

p -adic modular forms and the Hodge-Tate period map

Sean Howe

University of Chicago

AMS Summer Institute in Algebraic Geometry, 2015-07-28

Section 1

Introduction

Introduction

Goals of this talk

- ▶ To explain a simplified construction of spaces of overconvergent modular forms using Scholze's perfectoid modular curve at infinite level and the Hodge-Tate period map.
- ▶ To explore some new directions suggested by this construction.

Outline

Introduction

A brief history of geometric p -adic modular forms

The Hodge-Tate period map

Reduction of structure group on the flag variety

Some notation

- ▶ We fix a prime p and a tame level Γ away from p .
- ▶ Let \mathcal{Y}_{p^n} , resp. \mathcal{X}_{p^n} , be the modular curve of tame level Γ and full level p^n , resp its compactification, viewed as rigid analytic varieties over \mathbb{Q}_p .
- ▶ Let $\mathcal{Y}_{\text{ord}} \subset \mathcal{Y}$, resp. $\mathcal{X}_{\text{ord}} \subset \mathcal{X}$ be the *ordinary locus*, obtained by removing the finitely many open disks of supersingular reduction.
- ▶ Denoting \mathcal{E} the universal family of elliptic curves, let $\omega = (\text{Lie}\mathcal{E})^*$ be the standard modular bundle.
- ▶ Let $\widehat{\omega}$ be the completed line bundle corresponding to ω . This is the “correct” version of ω for comparison to étale objects, and there are more hats to come, but it’ll be ok to pretend hats aren’t there if you’re not familiar with the pro-étale site.

Section 2

A brief history of geometric p -adic modular forms

Katz's p -adic modular forms: an anachronistic summary

- ▶ Over \mathcal{X}_{ord} , p -adic Hodge theory furnishes a pro-étale reduction of structure group of $\widehat{\omega}$ from $\widehat{\mathcal{O}}^\times$ to $\mathbb{Z}_p^\times := \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times$.

- ▶ Concretely, there is a pro-étale \mathbb{Z}_p^\times torsor \mathcal{I} (the Igusa tower), coming from étale cohomology, and an isomorphism

$$\mathcal{I} \times_{\mathbb{Z}_p^\times} \widehat{\mathcal{O}}^\times \cong \text{Isom}(\widehat{\mathcal{O}}, \widehat{\omega})$$

- ▶ Works only over ordinary locus – uses that for an ordinary elliptic curve E the p -divisible group of E is an extension of an étale p -divisible group by $\widehat{\mathbb{G}}_m$.
- ▶ For any continuous character κ of \mathbb{Z}_p^\times , we obtain the sheaf of modular forms of weight κ over \mathcal{X}_{ord}

$$\widehat{\omega}^\kappa := \mathcal{I} \times_{\mathbb{Z}_p^\times, \kappa} \widehat{\mathcal{O}}$$

- ▶ p -adic modular forms are global sections

$$M^\kappa := H^0(\mathcal{X}_{\text{ord}}, \widehat{\omega}^\kappa)$$

Overconvergent modular forms

Definition

An *overconvergent* modular form of weight κ is a section of $\widehat{\omega}^\kappa$ over a neighborhood $\mathcal{X}_{\text{ord}}(\epsilon)$ of the ordinary locus.

Important, e.g., in studying p-adic families of modular forms.

Problem – The reduction of structure group to \mathbb{Z}_p^\times is only defined over \mathcal{X}_{ord} , so what is the definition of $\widehat{\omega}^\kappa$ on such a neighborhood for $\kappa \notin \mathbb{Z}$?

The first solution (Coleman)

- ▶ Eisenstein series vary in a natural p-adic family; can use this to transfer the definition from integral weight.
- ▶ Useful but unsatisfactory
 - ▶ Not geometric
 - ▶ Does not generalize well to other Shimura varieties

Geometric overconvergent modular forms I

The first geometric construction

(Pilloni, Andreatta-Iovita-Stevens)

- ▶ Construct neighborhoods $\mathcal{X}_{\text{ord}}(\epsilon)$ of \mathcal{X}_{ord} and reductions of structure group of ω to analytic groups $\mathbb{Z}_p^\times(\epsilon)$ with

$$\mathbb{Z}_p^\times \subset \mathbb{Z}_p^\times(\epsilon) \subset \mathcal{O}^\times$$

e.g.

$$\mathbb{Z}_p^\times(\epsilon) = \mathbb{Z}_p^\times \cdot (1 + p^n \mathcal{O}^+)$$

(a finite union of balls of radius $|p^n|$ indexed by $(\mathbb{Z}/p^n\mathbb{Z})^\times$.)

- ▶ Uses integral p-adic Hodge theory, canonical subgroups.
- ▶ A character κ of \mathbb{Z}_p^\times extends to an analytic neighborhood of \mathbb{Z}_p^\times , and thus we obtain overconvergent sheaves ω^κ .

