl}_n \mathbb{CP}^2$ from birational automorphisms of $\mathbb{CP}^2$:

Geiser, Bertini, de Jonquiére.

Classical example 1: Geiser involution

$\gamma \in \mathrm{Aut}(\mathrm{Bl}_P\,\mathbb{CP}^2)$ $\qquad P = \{7 \text{ points in general position}\}$



For any $q \in \mathbb{CP}^2 - P$,
$C_1, C_2 =$ cubic curve through $P \cup \{q\}$

$\gamma(q) = q'$ where $C_1 \cap C_2 = P \cup \{q, q'\}$

(well-defined by Cayley–Bacharach)

γ extends to an order two diffeomorphism of $\mathrm{Bl}_P\,\mathbb{CP}^2 \cong \mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$.

Mapping class groups of del Pezzo manifolds

Corollary (Special case of Freedman '82, Quinn '86)

$$\begin{aligned} \mathrm{Mod}(M_n) &\to \mathrm{Aut}(H_2(M_n), Q_{M_n}) \cong \mathrm{O}(1,n)(\mathbb{Z}) \\ [f] &\mapsto f_* : H_2(M_n) \to H_2(M_n) \end{aligned}$$

is an isomorphism.

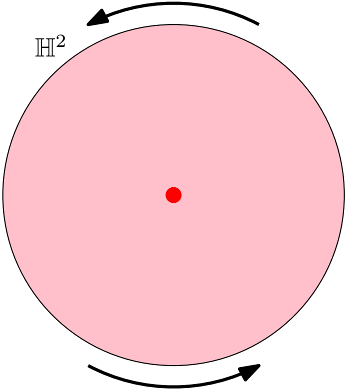
$$\underbrace{\mathrm{O}^+(1,n)(\mathbb{Z})}_{[\mathrm{O}(1,n)(\mathbb{Z}):\mathrm{O}^+(1,n)(\mathbb{Z})]=2} \leq \mathrm{O}^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^n)$$

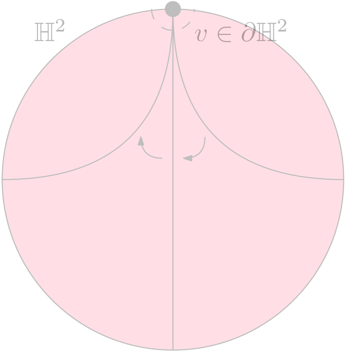
(via the hyperboloid model for $\mathbb{H}^n \subseteq \mathbb{R}^{n+1}$)

$O^+(1, n)(\mathbb{Z}) \leq \mathrm{Isom}(\mathbb{H}^n)$

Two specific examples:

1. Elliptic: $G = \mathbb{Z}/2\mathbb{Z}$
2. Parabolic: G fixes a unique point $v \in \partial\mathbb{H}^n$.





New mapping classes out of old

$M_n \cong M\#k\overline{\mathbb{CP}^2}$ and

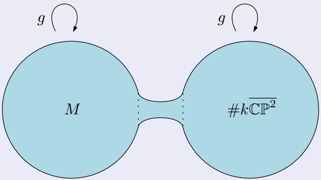
$$g_0 \curvearrowright H_2(M), \qquad g_1 \curvearrowright H_2(\#k\overline{\mathbb{CP}^2}).$$

$\rightsquigarrow H_2(M_n) \cong H_2(M) \oplus H_2(k\overline{\mathbb{CP}^2})$ and

$$g = g_0 \oplus g_1 \curvearrowright H_2(M) \oplus H_2(k\overline{\mathbb{CP}^2})$$

Definition

Any mapping class g constructed as above is called reducible.

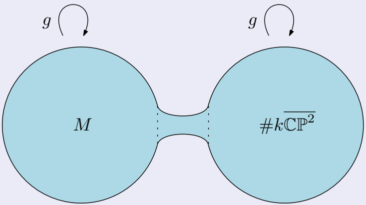


$G = \mathbb{Z}/2\mathbb{Z}$

Theorem (L. '22)

Let $1 \leq n \leq 8$ *and let* $g \in \mathrm{O}^+(1, n)(\mathbb{Z}) \leq \mathrm{Mod}(M_n)$ *have order* 2. *Exactly one of the following holds:*

1. g *is reducible.*



2. g *is realized by a de Jonquiéres* ($d > 2$)*, Geiser, or Bertini involution on* $\mathrm{Bl}_P\,\mathbb{CP}^2$ *for some* $P \subseteq \mathbb{CP}^2$.

Nielsen realization for $G = \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccc} & \overset{\exists ?}{\dashrightarrow} & \mathrm{Diff}^+(M_n) \\ & & \downarrow \\ \mathbb{Z}/2\mathbb{Z} & \hookrightarrow & \mathrm{Mod}(M_n) = \pi_0(\mathrm{Homeo}^+(M_n)) \end{array}$$

Corollary (L. '22)

Let M be a del Pezzo manifold. Any order-2 element $g \in \mathrm{Mod}(M)$ has a diffeomorphism representative of order 2.

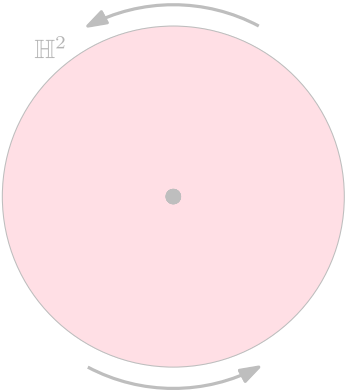
Proof idea.

