Center Manifolds and Hamiltonian Evolution Equations

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Equations: focusing nonlinear Klein-Gordon, Schrödinger, critical wave
Review of local well-posedness theory, global existence vs. finite-time blowup. Forward scattering set $S_+$
Stationary solutions, ground states, variational analysis
Some questions about $S_+$, and some answer
Payne-Sattinger theory: global dynamics below the ground state energy, functionals $J$ and $K$.
Raising the bar: energies above the ground state energy.
Stable, Unstable, Center manifolds
Hyperbolic dynamics, ejection lemma
One-pass theorem, absence of almost homoclinic orbits
Conclusion
Introduction

Energy subcritical equations:

\[ \Box u + u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^{1+1}_{t,x} \text{ (even), } \mathbb{R}^{1+3}_{t,x} \]
\[ i\partial_t u + \Delta u = |u|^2 u \quad \text{in radial } \mathbb{R}^{1+3}_{t,x} \]

Energy critical case:

\[ \Box u = |u|^{2^*-2} u \quad \text{in radial } \mathbb{R}^{1+d}_{t,x} \quad (1) \]

\[ d = 3, 5. \]

Goals: Describe transition between blowup/global existence and scattering, “Soliton resolution conjecture”. Results apply only to the case where the energy is at most slightly larger than the energy of the “ground state soliton”.

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Center Manifolds and Hamiltonian Evolution Equations
Basic well-posedness, focusing cubic NLKG in $\mathbb{R}^3$

$\forall\, u[0] \in \mathcal{H}$ there $\exists!$ strong solution $u \in C([0, T); H^1)$, $\dot{u} \in C^1([0, T); L^2)$ for some $T \geq T_0(\|u[0]\|_{\mathcal{H}}) > 0$. Properties:
continuous dependence on data; persistence of regularity; energy conservation:

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\dot{u}|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 - \frac{1}{4} |u|^4 \right) \, dx$$

If $\|u[0]\|_{\mathcal{H}} \ll 1$, then global existence; let $T^* > 0$ be maximal forward time of existence: $T^* < \infty \implies \|u\|_{L^3([0, T^*), L^6(\mathbb{R}^3))} = \infty$.
If $T^* = \infty$ and $\|u\|_{L^3([0, T^*), L^6(\mathbb{R}^3))} < \infty$, then $u$ scatters:
$\exists (\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}$ s.t. for $v(t) = S_0(t)(\tilde{u}_0, \tilde{u}_1)$ one has

$$(u(t), \dot{u}(t)) = (v(t), \dot{v}(t)) + o_{\mathcal{H}}(1) \quad t \to \infty$$

$S_0(t)$ free KG evol. If $u$ scatters, then $\|u\|_{L^3([0, \infty), L^6(\mathbb{R}^3))} < \infty$.

Finite prop.-speed: if $\ddot{u} = 0$ on $\{|x - x_0| < R\}$, then $u(t, x) = 0$ on $\{|x - x_0| < R - t, 0 < t < \min(T^*, R)\}$. 

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Finite time blowup, forward scattering set

$T > 0$, exact solution to cubic NLKG

$$\varphi_T(t) \sim c(T - t)^{-\alpha} \quad \text{as} \quad t \to T_+$$

$\alpha = 1$, $c = \sqrt{2}$.

Use finite prop-speed to cut off smoothly to neighborhood of cone $|x| < T - t$. Gives smooth solution to NLKG, blows up at $t = T$ or before.

**Small data:** global existence and scattering. **Large data:** can have finite time blowup.

Is there a criterion to decide finite time blowup/global existence?

**Forward scattering set:** $S(t) = \text{nonlinear evolution}$

$$S_+ := \left\{ (u_0, u_1) \in \mathcal{H} := H^1 \times L^2 \mid u(t) := S(t)(u_0, u_1) \exists \forall \text{ times and scatters to zero, i.e.,} \right\}$$

$$\|u\|_{L^3([0,\infty);L^6)} < \infty$$
Forward Scattering set

\( S_+ \) satisfies the following properties:
- \( S_+ \supset B_\delta(0) \), a small ball in \( \mathcal{H} \),
- \( S_+ \neq \mathcal{H} \),
- \( S_+ \) is an open set in \( \mathcal{H} \),
- \( S_+ \) is path-connected.

