The method of concentration compactness and dispersive Hamiltonian Evolution Equations

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Aalborg, August 2012

Overview

Goal: To describe recent advances in large data results for nonlinear wave equations

$$\Box u = F(u, Du), F(0) = DF(0) = 0, (u(0), \dot{u}(0)) = (f, g)$$

- Small data theory: F treated as perturbation. Local/Global well-posedness, conserved quantities (energy), symmetries (especially dilation), choice of spaces, algebraic properties of F (nullforms)
- Large data: local-in-time existence, energy subcritical problems: time of existence depends on energy of data, so can time-step. Problem: no information on long-term dynamics such as scattering (solutions are asymptotically free). Finite-time breakdown (blowup) of solutions may occur (type I and II). Classification of possible blowup dynamics
- Induction of energy to prove scattering for global solution: If false then there exists a minimal energy E_* where it fails. Construct critical solution (*minimal criminal*) u_* with energy E_*.

Overview

- u_* enjoys compactness properties modulo symmetries. Forward trajectory $(u_*(t), \partial_t u_*(t))$, $t \geq 0$ pre-compact in energy space.
  Idea: if not compact, then by the method of concentration compactness u_* decomposes into different solutions with strictly smaller energies than E_*. By induction hypothesis, each of these solutions has the desired property and by means of suitable perturbation theory one shows that u_* then also possess this property.
- Rigidity: Show that u_* with this property cannot exist. Kenig-Merle scheme
- Concentration compactness much more versatile, is not tied to induction on energy: key ingredient in the classification of blow-up behavior.

Sobolev imbedding in $\mathbb{R}^3$: $\|f\|_{L^p(\mathbb{R}^3)} \le C\|f\|_{H^1(\mathbb{R}^3)}, \quad 2 < p < 6$

What are the extremizers, optimal constant?

Variational problem:

$$\inf\left\{\|f\|_{H^1(\mathbb{R}^3)} \,\Big|\, \|f\|_{L^p(\mathbb{R}^3)} = 1\right\} = \mu > 0$$

Minimizing sequence

$$\{f_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^3), \quad \|f_n\|_p = 1, \quad \|f_n\|_{H^1(\mathbb{R}^3)} \to \mu$$

How to pass to a limit $f_n \to f_\infty$ *strongly in* $L^p(\mathbb{R}^3)$?

Loss of compactness due to translation invariance!

Claim for $p < 6$: there exists a sequence $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^3$ such that $\{f_n(\cdot - y_n)\}_{n=1}^\infty$ precompact in $L^p(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$.

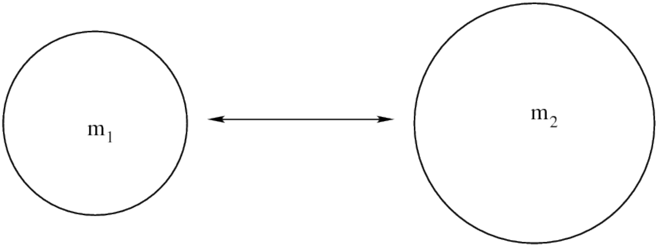


Figure: masses separating

Simplified model: Assume that $f_n = g_n + h_n$ where $\|g_n\|_p^p = m_1 > 0$ and $\|h_n\|_p^p = m_2 > 0$, $m_1 + m_2 = 1$, supports of g_n, h_n disjoint.

Then $\|f_n\|_{H^1}^2 = \|g_n\|_{H^1}^2 + \|h_n\|_{H^1}^2 \geq \mu^2(m_1^{2/p} + m_2^{2/p})$, $\quad 2/p < 1$

This is a contradiction since right-hand side $> \mu^2$.

A concentration-compactness decomposition

$\{f_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^3)$ a bounded sequence. Then $\forall j \geq 1$ there $\exists$ (up to subsequence) $\{x_n^j\}_{n=1}^\infty \subset \mathbb{R}^3$ and $V^j \in H^1$ such that

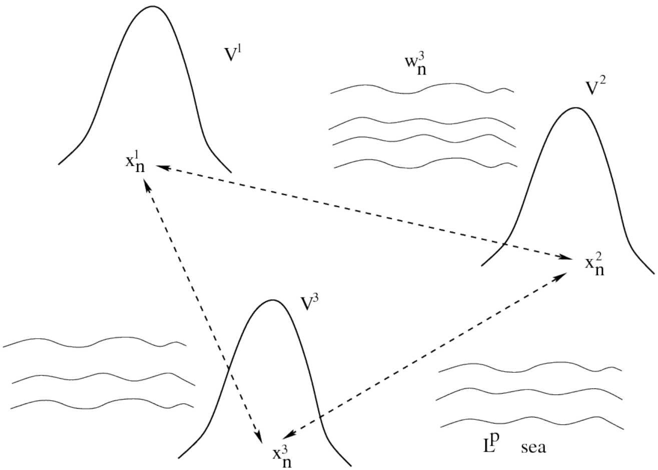
- for all $J \geq 1$ one has $f_n = \sum_{j=1}^J V^j(\cdot - x_n^j) + w_n^J$
- $\forall j \neq k$ one has $|x_n^j - x_n^k| \to \infty$ as $n \to \infty$
- $w_n^J(\cdot + x_n^j) \rightharpoonup 0$ for each $1 \leq j \leq J$ as $n \to \infty$
- $\limsup_{n\to\infty} \|w_n^J\|_{L^p(\mathbb{R}^3)} \to 0$ as $J \to \infty$ for all $2 < p < 6$

Moreover, as $n \to \infty$,

- $\|f_n\|_2^2 = \sum_{j=1}^J \|V^j\|_2^2 + \|w_n^J\|_2^2 + o(1)$
- $\|\nabla f_n\|_2^2 = \sum_{j=1}^J \|\nabla V^j\|_2^2 + \|\nabla w_n^J\|_2^2 + o(1)$

- P. Gérard 1998, more explicit form of P. L. Lions' concentration-compactness trichotomy for measures. Makes failure of compactness modulo symmetries explicit.
- immediately implies compactness claim for minimizing sequences: $V^j = 0$ for $j > 1$.
- only noncompact symmetry groups matter (no rotations)!