Geometric overconvergent modular forms II

A new geometric interpretation

(H., Chojecki-Hansen-Johansson)

- ▶ Work equivariantly at infinite level and construct analytic reductions of structure group on the period domain, which can then be pulled back via the period map and descended to finite level.
- ▶ Closely related to approach of Pilloni, A.-I.-S., but offers several conceptual advantages:
 - ▶ Construction of overconvergent torsors and overconvergent loci is on \mathbb{P}^1 , where geometry is simple.
 - ▶ Transfers cleanly to Hodge-type Shimura varieties, including when μ -ordinary is not ordinary.
 - ▶ Concrete interpretation of overconvergent modular forms as functions satisfying a transformation rule $(1/(a + bz)^\kappa)$.
 - ▶ Canonical subgroup disentangled from exposition (replaced by Hodge-Tate period map).
 - ▶ Leads to new spaces of p -adic automorphic forms and overconvergence phenomena in more general settings.

Section 3

The Hodge-Tate period map

The Hodge-Tate sequence

Let E be an elliptic curve over \mathbb{C}_p . There is a Hodge-Tate exact sequence (or Hodge-Tate filtration)

$$0 \rightarrow \mathrm{Lie}E(1) \rightarrow T_p E \otimes_{\mathbb{Z}_p} \mathbb{C}_p \rightarrow \omega_E \rightarrow 0$$

- ▶ Loosely, a point in $T_p E$ induces, via the Weil pairing, a map $\widehat{E} \rightarrow \widehat{\mathbb{G}_m}$, and the arrow $T_p E \otimes \mathbb{C}_p \rightarrow \omega_E$ is just pullback of dt/t along this map. The left arrow is its dual.
- ▶ If E has ordinary reduction then the kernel is spanned by the canonical line in $T_p E$, and in particular is defined over \mathbb{Q}_p .
- ▶ Over a more general base, the Hodge-Tate sequence is an exact sequence of sheaves on the pro-étale site

$$0 \rightarrow \widehat{\mathrm{Lie}\mathcal{E}}(1) \rightarrow T_p \mathcal{E} \otimes \widehat{\mathcal{O}} \rightarrow \widehat{\omega}_{\mathcal{E}} \rightarrow 0$$

The Hodge-Tate period map

Theorem (Scholze)

The tower $(\mathcal{X}_{p^n})_{n \in \mathbb{N}}$ can be studied via a single adic space \mathcal{X}_∞ over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, perfectoid over $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cyc}}, \mathbb{Z}_p^{\mathrm{cyc}})$.

($\mathcal{X}_\infty =$ “Moduli of elliptic curves equipped with a trivialization of the Tate module”)

The Hodge-Tate filtration induces a $GL_2(\mathbb{Q}_p)$ and Hecke equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{X}_\infty \rightarrow \mathbb{P}^1$$

such that $\widehat{\omega} = \pi_{\mathrm{HT}}^ \mathcal{O}(1)$. Furthermore,*

$$\overline{\mathcal{X}_{\mathrm{ord}, \infty}} = \pi_{\mathrm{HT}}^{-1}(\mathbb{P}^1(\mathbb{Q}_p))$$

The Hodge-Tate period map and reduction of structure group

- ▶ Should think $\mathcal{X} = GL_2(\mathbb{Z}_p) \backslash \mathcal{X}_\infty$.
- ▶ A reduction of structure group for $\widehat{\omega}$ over some $\mathcal{X}_{\text{ord}}(\epsilon) \subset \mathcal{X}$ is the same as a $GL_2(\mathbb{Z}_p)$ -equivariant reduction of structure group of $\widehat{\omega}$ over a $GL_2(\mathbb{Z}_p)$ -invariant $\mathcal{X}_{\text{ord}}^\infty(\epsilon) \subset \mathcal{X}^\infty$
- ▶ **Key observation:**
To produce overconvergent torsors, and thus overconvergent modular forms, it suffices to construct $GL_2(\mathbb{Z}_p)$ -equivariant reductions of structure group of $\mathcal{O}(1)$ on $GL_2(\mathbb{Z}_p)$ -invariant neighborhoods of $\mathbb{P}^1(\mathbb{Q}_p)$ in \mathbb{P}^1 , which can then be pulled back via π_{HT} and descended to finite level.

Example

- ▶ The restricted map

$$\pi_{\text{HT}} : \mathcal{X}_{\text{ord}, \infty} \rightarrow \mathbb{P}^1(\mathbb{Z}_p) = \mathbb{P}^1(\mathbb{Q}_p)$$

is the connected component map

- ▶ It is a map of adic spaces, where the profinite set $\mathbb{P}^1(\mathbb{Z}_p)$ has a natural structure of an adic space.
- ▶ Over $\mathbb{P}^1(\mathbb{Z}_p)$, the sheaf \mathcal{O}^\times is equal to the sheaf \mathbb{Q}_p^\times , and using this we can obtain a reduction of structure group of $\mathcal{O}(1)$ to \mathbb{Z}_p^\times !
- ▶ This reinterprets the Katz reduction of structure group over the ordinary locus.