Some natural questions:

1. Is \( S_+ \) bounded in \( \mathcal{H} \)?
2. Is \( \partial S_+ \) a smooth manifold or rough?
3. If \( \partial S_+ \) is a smooth mfld, does it separate regions of FTB/GE?
4. Dynamics starting from \( \partial S_+ \)? Any special solutions on \( \partial S_+ \)?
Stationary solutions, ground state

*Stationary solution* \( u(t, x) = \varphi(x) \) of NLKG, weak solution of

\[
- \Delta \varphi + \varphi = \varphi^3
\]  

(2)

Minimization problem

\[
\inf \{ \| \varphi \|_{H^1}^2 \mid \varphi \in H^1, \| \varphi \|_4 = 1 \}
\]

has radial solution \( \varphi_\infty > 0 \), decays exponentially, \( \varphi = \lambda \varphi_\infty \) satisfies (2) for some \( \lambda > 0 \).

Coffman: **unique ground state** \( Q \).

*Minimizes the stationary energy (or action)*

\[
J(\varphi) := \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} |\varphi|^2 - \frac{1}{4} |\varphi|^4 \right) dx
\]

amongst all nonzero solutions of (2). Dilation functional:

\[
K_0(\varphi) = \langle J'(\varphi) | \varphi \rangle = \int_{\mathbb{R}^3} (|\nabla \varphi|^2 + |\varphi|^2 - |\varphi|^4)(x) \, dx
\]
Some answers

**Theorem**

Let \( E(u_0, u_1) < E(Q, 0) + \varepsilon^2, (u_0, u_1) \in \mathcal{H}_{\text{rad}} \). In \( t \geq 0 \) for NLKG:

1. **finite time blowup**
2. **global existence and scattering to 0**
3. **global existence and scattering to \( Q \):**
   
   \[
   u(t) = Q + v(t) + O_{H^1}(1) \text{ as } t \to \infty, \text{ and }
   \]
   
   \[
   \dot{u}(t) = \dot{v}(t) + O_{L^2}(1) \text{ as } t \to \infty, \nabla v + v = 0, (v, \dot{v}) \in \mathcal{H}.
   \]

All 9 combinations of this trichotomy allowed as \( t \to \pm \infty \).

- Applies to \( \dim = 3 \), cubic power, or \( \dim = 1 \), all \( p > 5 \).
- Under **energy assumption** (EA) \( \partial S_+ \) is connected, smooth \( \text{mfld} \), which gives (3), separating regions (1) and (2). \( \partial S_+ \) contains \( (\pm Q, 0) \). \( \partial S_+ \) forms the **center stable manifold** associated with \( (\pm Q, 0) \).
- \( \exists \) 1-dimensional stable, unstable \( \text{mflds} \) at \( (\pm Q, 0) \). Stable \( \text{mfld} \): Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko.
Hyperbolic dynamics

\[ \dot{x} = Ax + f(x), \quad f(0) = 0, \quad Df(0) = 0, \quad \mathbb{R}^n = X_s + X_u + X_c, \]

\( A \)-invariant spaces, \( A \upharpoonright X_s \) has evals in \( \Re z < 0 \), \( A \upharpoonright X_u \) has evals in \( \Re z > 0 \), \( A \upharpoonright X_c \) has evals in \( i\mathbb{R} \).

If \( X_c = \{0\} \), \textbf{Hartmann-Grobman theorem}: conjugation to \( e^{tA} \).

If \( X_c \neq \{0\} \), \textbf{Center Manifold Theorem}: \( \exists \) local invariant mflds around \( x = 0 \), tangent to \( X_u, X_s, X_c \).

\[
X_s = \{ |x_0| < \varepsilon \mid x(t) \to 0 \text{ exponentially fast as } t \to \infty \} \\
X_u = \{ |x_0| < \varepsilon \mid x(t) \to 0 \text{ exponentially fast as } t \to -\infty \}
\]

Example:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} x + O(|x|^2)
\]

\( \text{spec}(A) = \{1, -1, i, -i\} \)
Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$.