The profiles V^j in the L^p sea



We fish for more profiles from the sea: $w_n^3(\cdot + y_n) \rightharpoonup V^4$

Euler-Lagrange equation

Pass to limit $f_n(\cdot - y_n) \to f_\infty$ in $H^1(\mathbb{R}^3)$, $\|f_\infty\|_p = 1$, $\|f_\infty\|_{H^1} = \mu$. Can assume $f_\infty \geq 0$.
Then $\exists \lambda > 0$ Lagrange multiplier

$$-\Delta f_\infty + f_\infty = \lambda |f_\infty|^{p-2} f_\infty$$

Remove $\lambda > 0$ since $p > 2$. Then $f_\infty = Q > 0$ solves

$$-\Delta Q + Q = |Q|^{p-2} Q \qquad (*)$$

$Q \in H^1$, $Q > 0$ **unique** up to translation (Kwong 1989, McLeod 93).

Q is exponentially decaying, radial, smooth. For $dim = 1$ explicit formula, only solutions to $(*)$ in $H^1(\mathbb{R})$ are $0, \pm Q$.

For $d > 1$ have infinitely many radial solutions to $(*)$ that change sign (nodal solutions). Berestycki, Lions, 1983.

What happens for $p = 6$?

Decomposition from above fails at $p = 6$ due to dilation symmetry. Correct setting is $\dot{H}^1(\mathbb{R}^3)$ since

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C\|f\|_{\dot{H}^1(\mathbb{R}^3)} = C\|\nabla f\|_2 \quad (\dagger)$$

Translation and scaling invariant, noncompact group actions.

$\{f_n\}_{n=1}^{\infty} \subset \dot{H}^1(\mathbb{R}^3)$ a bounded sequence. Then $\forall j \geq 1$ there $\exists$ (up to subsequence) $\{x_n^j\}_{n=1}^{\infty} \subset \mathbb{R}^3$, $\{\lambda_n^j\}_{n=1}^{\infty} \in \mathbb{R}^+$ and $V^j \in \dot{H}^1$ such that

- for all $J \geq 1$ one has $f_n = \sum_{j=1}^{J} \sqrt{\lambda_n^j} V^j(\lambda_n^j(\cdot - x_n^j)) + w_n^J$
- $\forall j \neq k$ one has $\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \lambda_n^j |x_n^j - x_n^k| \to \infty$ as $n \to \infty$
- $\limsup_{n\to\infty} \|w_n^J\|_{L^6(\mathbb{R}^3)} \to 0$ as $J \to \infty$.

Moreover, as $n \to \infty$,

$$\|\nabla f_n\|_2^2 = \sum_{j=1}^{J} \|\nabla V^j\|_2^2 + \|\nabla w_n^J\|_2^2 + o(1)$$

Minimizer for $p = 6$

Variational problem associated with (†)

$$\inf\left\{\|f\|_{\dot{H}^1(\mathbb{R}^3)} \,\Big|\, \|f\|_{L^6(\mathbb{R}^3)} = 1\right\} = \mu > 0$$

Minimizing sequence

$$\{f_n\}_{n=1}^{\infty} \subset \dot{H}^1(\mathbb{R}^3), \quad \|f_n\|_{L^6(\mathbb{R}^3)} = 1, \quad \|f_n\|_{\dot{H}^1(\mathbb{R}^3)} \to \mu$$

From the decomposition/minimization: Exactly one profile
$\exists\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^3, \{\lambda_n\}_{n=1}^{\infty} \in \mathbb{R}^+$ such that $\{\lambda_n^{1/2} f_n(\lambda_n(\cdot - y_n))\}_{n=1}^{\infty}$ precompact in $L^6(\mathbb{R}^3)$ and $\dot{H}^1(\mathbb{R}^3)$.
$\lambda_n^{1/2} f_n(\lambda_n(\cdot - y_n)) \to f_\infty$, Euler-Lagrange equation for $\varphi = cf_\infty$

$$\Delta\varphi + \varphi^5 = 0$$

Only *radial* solutions are $\pm W$, 0 up to dilation symmetry, where

$$W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$$

Let

$$\mathcal{L}(u, \partial_t u) := \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\Big(-u_t^2 + |\nabla u|^2 \Big)(t,x)\, dtdx \qquad (1)$$

Substitute $u = u_0 + \varepsilon v$. Then

$$\mathcal{L}(u, \partial_t u) = \mathcal{L}_0 + \varepsilon \int_{\mathbb{R}^{1+d}_{t,x}} (\Box u_0)(t,x) v(t,x)\, dtdx + O(\varepsilon^2)$$

where $\Box = \partial_{tt} - \Delta$.
Thus u_0 is a critical point of $\mathcal{L}$ if and only if $\Box u_0 = 0$.

Significance:

- Underlying symmetries ⇒ invariances ⇒ Conservation laws
  Conservation of energy, momentum, angular momentum
- Lagrangian formulation has a universal character, and is flexible, versatile.

