The p-adic Jacquet-Langlands correspondence and a question of Serre

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Outline

Serre's mod p Jacquet-Langlands

A p-adic Jacquet-Langlands

Proof (highlights)

Section 1

Serre's mod p Jacquet-Langlands

Some notation

- p is a fixed prime
- X is the compactified modular curve with level

$$K = GL_2(\mathbb{Z}_p) \cdot K^p, \;\; K^p \subset GL_2(\mathbb{A}_f^{(p)})$$

- ω/X is the modular sheaf.
- D is the quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$.
- $\mathcal{O}$ is the maximal order in $D(\mathbb{Q}_p)$, and

$$\psi : \mathcal{O}^\times \to \mathbb{F}_{p^2}^\times$$

is the natural map (reduction mod uniformizer).

Theorem (Serre, 1987)

The Hecke eigensystems appearing in the space of mod p modular forms

$$\oplus_{K^p,k} H^0(X_{\overline{\mathbb{F}_p}}, \omega^k)$$

are the same as those appearing in the space of quaternionic mod p automorphic forms

$$\mathrm{Cont}(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f), \overline{\mathbb{F}_p})$$

Furthermore, a Hecke eigensystem in the latter corresponding to a form of weight k can be chosen so that the action of $\mathcal{O}^\times$ is by ψ^k.

Proof sketch.

For a fixed level $K = GL_2(\mathbb{Z}_p)K^p$, the set of super-singular curves over $\overline{\mathbb{F}_p}$ with level K-structure and a $\mathbb{F}_{p^2}$-rational differential non-vanishing is in bijection with

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/\ker\psi \cdot K^p.$$

A modular form is just a function on elliptic curves with non-vanishing differential (homogeneous of weight k), so evaluating on this set we obtain functions.

$\square$

Section 2

A p-adic Jacquet-Langlands

Serre's question

Question [Serre]. *(26) Analogues p-adiques. Au lieu de regarder les fonctions localement constantes sur* $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)$*, à valeurs dans* $\mathbb{C}$*, il serait plus amusant de regarder celles à valeurs p-adiques [...] localement constants par rapport á la variable dans* $D^\times(\mathbb{A}_f^{(p)})$ *et continues (ou analytiques, ou davantage) par rapport a la variable dans* $D^\times(\mathbb{Q}_p)$*...*
Y aurait-il des representations galoisiennes p-adiques associés à de telles fonctions, supposés fonctions propres des opérateurs de Hecke? Peut-on interpreter les constructions de Hida (et Mazur) dans un tel style? Je n'en ai aucune idée.

Theorem (H.)

The Hecke eigensystems appearing in the space of overconvergent modular forms of level K^p also appear in the space of p-adic automorphic functions on $D^\times$,

$$\mathrm{Cont}(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^{(p)})/K^p, \overline{\mathbb{Q}_p}).$$

Furthermore, if f is an overconvergent eigenform of weight

$$\kappa : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}_p}^\times$$

and we fix a quadratic extension $E/\mathbb{Q}_p$ and an embedding $E \hookrightarrow D(\mathbb{Q}_p)$, the automorphic function on $D^\times$ corresponding to f can be chosen such that the action of $R^\times$ for a sufficiently small order

$$R \subset E \subset D(\mathbb{Q}_p)$$

is by the character $\kappa(r)$

Remark

- *In the second part, κ is thought of a character of some analytic neighborhood of $\mathbb{Z}_p^\times$ in $\mathbb{G}_m$ containing R.*
- *Full result is stronger: gives a Hecke, R-equivariant injection from M_κ^w to automorphic functions for any fixed radius of overconvergence w.*
- *Does not require finite slope.*
- *Does not use interpolation of the classical correspondence for integral weight forms – construction is completely explicit, as in Serre's mod p correspondence.*

Comparison with other p-adic Jacquet-Langlands results

- Knight and Scholze have both (independently) constructed a local p-adic Jacquet-Langlands correspondence using the cohomology of the Drinfeld/Lubin-Tate towers and shown local-global compatibility for the completed cohomology of Shimura curves.
  - Conjecturally their correspondences agree.
  - Their construction gives little information about the division algebra representation (for example, away from the deRham case is the division algebra representation non-zero?).
  - It would be natural to conjecture a local-global compatibility statement between our global correspondence and their local correspondence. This global construction would then be a useful way to study the division algebra representations appearing in the local correspondence.
- Other work (e.g. Chenevier, Newton) considers D split at p.

Generalizations

- Correspondence between arbitrary totally definite quaternion algebras over totally real fields and quaternionic Shimura curves.
- Part of a larger program to construct explicit functorialities between spaces of p-adic automorphic forms on Shimura varieties which includes, e.g., a direct comparison of overconvergent modular forms and completed cohomology of modular curves. [Work in progress.]

Section 3

Proof (highlights)

Main ingredients

- A new construction of overconvergent modular forms.
  - Key point – use the pro-étale site and the Hodge-Tate period map to reduce to studying reduction of structure group of $O(1)$ on $\mathbb{P}^1$. Gives a very general method for understanding overconvergent automorphic forms and Hecke operators at p for abelian-type Shimura varieties, including, e.g., non-Hodge type and those where μ-ordinary is not ordinary.
  - Due independently also to Chojecki-Hansen-Johansson for Shimura curves over $\mathbb{Q}$.

Main ingredients (continued)

- p-adic uniformization at (supersingular) formal CM points.
  - There exist (supersingular) formal CM points with Hodge-Tate period arbitrarily close to $\mathbb{P}^1(\mathbb{Q}_p)$ (and thus in arbitrary neighborhoods of the ordinary locus).
  - Can evaluate at a formal CM point to get a function on a twisted version of the double coset we are interested in.
  - Working at formal CM points gives a reciprocity law that can be used to obtain information about the $D^\times$ action.

End matter

Bibliography:

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More info:

- Contact: seanpkh at gmail dot com
- Slides available at math.uchicago.edu/$\sim$seanpkh
- Article on arXiv soon!