- $\langle L_+ Q | Q \rangle = -2 \| Q \|_4^4 < 0$
- $L_+ \rho = -k^2 \rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: $L_+$ has no eigenvalues in $(0, 1]$, no threshold resonance (delicate!)

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}$$

Then $\text{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ simple evals. Formally: $X_s = P_1L^2$, $X_u = P_{-1}L^2$. $X_c$ is the rest.
The invariant manifolds

Figure: Stable, unstable, center-stable manifolds
Variational characterization

\[ J(Q) = \inf \{ J(\varphi) \mid \varphi \in H^1 \setminus \{0\}, K_0(\varphi) = 0 \} \]
\[ = \inf \{ J(\varphi) - \frac{1}{4}K_0(\varphi) \mid \varphi \in H^1 \setminus \{0\}, K_0(\varphi) \leq 0 \} \] (3)

**Note:** if minimizer \( \exists \varphi_\infty \geq 0 \) (radial), then Euler-Lagrange:
\[ J'(\varphi_\infty) = \lambda K_0'(\varphi_\infty), K_0(\varphi_\infty) = 0. \] So
\[ 0 = K_0(\varphi_\infty) = \langle J'(\varphi_\infty)|\varphi_\infty \rangle = \lambda \langle K_0'(\varphi_\infty)|\varphi_\infty \rangle = -2\lambda \| \varphi_\infty \|_4^4 \]
\[ \lambda = 0 \implies J'(\varphi_\infty) = 0 \implies \varphi_\infty = Q. \]

- Energy near \( \pm Q \) a "saddle surface": \( x^2 - y^2 \leq 0 \)
- Better analogy \( q(\xi) = -\xi_0^2 + \sum_{j=1}^{\infty} \xi_j^2 \) in \( \ell^2(\mathbb{Z}_0^+) \), "needle like"
- Similar picture for \( E(u, \dot{u}) < J(Q) \). Solution trapped by \( K \geq 0, K < 0 \) in that set.
Schematic depiction of $J, K_0$

Figure: The splitting of $J(u) < J(Q)$ by the sign of $K = K_0$

- Energy near $\pm Q$ a "saddle surface": $x^2 - y^2 \leq 0$
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- Similar picture for $E(u, \dot{u}) < J(Q)$. Solution trapped by $K \geq 0, K < 0$ in that set.
Payne-Sattinger theory I

\[ j_\varphi(\lambda) := J(e^\lambda \varphi), \varphi \neq 0 \text{ fixed.} \]

Normalize so that \( \lambda_* = 0 \). Then \( \partial_\lambda j_\varphi(\lambda)\big|_{\lambda=\lambda_*} = K_0(\varphi) = 0 \).

“Trap” the solution in the well on the left-hand side: need

\[ E < \inf\{j_\varphi(0) \mid K_0(\varphi) = 0, \varphi \neq 0\} = J(Q) \text{ (lowest mountain pass).} \]

Expect global existence in that case.
Invariant decomposition of $E < J(Q)$:

\[ \mathcal{P}S_+ := \{(u_0, u_1) \in \mathcal{H} \mid E(u_0, u_1) < J(Q), \ K_0(u_0) \geq 0\} \]

\[ \mathcal{P}S_- := \{(u_0, u_1) \in \mathcal{H} \mid E(u_0, u_1) < J(Q), \ K_0(u_0) < 0\} \]

In $\mathcal{P}S_+$ global existence in $\mathbb{R}$: $K_0(u(t)) \geq 0$ implies

\[ \|u(t)\|_{\mathcal{H}^1}^4 \leq \|u(t)\|_{\mathcal{H}^1}^2 \implies E \geq \frac{1}{4}\|u(t)\|_{\mathcal{H}^1}^2 + \frac{1}{2}\|\dot{u}(t)\|_2^2 \simeq E \]

In $\mathcal{P}S_-$ finite time blowup in both positive and negative times. Convexity argument: $y(t) := \|u(t)\|_{L^2}^2$ satisfies $K_0(u(t)) < -\delta$,

\[ \ddot{y} = 2[\|\dot{u}\|_2^2 - K_0(u(t))] \]
\[ = 6\|\dot{u}\|_2^2 - 8E(u, \dot{u}) + 2\|u\|_{\mathcal{H}^1}^2 \]

\[ \partial_{tt}(y^{-\frac{1}{2}}) = -\frac{1}{2}y^{-\frac{5}{2}}[y\ddot{y} - \frac{3}{2}\dot{y}^2] < 0 \]

So finite time blowup.
**Corollary:** $Q$ unstable.