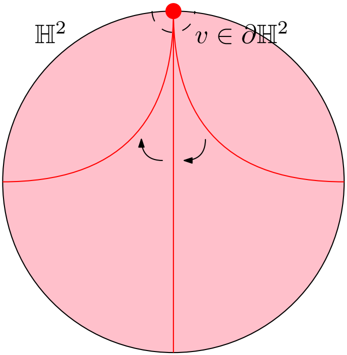
1. Reducible: construct a diffeomorphism of order 2 inductively by equivariant connected sums,
2. Otherwise, g is realized by one of three classical involutions. □

$O^+(1, n)(\mathbb{Z}) \leq \mathrm{Isom}(\mathbb{H}^n)$

Two specific examples:

1. Elliptic: $G = \mathbb{Z}/2\mathbb{Z}$
2. Parabolic: G fixes a unique point $v \in \partial\mathbb{H}^n$.





Properties of parabolic case

1. Fixed isotropic class:

$$A \in \mathrm{O}^+(1,n)(\mathbb{Z}) \text{ parabolic} \quad \longleftrightarrow \quad \exists \text{ isotropic } v \in H_2(M_n;\mathbb{Z})_{\neq 0} \text{ such that } A \cdot v = v.$$

2. Short exact sequence:

$$0 \to \underbrace{\Lambda}_{\cong \mathbb{Z}^{n-1}} \to \mathrm{Stab}(v) \xrightarrow{q} \underbrace{\mathrm{O}(v^\perp / v)}_{\text{finite}} \to 0$$

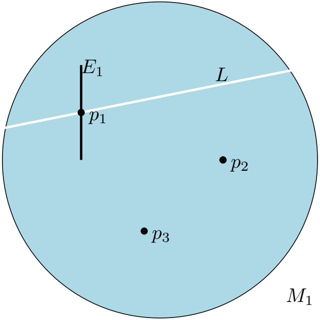
   with Λ parabolic.

3. For $n \leq 8$, there exists a unique $\mathrm{O}(1,n)(\mathbb{Z})$-orbit of primitive, isotropic vectors v.

Conic bundles

$$\pi : M_n \to \mathbb{CP}^1$$

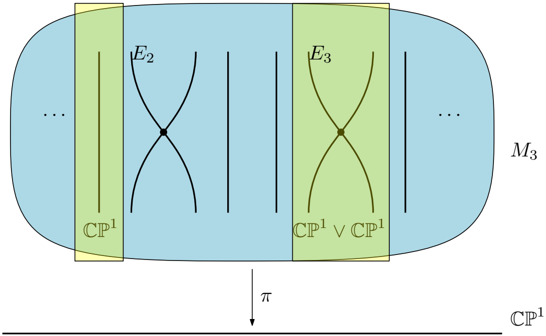
"Project onto E_1."



1. $b : M_n \to M_1$
   Blow down $E_2, \ldots, E_n$
2. $p : M_1 \to \mathbb{CP}^1$
   Space of lines through
   $p_1 = E_1 \cong \mathbb{CP}^1$
   $p|_L \equiv [L] \in \mathbb{CP}^1$
3. $p : M_1 \to \mathbb{CP}^1$: Unique nontrivial $\mathbb{S}^2$-bundle over $\mathbb{S}^2$

$v \in \partial\mathbb{H}^n$ as fiber class of conic bundles

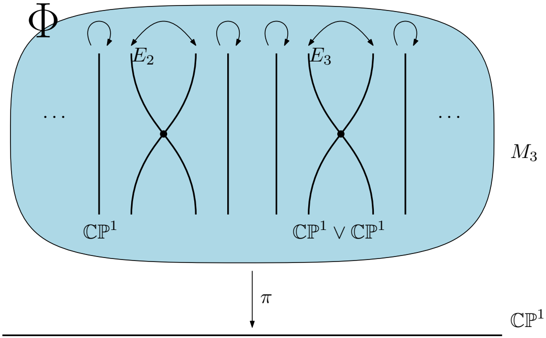
$$\pi : M_n \to \mathbb{CP}^1$$



1. Smooth fibers $F \cong \mathbb{CP}^1$ with $[F] = v$,
2. $(n-1)$-many singular fibers $\cong \mathbb{CP}^1 \vee \mathbb{CP}^1$.

Classical example 2: de Jonquiéres involution

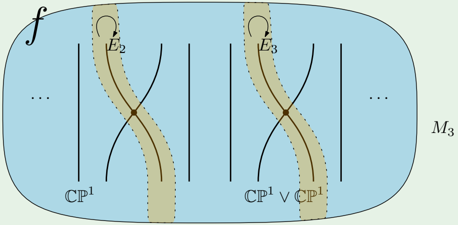
$$\pi : M_n \to \mathbb{CP}^1$$



$\Phi \in \mathrm{Aut}(\mathrm{Bl}_P\,\mathbb{CP}^2)$, $\quad P =$ certain subset of n-many points, n odd

More examples:

Example



1. $\text{supp}(f) =$ normal neighborhoods of $E_2, \dots, E_n$
2. $f|_{E_k} =$ orientation-reversing reflection

Example

$$[f \circ \underbrace{\Phi}_{\text{de Jonquiéres}}] \in \Lambda \cong \mathbb{Z}^{n-1}$$

Theorem (L., in progress)

The subgroup $\Lambda \cong \mathbb{Z}^{n-1} \leq \mathrm{Mod}(M_n)$ is realized by diffeomorphisms.

$$
\begin{array}{ccc}
 & & \mathrm{Diff}^+(M_n) \\
 & \overset{s}{\nearrow} & \downarrow \\
\Lambda & \hookrightarrow & \mathrm{Mod}(M_n) = \pi_0(\mathrm{Homeo}^+(M_n))
\end{array}
$$

All $\varphi \in s(\mathbb{Z}^{n-1})$ preserve the fibers of π away from a neighborhood of all singular fibers.