Point counting, stability, and the complement of the discriminant locus

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Outline

Definition

Let $\mathbb{F}_q$ be a finite field. A *degree d hypersurface* in $\mathbb{P}^n_{\mathbb{F}_q}$ is the vanishing set of a nonzero degree d homogeneous polynomial $F \in \mathbb{F}_q[X_0, ..., X_n]$. It is *smooth* if the partials $\partial F / \partial X_i$ do not all vanish at any point in $\mathbb{P}^n(\overline{\mathbb{F}_q})$ (for a given hypersurface the choice of F is unique up to scaling, and thus this condition is independent of the equation chosen).

Question: How many **smooth** hypersurfaces of degree d are there in $\mathbb{P}^n_{\mathbb{F}_q}$?

- The answer depends on q, d, and n.
- There are $q^{\binom{n+d}{d}}$ homogeneous polynomials of degree d in $n+1$ variables, and thus $(q^{\binom{n+d}{d}} - 1)/(q-1)$ degree d hypersurfaces in $\mathbb{P}^n_{\mathbb{F}_q}$.

Question: What is the probability that a random degree d homogeneous polynomial in $\mathbb{F}_q[X_0, ..., X_n]$ is smooth (that is, that the corresponding hypersurface is smooth)?

- This is a normalized version of the previous question.
- For fixed q and n, we should only expect a nice answer asymptotically as $d \to \infty$.

Theorem (Poonen)

As $d \to \infty$, the probabilty that a random degree d homogeneous polynomial in $\mathbb{F}_q[X_0, ..., X_n]$ is smooth approaches

$$\zeta_{\mathbb{P}^n_{\mathbb{F}_q}}(n+1)^{-1} = \prod_{1 \leq k \leq n} (1 - 1/q^k)$$

- Poonen's theorem is much more general – for example, it also answers the analogous question for hypersurface sections of arbitrary quasi-projective varieties.
- Proof by sieving

Taking $n = 1$ and replacing $\mathbb{P}^1$ with $\mathbb{A}^1$, we obtain a particularly simple version of this problem:

Theorem

For $d \geq 2$, the probability that a degree d monic polynomial in one variable over $\mathbb{F}_q$ ($f = x^d + a_{d-1}x^{d-1} + ... + a_0 \in \mathbb{F}_q[x]$) is squarefree is

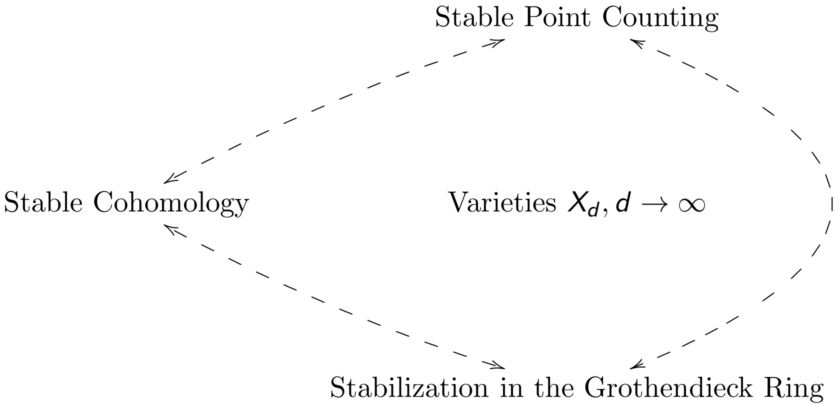
$$\zeta_{\mathbb{A}^1}(2)^{-1} = 1 - 1/q$$

- Smooth = squarefree.
- Here the statement is literally true for all $d \geq 2$, not just in the limit.

Three proofs:

1. Sieving.
   - *The probability that a random $n \in \mathbb{Z}$ is squarefree is $\zeta_{\mathbb{Z}}(2)^{-1}$.*
     "Proof:"
     $$\prod_p (1 - 1/p^2) = \zeta_{\mathbb{Z}}(2)^{-1}$$
2. Computation in the Grothendieck ring.
   $$[\text{Space of monic polynomials of degree d}] = [\mathbb{A}^d] - [\mathbb{A}^{d-1}]$$
3. Stable cohomology.
   - Space of monic squarefree polynomials $= \mathrm{Conf}^d \mathbb{A}^1$!
     (polynomial $\leftrightarrow$ roots).
   - The Grothendieck-Lefschetz formula lets us compute the count from the étale cohomology of $\mathrm{Conf}^d \mathbb{A}^1$!
   - Étale cohomology = Singular cohomology of $\mathrm{Conf}^d \mathbb{C}$!
   - $H^i(\mathrm{Conf}^d, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 1 \\ 0 & \text{otherwise!} \end{cases}$

The Stable Sometimes-Triangle:



Philosophy: There are many obstacles to moving from one vertex to another, and one can easily cook up examples to show that the arrows can't always be filled in. Nevertheless, for "natural" examples, we should expect the data on the three vertices to be consistent!

Back to hypersurfaces...