$v_j = \lambda_j \rho + w_j, \ j = 0, 1, \ w_j \perp \rho, \ \omega = \sqrt{L + P_\rho}$

$$E(Q + v_0, v_1) = J(Q) + \frac{1}{2}(\langle L + v_0 | v_0 \rangle + \| v_1 \|^2_2) + O(\| v_0 \|^3_{H^1})$$

$$= J(Q) + \frac{1}{2}(\lambda_1^2 - k^2 \lambda_0^2) + \frac{1}{2}(\| \omega w_0 \|^2_2 + \| w_1 \|^2_2) + O(\| v_0 \|^3_{H^1})$$

$$K_0(Q + v_0) = -2\langle Q^3 | v_0 \rangle + O(\| v_0 \|^2_{H^1})$$

**Specialize:** \( v_0 = \varepsilon \rho, \ v_1 = 0 \):

$$E(Q + v_0, 0) = J(Q) - \frac{k^2}{2} \varepsilon^2 + O(\varepsilon^3) < J(Q)$$

$$K_0(Q + v_0) = -2\varepsilon\langle Q^3 | \rho \rangle + O(\varepsilon^2)$$

So \( \text{sign}(K_0) \) determined by \( \text{sign}(\varepsilon) \).
Numerical 2-dim section through $\partial S_+$ (with R. Donninger)

Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, $(A, B)$ vary in $[-9, 2] \times [-9, 9]$
- **RED**: global existence, **WHITE**: finite time blowup, **GREEN**: $PS_-$, **BLUE**: $PS_+$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)

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Beyond \( J(Q) \), center-stable manifold (radial)

Solve NLKG with \( u = \pm (Q + v) \rightarrow \ddot{v} + L_+ v = N(Q, v) \rightarrow \)

\[
\dot{\lambda}_+ - k \lambda_+ = \frac{1}{2k} N_\rho(Q, v) \quad (4)
\]

\[
\dot{\lambda}_- + k \lambda_- = -\frac{1}{2k} N_\rho(Q, v) \quad (5)
\]

\[
\ddot{\gamma} + L_+ \gamma = P^\perp_\rho N(Q, v) \quad (6)
\]

\( P_\rho N(Q, v) = N_\rho(Q, v) \rho, \; v = \lambda \rho + \gamma \). ODE \( \ddot{\lambda} - k^2 \lambda = N_\rho(Q, v) \) is diagonalized by

\[
\lambda_\pm = \frac{1}{2} (\lambda \pm k^{-1} \dot{\lambda})
\]

(4) corresponds to eval \( k \) of \( A = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \); (5) eval \(-k\); (6) to essential spectrum \( iR \setminus (-i, i) \) of \( A \). “Stabilize” exponential growth in (4): if \( N_\rho \equiv 0 \), means \( \lambda_+(0) = 0 \). In general:
Solving the system (4)-(6)

**Stability condition:**

\[
0 = \lambda_+(0) + \frac{1}{2k} \int_0^\infty e^{-sk} N_\rho(Q, v)(s) \, ds
\]  

(7)

yields (recall \( v = \lambda \rho + \gamma \))

\[
\lambda(t) = e^{-kt} \left[ \lambda(0) + \frac{1}{2k} \int_0^\infty e^{-ks} N_\rho(s) \, ds \right] + \frac{1}{2k} \int_0^\infty e^{-k|t-s|} N_\rho(s) \, ds
\]

\[
\ddot{\gamma} + L_+ \gamma = P_\rho^\perp N
\]

Solve via Strichartz estimates for \( \partial_{tt} + L_+ \).  **Conclusion:**

\( \exists M \ni (\pm Q, 0) \) small smooth, codim 1 mfld, \((u_0, u_1) \in M \Rightarrow u = Q + v + o_{H^1}(1) \) as \( t \to \infty \), \( v \) free KG wave, \( M \) parametrized by \((\lambda(0), \gamma_\infty(0))\), where \( \gamma_\infty \) is the scattering solution of \( \gamma \).  **Energy partition:** \( E(u, \dot{u}) = J(Q) + E_0(\gamma_\infty, \dot{\gamma}_\infty) \)  **M unique:** if \( u \)