Let (M, g) be a Riemannian manifold, and $u : \mathbb{R}^{1+d}_{t,x} \to M$ smooth. What does it mean for u to satisfy a wave equation?

Lagrangian

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\Big(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2\Big)\, dtdx$$

Critical points $\mathcal{L}'(u, \partial_t u) = 0$ satisfy “manifold-valued wave equation”.

$M \subset \mathbb{R}^N$ imbedded, this equation is

$$\Box u \perp T_u M \;\text{ or }\; \Box u = A(u)(\partial u, \partial u),$$

A being the second fundamental form.

For example, $M = \mathbb{S}^{n-1}$, then

$$\Box u = u(|\partial_t u|^2 - |\nabla u|^2)$$

Note: Nonlinear wave equation, null-form! Harmonic maps are solutions.

Wave maps 2

Intrinsic formulation: $D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0$, in coordinates

$$-u^i_{tt} + \Delta u^i + \Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0$$

$\eta = (-1, 1, 1, \ldots, 1)$ Minkowski metric

- Similarity with geodesic equation: $u = \gamma \circ \varphi$ is a wave map provided $\Box\varphi = 0$, γ a geodesic.
- Energy conservation: $E(u, \partial_t u) = \int_{\mathbb{R}^d} \Big(|\partial_t u|_g^2 + \sum_{j=1}^d |\partial_j u|_g^2 \Big)\, dx$ is conserved in time.
- Cauchy problem:

$$\Box u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

smooth data. Does there exist a smooth local or global-in-time solution?

Local: Yes. Global: depends on the dimension of Minkowski space and the geometry of the target.

Criticality, dimension

If $u(t,x)$ is a wave map, then so is $u(\lambda t, \lambda x)$ $\forall \lambda > 0$.

Data in the Sobolev space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$. For which s is this space invariant under the natural scaling?. Answer: $s = \frac{d}{2}$.

Scaling of the energy: $u(t,x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$ same as $\dot{H}^1 \times L^2$.

- Subcritical case: $d = 1$ the natural scaling is associated with less regularity than that of the conserved energy. Expect global existence. Logic: local time of existence only depends on energy of data, which is preserved.
- Critical case: $d = 2$. Energy keeps the balance with the natural scaling of the equation. For $\mathbb{S}^2$ can have finite-time blowup, whereas for $\mathbb{H}^2$ have global existence. Krieger-S.-Tataru 06, Krieger-S. 09, Rodnianski-Raphael 09, Sterbenz-Tataru 09, T. Tao.
- Supercritical case: $d \geq 3$. Poorly understood. Self-similar blowup $Q(r/t)$ for sphere as target, Shatah 80s. Also negatively curved manifolds possible in high dimensions: Cazenve, Shatah, Tahvildar-Zadeh 98.

Basic mathematical questions (for nonlinear problems)

- Wellposedness: Existence, uniqueness, continuous dependence on the data, persistence of regularity. At first, one needs to understand this locally in time.
- Global behavior: Finite time break down (some norm, such as L^∞, becomes unbounded in finite time)? Or global existence: smooth solutions for all times for smooth data?
- Blow up dynamics: If the solution breaks down in finite time, can one describe the mechanism by which it does so? For example, via energy concentration at the tip of a light cone? Often, symmetries (in a wider sense) play a crucial role here.
- Scattering to a free wave: If the solutions exists for all $t \geq 0$, does it approach a free wave? $\Box u = N(u)$, then $\exists v$ with $\Box v = 0$ and $(\vec{u} - \vec{v})(t) \to 0$ as $t \to \infty$ in a suitable norm? Here $\vec{u} = (u, \partial_t u)$. If scattering occurs, then we have local energy decay.

Basic questions 2

- Special solutions: If a global solution does not approach a free wave, does it scatter to something else? A stationary nonzero solution, for example? Focusing equations often exhibit nonlinear bound states.
- Stability theory: If special solutions exist such as stationary or time-periodic ones, are they orbitally stable? Are they asymptotically stable?
- Multi-bump solutions: Is it possible to construct solutions which asymptotically split into moving "solitons" plus radiation? Lorentz invariance dictates the dynamics of the single solitons.
- Resolution into multi-bumps: Do all solutions decompose in this fashion (as in linear *asymptotic completeness*)? Suppose solutions $\exists$ for all $t \geq 0$: either scatter to a free wave, or the energy collects in "pockets" formed by such "solitons"? Quantization of energy.

In $\mathbb{R}^3$, Cauchy problem $\Box u = 0$, $u(0) = 0, \partial_t u(0) = g$ has solution

$$u(t,x) = t \fint_{tS^2} g(x+y)\,\sigma(dy)$$

If g supported on $B(0,1)$, then $u(t,x)$ supported on $\big||t|-|x|\big| \le 1$.
Huygens' principle. Decay of the wave:

$$\|u(t,\cdot)\|_\infty \le Ct^{-1}\|Dg\|_1 \quad (*)$$

In general dimensions the decay is $t^{-\frac{d-1}{2}}$.