Section 4

Reduction of structure group on the flag variety

New analytic moduli

- ▶ Let $U \subset \mathbb{P}^1$ be a reasonable subset, e.g. U a connected affinoid.
- ▶ There is a moduli space $\mathcal{X}_{\infty,U} = \pi_{HT}^{-1}(U)$ of elliptic curves with a basis for the Tate module such that the Hodge-Tate filtration with respect to that basis lies in U .
- ▶ The set U is preserved by a finite index subgroup Γ_U of $GL_2(\mathbb{Z}_p)$, and thus defines a new finite level analytic moduli space \mathcal{X}_U inside the classical modular curve X_{Γ_U} .
- ▶ For appropriate U , $\mathcal{O}(1)$ admits a Γ_U -equivariant reduction of structure group over U , and thus $\widehat{\omega}$ admits a reduction of structure group over \mathcal{X}_U .

Example

- ▶ $U = GL_2(\epsilon) \cdot \infty$, where

$$GL_2(\epsilon) = GL_2(\mathbb{Z}_p) \cdot (\text{Id} + p^n M_2(\mathcal{O}^+))$$

$$GL_2(\mathbb{Z}_p) \subset GL_2(\epsilon) \subset GL_2(\mathcal{O})$$

- ▶ $\Gamma_U = GL_2(\mathbb{Z}_p)$
- ▶ \mathcal{X}_U is a neighborhood of the ordinary locus inside $\mathcal{X}_{\Gamma_U} = \mathcal{X}$
- ▶

$$GL_2(\epsilon) \rightarrow GL_2(\epsilon)/P \cap GL_2(\epsilon) = U$$

gives a $GL_2(\mathbb{Z}_p)$ -equivariant reduction of structure group of $\mathcal{O}(1)$ to $\mathbb{Z}_p^\times \cdot (1 + p^n \mathcal{O}^+)$.

- ▶ $U = \{[1 : z] \mid |z| \leq |p^n|\}$, a ball around ∞
 - ▶ $\Gamma_U = \Gamma_0(p^n)$
 - ▶ \mathcal{X}_U is a neighborhood of the ordinary locus at level 1, lifted to a subset of $\mathcal{X}_{\Gamma_0(p^n)}$ via the canonical subgroup.

Example

- ▶ $U = \{[1 : z] \mid \inf_{a \in p\mathbb{Z}_p} (|z - a|) \leq p^w\}$ for $w \in \mathbb{Q}^\times, w \geq 1$.
 - ▶ $\Gamma_U = \Gamma_0(p)$
 - ▶ $\mathcal{X}_U \subset \mathcal{X}_{\Gamma_0(p)}$ is the w -ordinary locus, as defined by Chojecki-Hansen-Johannson.
- ▶ $U = GL_2(\epsilon) \cdot x$ for $x \in \mathbb{P}^1(K) \setminus \mathbb{P}^1(\mathbb{Q}_p), [K : \mathbb{Q}_p] = 2$.
 - ▶ $\Gamma_U = GL_2(\mathbb{Z}_p)$
 - ▶ $\mathcal{X}_U \subset \mathcal{X}$ is a neighborhood of a set of curves whose p -divisible groups have isogenies by K (in particular, this lives inside the super-singular locus).
 - ▶ Over $\mathcal{X}_U, \widehat{\omega}$ admits a reduction of structure group to an analytic neighborhood of \mathcal{O}_K^\times
 - ▶ On these neighborhoods we obtain “overconvergent modular forms” of weights given by characters of \mathcal{O}_K^\times !

A broader perspective

This approach leads to canonically defined analytic loci and reduction of structure group phenomena in much greater generality:

- ▶ Hodge-type Shimura varieties
- ▶ Canonical loci and over-convergent Hodge bundles for *any* smooth proper family of rigid analytic varieties.
- ▶ Any time you have an étale torsor for a lattice in an algebraic group G over a finite extension of \mathbb{Q}_p and a reduction of structure group after pushing out to the corresponding $G(\hat{\mathcal{O}})$ torsor.

Related work

- ▶ Chojecki-Hansen-Johansson, arXiv: 1507.04875
 - ▶ Use interpretation of overconvergent modular forms as functions at infinite level satisfying a transformation property (coming from a canonical trivialization of $\mathcal{O}(1)$ over one of the loci defined before)
 - ▶ Use the flag variety and the Hodge-Tate period map to define overconvergent loci.
 - ▶ Prove a strong version of the Andreatta-Iovita-Stevens overconvergent Eichler-Shimura isomorphism for compact Shimura curves over \mathbb{Q} .
 - ▶ Don't use the analytic torsors perspective or explore more general reduction of structure group phenomena.
- ▶ Hansheng Diao and Fucheng Tan

That's all! Thanks for coming!

- ▶ Contact: seanpkh@gmail.com
- ▶ Slides available at math.uchicago.edu/~seanpkh