\( \forall t \geq 0, \, \text{dist}((u, \dot{u}), (\pm Q, 0)) \) small \( \forall t \geq 0, \Rightarrow (u, \dot{u}) \in M. \)
Stable and unstable manifolds

If \((u, \dot{u}) \to (Q, 0)\) as \(t \to \infty\), then \(E(\vec{u}) = J(Q) \Rightarrow \gamma_\infty \equiv 0\). So \(\vec{u}\) parametrized by \(\lambda(0)\).

**Three cases:** \(\lambda > 0, \lambda \equiv 0, \lambda < 0\).

Main \((\lambda, \gamma)\)-system \(\Rightarrow \lambda(t)\) decays exponentially as \(t \to \infty\).

Duyckaerts-Merle type solutions: \(W_\pm(t - t_0)\).

as \(t \to -I\), \(W_+\) blows up in finite time, \(W_-\) scatters to 0.

**Remark:** Construction more involved in the presence of symmetries (non-radial NLKG, radial or nonradial NLS). **Beceanu’s linear estimates:** \(\mathcal{H} = \mathcal{H}_0 + V\) matrix NLS Hamiltonian, \(Z = P_cZ\),

\[
\mathcal{H} = \begin{pmatrix}
\Delta - \mu & 0 \\
0 & -\Delta + \mu
\end{pmatrix} + \begin{pmatrix}
W_1 & W_2 \\
-W_2 & W_1
\end{pmatrix}
\]

\(i\partial_t Z - iv(t)\nabla Z + A(t)\sigma_3 Z + \mathcal{H}Z = F\), \(Z(0)\) given,

\[\|A\|_\infty + \|v\|_\infty < \epsilon\), no eigenvalues or resonances of \(\mathcal{H}\) in \((-\infty, -\mu] \cup [\mu, \infty)\). Then

\[
\|Z\|_{L_t^{\infty} L_x^2 \cap L_t^2 L_x^{6,2}} \leq C \left(\|Z(0)\|_2 + \|F\|_{L_t^1 L_x^2 + L_t^2 L_x^{6/5,2}}\right)
\]
Unstable dynamics off the center-stable mfld $\mathcal{M}$

$\mathcal{M}$ is repulsive (restatement of uniqueness of $\mathcal{M}$).

**Goal:** *Stabilize* $\text{sign}(K_0(u(t))), \text{sign}(K_2(u(t)))$.  

**Virial functional:**

$$K_2(u) = \langle J'(u) | Au \rangle = \partial_\lambda |_{\lambda=0} J(e^{\frac{3\lambda}{2}} u(e^{\lambda} \cdot)), \ A = \frac{1}{2} (x \cdot \nabla + \nabla \cdot x),$$

“Stabilize”: $u(t)$ defined on $[0, T_*)$, then $\text{sign}(K(u(t)) \geq 0$ or $< 0$ on $(T_{**}, T_*)$.

*Figure:* Sign of $K = K_0$ upon exit
Lemma (Ejection Lemma)

\[ \exists 0 < \delta_X \ll 1 \text{ s.t.: } u(t) \text{ local solution of NLKG3 on } [0, T] \text{ with } \]
\[ R := d_Q(\tilde{u}(0)) \leq \delta_X, \quad E(\tilde{u}) < J(Q) + R^2 / 2 \]

and for some \( t_0 \in (0, T) \), one has the ejection condition:

\[ d_Q(\tilde{u}(t)) \geq R \quad (0 < \forall t < t_0). \quad (8) \]

Then \( d_Q(\tilde{u}(t)) \nearrow \) until it hits \( \delta_X \), and

\[ d_Q(\tilde{u}(t)) \simeq -s \lambda(t) \simeq -s \lambda_+(t) \simeq e^{kt} R, \]
\[ |\lambda_-(t)| + \|\tilde{\gamma}(t)\|_E \lesssim R + d_Q^2(\tilde{u}(t)), \]
\[ \min_{s=0,2} s K_s(u(t)) \gtrsim d_Q(\tilde{u}(t)) - C_* d_Q(\tilde{u}(0)), \]

for either \( s = +1 \) or \( s = -1 \).
Variational structure above $J(Q)$ (Noneffective!)