$(*)$ not suitable for nonlinear problems, since the spaces are not invariant. Energy based variant

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^3)} \lesssim \|(u(0),\dot u(0))\|_{\dot H^1\times L^2(\mathbb{R}^3)} + \|\Box u\|_{L_t^1 L_x^2(\mathbb{R}^3)}$$

where $\frac{1}{p}+\frac{3}{q}=\frac{1}{2}$. **Strichartz estimates**
For example, $L_t^\infty L_x^6(\mathbb{R}^{1+3})$, $L_{t,x}^8(\mathbb{R}^{1+3})$. $L_t^2 L_x^\infty$

Domain of influence

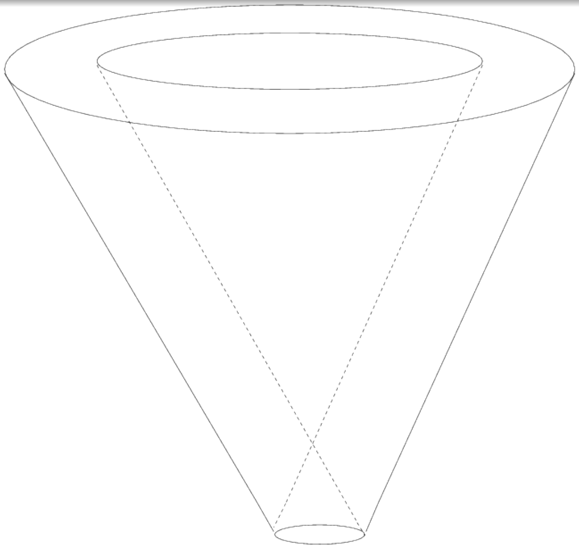


Figure: Huygens principle

A nonlinear Klein-Gordon equation 1

Consider in $\mathbb{R}^{1+3}_{t,x}$

$$\Box u + u + u^3 = 0, \quad (u(0), \dot{u}(0)) = (f,g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3)$$

Conserved energy

$$E(u,\dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{4}|u|^4 \Big)\, dx$$

With $S(t)$ the linear propagator of $\Box + 1$ we have

$$\vec{u}(t) = (u,\dot{u})(t) = S(t)(f,g) - \int_0^t S(t-s)(0,u^3(s))\, ds$$

whence by a simple energy estimate, $I = (0,T)$

$$\begin{aligned} \|\vec{u}\|_{L^\infty(I;\mathcal{H})} &\lesssim \|(f,g)\|_{\mathcal{H}} + \|u^3\|_{L^1(I;L^2)} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u\|^3_{L^3(I;L^6)} \\ &\lesssim \|(f,g)\|_{\mathcal{H}} + T\|\vec{u}\|^3_{L^\infty(I;\mathcal{H})} \end{aligned}$$

Contraction for small T implies local wellposedness for $\mathcal{H}$ data.

A nonlinear Klein-Gordon equation 2

T depends only on $\mathcal{H}$-size of data. From energy conservation we obtain global existence by time-stepping.

Asymptotic state of the solution? Behaves like a free wave?

Scattering (as in linear theory): $\|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \to 0$ as $t \to \infty$ where $\Box v + v = 0$ energy solution.

$$\vec{v}(0) := \vec{u}(0) - \int_0^\infty S(-s)(0, u^3)(s)\, ds \text{ provided } \|u^3\|_{L_t^1 L_x^2} < \infty$$

Where is finiteness of $\|u\|_{L_t^3 L_x^6}$ coming from? Requires *dispersion!* Strichartz estimate uniformly in intervals I

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} + \|u\|_{L^3(I;L^6)} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u\|^3_{L^3(I;L^6)}$$

Small data scattering! $\|\vec{u}\|_{L^3(I;L^6)} \lesssim \|(f,g)\|_{\mathcal{H}} \ll 1$ for all I. So $I = \mathbb{R}$ as desired.

Large data scattering valid; induction on energy, concentration compactness (Bourgain, Bahouri-Gerard, Kenig-Merle).

Scattering blueprint

Let $\vec{u}$ be nonlinear solution with data $(u_0, u_1) \in \mathcal{H}$. Forward scattering set

$$\mathcal{S}_+ = \{(u_0, u_1) \in \mathcal{H} \mid \vec{u}(t) \ \exists \text{ globally, scatters as } t \to +\infty\}$$

We claim that $\mathcal{S}_+ = \mathcal{H}$. This is proved via the following outline:

- (Small data result): $\|(u_0, u_1)\|_{\mathcal{H}} < \varepsilon$ implies $(u_0, u_1) \in \mathcal{S}_+$
- (Concentration Compactness): If scattering fails, i.e., if $\mathcal{S}_+ \neq \mathcal{H}$, then construct $\vec{u}_*$ of minimal energy $E_* > 0$ for which $\|u_*\|_{L_t^3 L_x^6} = \infty$. There exists $x(t)$ so that the trajectory

$$K_+ = \{\vec{u}_*(\cdot - x(t), t) \mid t \geq 0\}$$

  is pre-compact in $\mathcal{H}$.
- (Rigidity Argument): If a forward global evolution $\vec{u}$ has the property that K_+ pre-compact in $\mathcal{H}$, then $u \equiv 0$.

This scheme was introduced by Kenig-Merle 2006, based on Bahouri-Gérard decomposition 1998; Merle-Vega.