Definition

Let $X_{d,n}$ be the space of smooth degree d homogeneous polynomials in $n+1$ variables, where smooth means the hypersurface in $\mathbb{P}^n$ defined by $f = 0$ is smooth.

- $X_{d,n}$ is an open subvariety of $\mathbb{A}^{\binom{n+d}{d}}$; it is the complement of the *discriminant locus*, the set of polynomials defining singular hypersurfaces.
- The question "what is the probability a random homogeneous polynomial is smooth?" is a question about $\#X_{d,n}(\mathbb{F}_q)$ as $d \to \infty$.

Theorem (Poonen)

$\lim_{d\to\infty} \#X_{d,n}(\mathbb{F}_q)/q^{\binom{n+d}{d}} = \zeta_{\mathbb{P}^n}(n+1)^{-1} = \prod_{1\leq k\leq n}(1-1/q^k)$

Theorem (Tommasi)

The rational cohomology of $X_{d,n}$ stabilizes to the cohomology of GL_{n+1}, i.e., for any i and for $d >> 0$, $H^i(X_{d,n},\mathbb{Q}) \cong H^i(GL_{n+1},\mathbb{Q})$.

- The isomorphism is induced by the orbit map, so also get, e.g., Hodge structures, cup product on stable cohomology.
- In general, can't move from stable cohomology to point counting (this side of the triangle is missing).
  - Don't have control over unstable cohomology
  - Don't have control over primes where base change is valid.
- Nevertheless, If we plug Tommasi's result naively into the Grothendieck-Lefschetz formula, we do recover Poonen's result.

Question: What is the distribution of the number of $\mathbb{F}_{q^k}$-rational points on a random smooth degree d hypersurface in $\mathbb{P}^n_{\mathbb{F}_q}$?

- So far, we've talked about what the denominator should be in this kind of problem – that is, the number of smooth hypersurfaces.

Theorem (Poonen)

As $d \to \infty$, the number of $\mathbb{F}_q$-rational points on a random smooth hypersurface of degree d approaches a binomial random variable, the sum of $\#\mathbb{P}^n(\mathbb{F}_q)$ independent random variables that are 1 with probability

$$p = (q^n - 1)/(q^{n+1} - 1)$$

- Poonen's sieving results also give the distribution of the number of $\mathbb{F}_{q^k}$-rational points for any k.

Question: What can we say about this problem using stable cohomology? And vice versa?

- $$H^*(X_{d,n}, \mathbb{Q}) \leftrightarrow \text{Number of smooth hypersurfaces}$$
  $$H^*(X_{d,n}, \mathcal{V}) \leftrightarrow \text{Distribution of points on hypersurfaces}$$
  For certain local systems $\mathcal{V}$ on $X_{d,n}$ (coming from the primitive cohomology of the universal family).
- For monic polynomials, Church, Ellenberg, and Farb use the theory of representation stability to show that the cohomology of $\mathrm{Conf}^d \mathbb{A}^1$ with twisted coefficients stabilizes and obtain results on the distribution of roots of squarefree polynomials over $\mathbb{F}_q$.

Conjecture

Let $\mathcal{V}$ be a local system on $X_{d,n}$ obtained by applying a Schur functor to the primitive cohomology $\mathcal{P}$ of the universal hypersurface over $X_{d,n}$. Then $H^*(X_{d,n}, \mathcal{V})$ stabilizes as $d \to \infty$ and is stably of Tate type.

- Using Poonen's results in specific cases, we can formulate precise conjectures on the stable cohomology, e.g.:

Conjecture

$H^*(X_{d,n}, \mathcal{P})$ stabilizes to 0.

- The corresponding statistic is that the average number of $\mathbb{F}_q$-rational points on a smooth degree d hypersurface over $\mathbb{F}_q$ is zero.

Current status:

- Many instances of the conjecture should be accessible through the techniques of Tommasi (applied to certain related auxiliary spaces), but still working on this.
- There should be a cleaner and more complete statement of the conjecture in the spirit of representation stability results (much is known about the monodromy groups and so we can relate local systems using highest weight theory).
- Working on obstacles to actually recovering point counting results from stable cohomology (comparison of asymptotics).
- Generalization to ample line bundles on quasi-projective varieties (Poonen's results work here).
- Relation to motivic results and conjectures (Vakil-Wood).

Moduli of curves

- Looijenga (building on many others): Computation of the stable cohomology with twisted (symplectic) coefficients for the mapping class group.
- Achter, Erman, Kedlaya, Wood, Zurieck-Brown: A heuristic determination of the distribution of points on a random curve – they use Grothendieck-Lefschetz to compute the moments of the distribution, but only under the hypothesis that unstable cohomology doesn't contribute.
- This is precisely the opposite situation of hypersurfaces! For moduli of curves all of the results on stable cohomology are known, but because sieving techniques are unavailable (there is no natural ambient space!), the point-counting results are not.

That's all!

- Questions?
- Contact: s e a n p k h ATSYMBOL g m a i l . c o m