$E := E(u, u_t) > J(Q) + \varepsilon^2 =: J$

Figure: Signs of $K = K_0$ away from $(\pm Q, 0)$

$\forall \delta > 0 \ \exists \varepsilon_0(\delta), \kappa_0, \kappa_1(\delta) > 0$ s.t. $\forall \tilde{u} \in \mathcal{H}$ with $E(\tilde{u}) < J(Q) + \varepsilon_0(\delta)^2$, $d_Q(\tilde{u}) \geq \delta$, one has following dichotomy:

$K_0(u) \leq -\kappa_1(\delta)$ and $K_2(u) \leq -\kappa_1(\delta)$, or

$K_0(u) \geq \min(\kappa_1(\delta), \kappa_0 \| u \|_{H^1}^2)$ and $K_2(u) \geq \min(\kappa_1(\delta), \kappa_0 \| \nabla u \|_{L^2}^2)$. 

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One-pass theorem I

**Crucial no-return property:** Trajectory does not return to balls around \((\pm Q, 0)\). Suppose it did; Use *virial identity*

\[
\partial_t \langle w \dot{u} | Au \rangle = -K_2(u(t)) + \text{error}, \quad A = \frac{1}{2} (x \nabla + \nabla x) \quad (9)
\]

where \(w = w(t, x)\) is a space-time cutoff that lives on a rhombus, and the “error” is controlled by the external energy.

*Figure:* Space-time cutoff for the virial identity
One-pass theorem II

Finite propagation speed $\Rightarrow$ error controlled by free energy outside large balls at times $T_1, T_2$.

Integrating between $T_1, T_2$ gives contradiction; the bulk of the integral of $K_2(u(t))$ here comes from exponential ejection mechanism near $(\pm Q, 0)$.

Figure: Possible returning trajectories
One-pass theorem III

After integration of virial:

\[ \langle w \dot{u} | Au \rangle \bigg|_{T_1}^{T_2} = \int_{T_1}^{T_2} \left[ -K_2(u(t)) + \text{error} \right] dt \]

where \( T_1, T_2 \) are exit, and first re-entry times into \( R \)-ball.

Left-hand side: absolute value

\[ \lesssim R + SR^2 \lesssim R \quad \text{inner radius} \]

were \( S \simeq |\log R| \) size of base (\( Q \ll R \) outside that ball).

Right-hand side: lower bound on \( |K_2(u(t))| \) outside \( \delta_* \)-ball by variational lemma.

Exponentially increasing dynamics gives

\[ \int_{T_1}^{T_1^*} |K_2(u(t))| \ dt \gtrsim \delta_* \quad \text{outer radius} \]

where \( T_1^* \) exit-time from \( \delta_* \)-ball.
One-pass theorem IV

Some further issues:

- For trajectories of type I, this argument works; for type II, use ejection lemma at minimum point $M$.
- In the $K(u(t)) < 0$ region the above argument is sufficient, since error can be made small compared to $\kappa(\delta_*)$ by taking $R$ small (and thus $S$ large).
- In the $K(u(t)) \geq 0$ case, one has a possible complication due to $\int_{T_1}^{T_2} \| \nabla u(t) \|_{L^2}^2 dt$ being too small. In that case error becomes a problem (since we have no control over $T_2 - T_1$).
- Overcome that by showing $\exists \mu_0 > 0$ s.t.: if for some $\mu \in (0, \mu_0]$

$$\| \vec{u} \|_{L^\infty_t(0,2;H)} \leq M, \quad \int_0^2 \| \nabla u(t) \|_{L^2}^2 dt \leq \mu^2$$

then $u$ exists globally and scatters to 0 as $t \to \pm \infty$, $\| u(t) \|_{L^3_tL^6_x(\mathbb{R} \times \mathbb{R}^3)} \ll \mu^{1/6}$. 
Further results I

- **Nonradial NLKG3**: use relativistic energy (Lorentz invariant)

  \[ E_m(\vec{u})^2 = E(\vec{u})^2 - |P(\vec{u})|^2 \]

  where \( P(\vec{u}) \) is the conserved momentum. This works if \(|E| > |P|\), the other case being reduced to Payne-Sattinger.