Bahouri-Gérard: symmetries vs. dispersion

Let $\{u_n\}_{n=1}^{\infty}$ free Klein-Gordon solutions in $\mathbb{R}^3$ s.t.

$$\sup_n \|\vec{u}_n\|_{L_t^\infty \mathcal{H}} < \infty$$

$\exists$ free solutions v^j bounded in $\mathcal{H}$, and $(t_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}^3$ s.t.

$$u_n(t,x) = \sum_{1 \le j < J} v^j(t + t_n^j, x + x_n^j) + w_n^J(t,x)$$

satisfies $\forall\, j < J$, $\vec{w}_n^J(-t_n^j, -x_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$, and

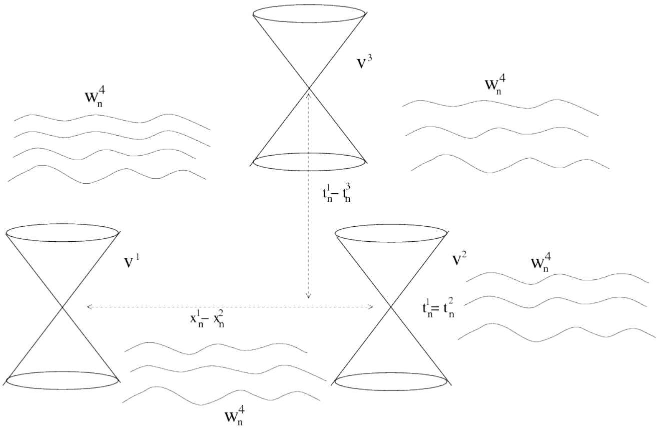
- $\lim_{n\to\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty \;\forall\, j \neq k$
- dispersive errors w_n^k vanish asymptotically:

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^J\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R}\times\mathbb{R}^3)} = 0 \quad \forall\, 2 < p < 6$$

- orthogonality of the energy:

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1 \le j < J} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{w}_n^J\|_{\mathcal{H}}^2 + o(1)$$

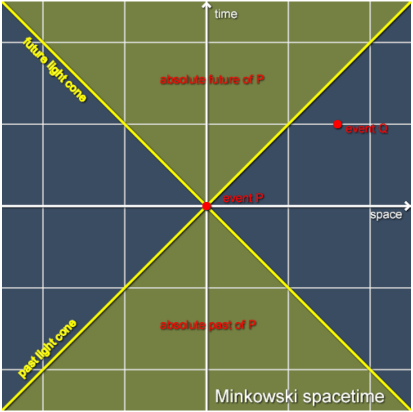
Profiles and Strichartz sea



We can extract further profiles from the Strichartz sea if w_n^4 does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n\to\infty} \|w_n^4\|_{L_t^\infty L_x^p(\mathbb{R}^3)} > 0$.

Lorentz transformations

$$\begin{bmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Remarks on Bahouri-Gérard

- Noncompact symmetry groups: space-time translations and Lorentz transforms.
  Compact symmetry groups: Rotations

  Why do Lorentz transforms not appear in the profiles?

  Energy bound compactifies them!
- Dispersive error w_n^J is not an energy error!
- In the radial case only need time translations

Critical element u_*

Key observation in the Kenig-Merle scheme: Can have only one profile due to minimality of the energy E_*.

- Critical sequence $\vec{u}_n(0) \in \mathcal{H}$, s.t. $E(\vec{u}_n(0)) \to E_*$ and $\|u_n\|_{L^3_t(\mathbb{R};L^6_x(\mathbb{R}^3))} \to \infty$ as $n \to \infty$.
- Apply B-G decomposition to $\{\vec{u}_n(0)\}_n$.
- Suppose $v^1 \neq 0$, $v^2 \neq 0$. Then $E(\vec{v}^j(\cdot + t_n^j)) < E_*$ for all j. Pass to *nonlinear profiles* V^j

$$\|\vec{v}^j(t_n^j) - \vec{V}^j(t_n^j)\|_{\mathcal{H}} \to 0 \ \text{ as } \ n \to \infty$$

$E(V^j) < E_*$ and V^j global solution, scatters.

- Pick J so large that $\|w_n^J\|_{L^3_t L^6_x} < \varepsilon$. Perturbation theory implies that we can glue all V^j together with w_n^J so as to imply

$$\|u_n\|_{L^3_t L^6_x} \leq M < \infty \quad \forall\ n$$

Contradiction! So have at most one profile. This gives compactness as in the elliptic case up to the symmetries.

- Gives compactness of forward/backward trajectory. Again

Rigidity argument, radial case

Radial case, $u_*(t)$ has precompact forward trajectory in $H^1 \times L^2(\mathbb{R}^3)$.

Virial identity, $A = \frac{1}{2}(x\nabla + \nabla x)$

$$\partial_t \langle \chi \dot{u}_* \mid A u_* \rangle = -\int_{\mathbb{R}^3} (|\nabla u_*|^2 + \frac{3}{4}|u_*|^4)\, dx + \mathrm{error}$$

$\chi(t,x)$ cutoff to $|x| \le R$, error is uniformly small due to compactness.

Integrate in time:

$$\langle \chi \dot{u}_* \mid A u_* \rangle \Big|_0^T = -\int_0^T \Big[\int_{\mathbb{R}^3} (|\nabla u_*|^2 + \frac{3}{4}|u_*|^4)\, dx + \mathrm{error} \Big](t)\, dt$$

LHS $= O(R \times Energy(\vec{u}_*))$, RHS $\ge T \times Energy(\vec{u}_*)$.

Contradiction for large T if $u_* \neq 0$.

Rigidity argument, nonradial case

There exists a path $x(t)$ s.t. $\vec{u}_*(t, \cdot - x(t))$ is relatively compact for $t \geq 0$ in $H^1 \times L^2$.

We know $|x(t)| \leq Ct$ by finite propagation speed. If optimal, would destroy virial argument.

Key observation: u_* has vanishing momentum

$$P(\vec{u}_*) = \langle \dot{u}_* \mid \nabla u_* \rangle = 0$$

Idea: If not, then by means of a Lorentz transform could lower the energy while retaining the property that the solution does not scatter. Contradiction to minimality of the energy!

So conclude that $x(t) = o(t)$. Virial argument applies as before.