  For the orbital stability form of 9-set theorem restrict to normalized solutions, i.e., with \( P(\vec{u}) = 0 \). Center-stable mfllds: Instead of \( Q \), need to work with 6-parameter family of ground states (translated, “boosted”). \( Q \) gets squashed by Lorentz contraction. Need a variant of Beceanu’s linear dispersive estimates.

- **NLS equation**: only radial; two modulation parameters for \( Q \):

  phase, mass \( e^{i\alpha^2 t + \gamma} \alpha Q(\alpha x) \). We “mod out” these symmetries (at least for the orbital stability part which does not involve the center-stable manifold); \( \alpha \) is controlled by the mass of the solution, for the phase write \( u = e^{i\theta}(Q + v) \).
Further results II

- **NLS equation**: Major difference in the one-pass theorem from NLKG: absence of finite propagation speed. So crucial virial argument is different; no time-dependent cutoffs. $K(u(t)) < 0$ case (for blowup and one-pass theorem) treated by a variant of the Ogawa-Tsutsumi argument. More difficult to treat $K(u(t)) \geq 0$. Use the following Morawetz identity due to Nakanishi, 1999:

$$\partial_t \langle |u| \frac{t}{4\lambda} u + i \frac{r}{2\lambda} u_r \rangle$$

$$= \int_{\mathbb{R}^3} \left\{ \frac{t^2}{\lambda^3} |\nabla Mu|^2 - \frac{|u|^4}{4} \left[ \frac{2}{\lambda} + \frac{t^2}{\lambda^3} \right] + \frac{15t^4}{4\lambda^7} |u|^2 \right\} dx,$$

where $\lambda := \sqrt{t^2 + r^2}$ and $M := e^{i|x|^2/(4t)}$. Right-hand side can be rewritten in terms of $K(u) = \|\nabla u\|_2^2 - \frac{1}{4} \|u\|_4^4$ and expressions which are integrable in time.
\[\ddot{u} - \Delta u = |u|^{2^*-2} u, \quad u(t, x) : \mathbb{R}^{1+d} \to \mathbb{R}, \quad 2^* = \frac{2d}{d-2} \quad (d = 3 \text{ or } 5),\]

Static Aubin, Talenti solutions

\[W_\lambda = T_\lambda W, \quad W(x) = \left[1 + \frac{|x|^2}{d(d-2)}\right]^{1-\frac{d}{2}},\]

\(T_\lambda\) is \(H^1\) preserving dilation

\[T_\lambda \varphi = \lambda^{d/2-1} \varphi(\lambda x)\]

Positive radial solutions of the static equation

\[-\Delta W - |W|^{2^*-2} W = 0\]

Variational structure:

\[J(\varphi) := \int_{\mathbb{R}^d} \left[\frac{1}{2} |\nabla \varphi|^2 - \frac{1}{2^*} |\varphi|^{2^*}\right] \, dx\]

\[K(\varphi) := \int_{\mathbb{R}^d} \left[|\nabla \varphi|^2 - |\varphi|^{2^*}\right] \, dx\]
Critical wave equation II

Radial $\dot{H}^1 \times L^2$, $E(\varphi) < J(W) + \varepsilon^2$, outside soliton tube

$$\{ \pm \tilde{W}_\lambda | \lambda > 0 \} + O(\varepsilon)$$

There exists four open disjoint sets which lead to all combinations of FTB/GE and scattering to 0 as $t \to \pm I$.

**NOTE:**

- We do not have a complete description of all solutions with energy $E(\varphi) < J(W) + \varepsilon^2$.
- We do not know if the center-stable manifold exists in $\dot{H}^1 \times L^2$ (but in 05 Krieger-S. showed that there is such an object in a stronger non-invariant topology).
- Inside the soliton tube there exist blowup solutions, as found by Krieger-S.-Tataru. Duykaerts-Kenig-Merle showed that all type II blowup are of the KST form, as long as energy only slightly above $J(Q)$. So trapping by the soliton tube cannot mean scattering to $\{ W_\lambda \}$ as it did in the subcritical case.