Grand conclusion: solutions of $\Box u + u + u^3 = 0$, arbitrary data in $H^1 \times L^2(\mathbb{R}^3)$, scatter to a free energy solution as $t \to \pm\infty$.

The focusing NLKG equation

The focusing NLKG

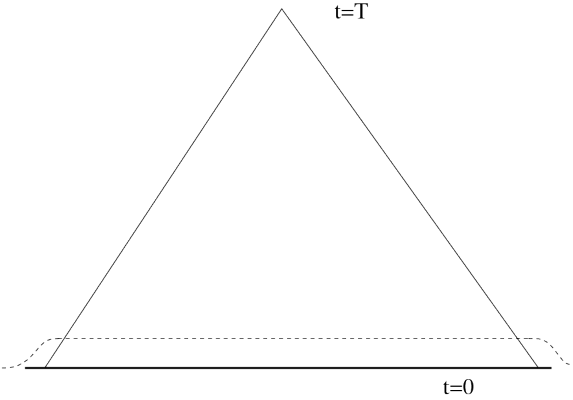
$$\Box u + u = \partial_{tt} u - \Delta u + u = u^3$$

has indefinite conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4 \right) dx$$

- Local wellposendness for $H^1 \times L^2(\mathbb{R}^3)$ data
- Small data global existence and scattering
- Finite time blowup $u(t) = \sqrt{2}(T - t)^{-1}(1 + o(1))$ as $t \to T-$
  Cutoff to a cone using finite propagation speed to obtain finite energy solution.
- stationary solutions $-\Delta\varphi + \varphi = \varphi^3$, ground state $Q(r) > 0$

Cutoff for the blowup construction



Dashed line is a smooth cutoff which $= 1$ on $|x| \leq T$.

Payne-Sattinger theory 1

Criterion: finite-time blowup/global existence?

Yes, provided the energy is less than the ground state energy

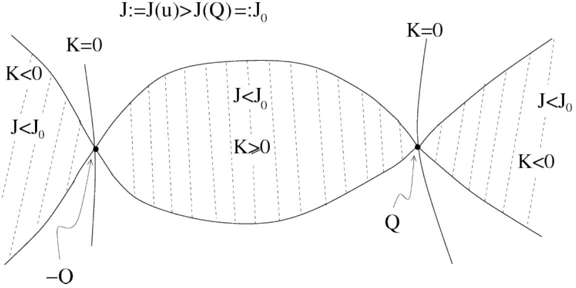


Figure: The saddle structure of the energy near the ground state

$$J(\varphi) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\Big)\, dx$$

$$K(\varphi) = \int_{\mathbb{R}^3} \Big(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\Big)\, dx$$

Uniqueness of Q is the foundation!

Payne-Sattinger theory 2

$j_\varphi(\lambda) := J(e^\lambda \varphi)$, $\varphi \neq 0$ fixed.

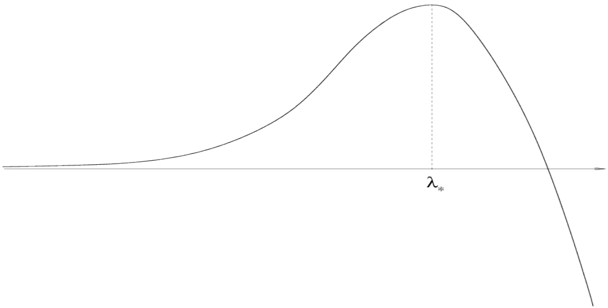


Figure: Payne-Sattinger well

Normalize so that $\lambda_* = 0$. Then $\partial_\lambda j_\varphi(\lambda)\big|_{\lambda=\lambda_*} = K_0(\varphi) = 0$.
"Trap" the solution in the well on the left-hand side: need $E < \inf\{j_\varphi(0) \mid K_0(\varphi) = 0, \varphi \neq 0\} = J(Q)$ (lowest mountain pass).
Expect global existence in that case.

Above the ground state energy, Nakanishi-S. 2010

Theorem

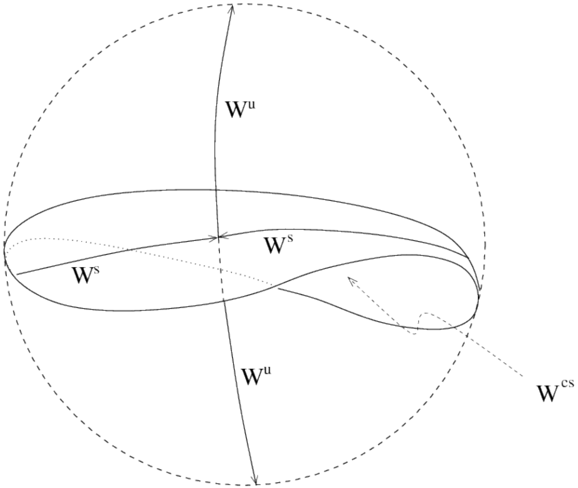
Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\rm rad}$. In $t \geq 0$ for NLKG:

1. *finite time blowup*
2. *global existence and scattering to 0*
3. *global existence and scattering to Q:* $u(t) = Q + v(t) + o_{H^1}(1)$ *as* $t \to \infty$, *and* $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 combinations of this trichotomy allowed as $t \to \pm\infty$.

- Applies to $\dim = 3$, $|u|^{p-1}u$, $7/3 < p < 5$, or $\dim = 1$, $p > 5$.
- Third alternative forms the **center stable manifold** associated with $(\pm Q, 0)$. Linearized operator $L_+ = -\Delta + 1 - 3Q^2$ has spectrum $\{-k^2\} \cup [1, \infty)$ on $L^2_{rad}(\mathbb{R}^3)$. Gap $[0, 1)$ difficult to verify, Costin-Huang-S., 2011.
- $\exists$ 1-dim. stable, unstable mflds at $(\pm Q, 0)$. Stable mfld: Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko 2009

The invariant manifolds



Ball in $H^1 \times L^2$ (radial), centered at $(Q, 0)$. Center-stable manifold separates blowup in finite positive time from existence for all times and scattering to a free wave.

Numerical 2-dim section through $\partial\mathcal{S}_+$ (with R. Donninger)

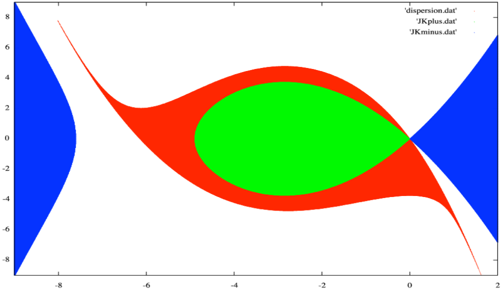


Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, (A, B) vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_-$, BLUE: $\mathcal{PS}_+$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)

Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$

- $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$
- $L_+\rho = -k^2\rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: L_+ has no eigenvalues in $(0,1]$, no threshold resonance (delicate!) Use Kenji Yajima's L^p-boundedness for wave operators.

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q,v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q,v) \end{pmatrix}$$

Then $\mathrm{spec}(A) = \{k, -k\} \cup i[1,\infty) \cup i(-\infty,-1]$ with $\pm k$ simple evals.
Formally: $X_s = P_1 L^2$, $X_u = P_{-1} L^2$, X_c is the rest.

Spectrum of matrix Hamiltonian

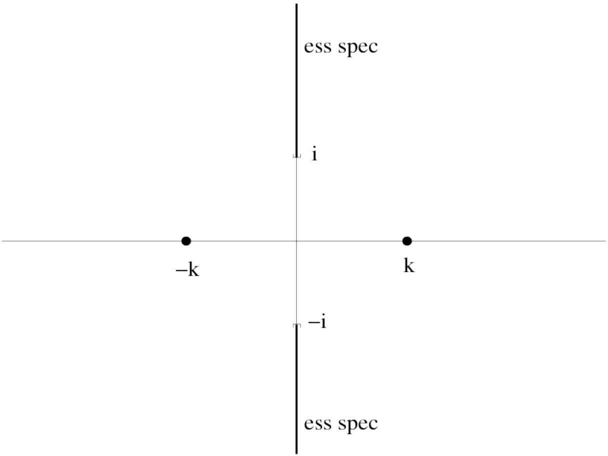
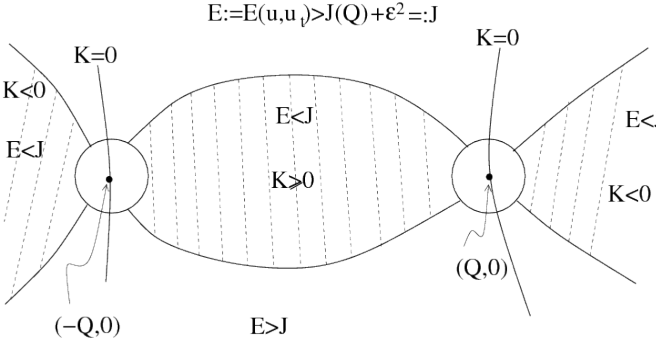


Figure: Spectrum of nonselfadjoint linear operator in phase space

Variational structure above $E(Q, 0)$



- Solution can pass through the balls. Energy is no obstruction anymore as in the Payne-Sattinger case.
- Key to description of the dynamics: One-pass (no return) theorem. The trajectory can make only one pass through the balls.
- Point: Stabilization of the sign of $K(u(t))$.

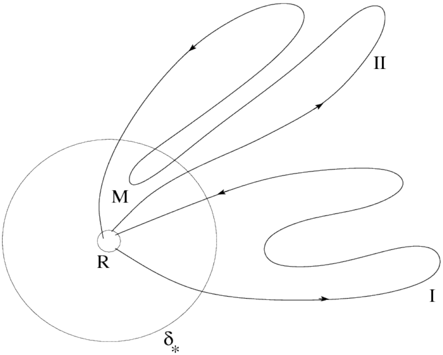


Figure: Possible returning trajectories

Such trajectories are excluded by means of an indirect argument using a variant of the virial argument that was essential to the rigidity step of Kenig-Merle.

Equivariant wave maps

$u : \mathbb{R}^{1+2}_{t,x} \to S^2$ satisfies WM equation

$$\Box u \perp T_u S^2 \Leftrightarrow \Box u = u(|\partial_t u|^2 - |\nabla u|^2)$$

as well as equivariance assumption $u \circ R = R \circ u \;\forall\; R \in SO(2)$

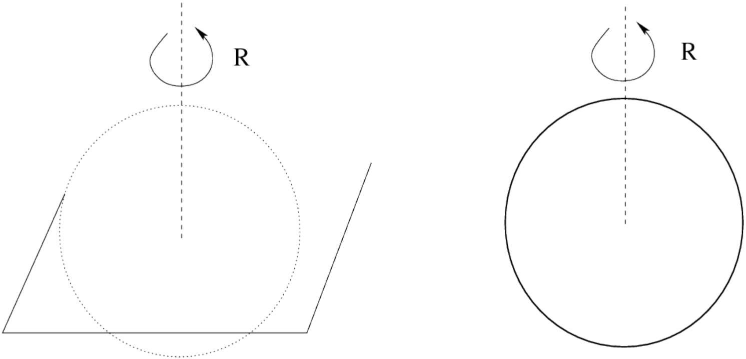


Figure: Equivariance and Riemann sphere

Equivariant wave maps 2

$u(t,r,\phi) = (\psi(t,r),\phi)$, spherical coordinates, ψ angle from north pole satisfies

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi,\dot\psi)(0) = (\psi_0,\psi_1)$$

- Conserved energy

$$E(\psi,\dot\psi) = \int_0^\infty \Big(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\Big)\, r\,dr$$

- $\psi(t,\infty) = n\pi, n \in \mathbb{Z}$, homotopy class = degree = n
- stationary solutions = harmonic maps = $0, \pm Q(r/\lambda)$, where $Q(r) = 2\arctan r$. This is the identity $\mathbb{S}^2 \to \mathbb{S}^2$ with stereographic projection onto $\mathbb{R}^2$ as domain (conformal map!).

Large data results for equivariant wave maps 1

Theorem (Côte, Kenig, Lawrie, S. 2012)

Let (ψ_0, ψ_1) be smooth data.

1. *Let $E(\psi_0, \psi_1) < 2E(Q, 0)$, degree 0. Then the solution exists globally, and scatters (energy on compact sets vanishes as $t \to \infty$). For any $\delta > 0$ there exist data of energy $< 2E(Q, 0) + \delta$ which blow up in finite time.*
2. *Let $E(\psi_0, \psi_1) < 3E(Q, 0)$, degree 1. If the solution $\psi(t)$ blows up at time $t = 1$, then there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $E(\vec{\varphi}) = E(\vec{\psi}) - E(Q, 0)$, and a decomposition*

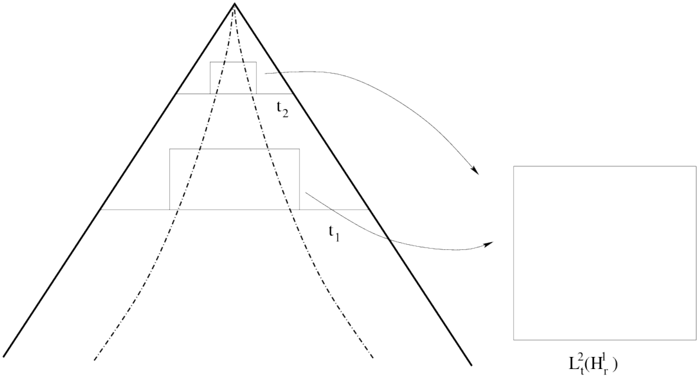
$$\vec{\psi}(t) = \vec{\varphi} + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + \vec{\epsilon}(t) \quad (\star)$$

s.t. $\vec{\epsilon}(t) \in \mathcal{H}_0$, $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to 1$.

Large data results for equivariant wave maps 2

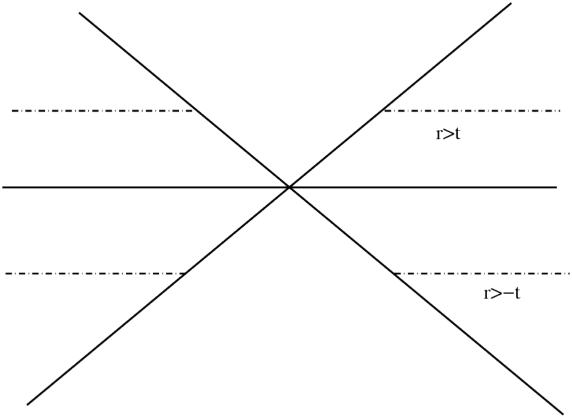
- For degree 1 have an analogous classification to $(\star)$ for global solutions.
- Côte, Kenig, Merle 2006 proved the degree 0 result for $E < E(Q,0) + \delta$. Proof proceeds via the small data scattering/concentration-compactness/rigidity scheme.
- Duyckaerts, Kenig, Merle established classification results for $\Box u = u^5$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$ instead of Q.
- Construction of $(\star)$ by Krieger-S.-Tataru, Donninger-Krieger $\lambda(t) = t^{-1-\nu}$
- Crucial role is played by Michael Struwe's bubbling off theorem (equivariant): if blowup happens, then there exists a sequence of times approaching blowup time, such that a rescaled version of the wave map approaches locally in energy space a harmonic map of positive energy.

Cuspidal energy concentration



Rescalings converge in $L^2_{t,r}$-sense to a stationary wave map of positive energy, i.e., a harmonic map.

Exterior energy



$\Box u = 0, \; u(0) = f \in \dot{H}^1(\mathbb{R}^d), \; u_t(0) = g \in L^2(\mathbb{R}^d)$ radial

Duyckaerts-Kenig-Merle: for all $t \geq 0$ or $t \leq 0$ have

$E_{\mathrm{ext}}(\vec{u}(t)) \geq cE(f, g)$ provided dimension odd. $c > 0$, $c = \frac{1}{2}$

Heuristics: incoming vs. outgoing data.

Exterior energy: even dimensions

Côte-Kenig-S.: This **fails in even dimensions**.

$d = 2, 6, 10, \ldots$ holds for data $(0, g)$ but fails in general for $(f, 0)$.
$d = 4, 8, 12, \ldots$ holds for data $(f, 0)$ but fails in general for $(0, g)$.

Fourier representation, Bessel transform, dimension d reflected in the phase of the Bessel asymptotics, computation of the asymptotic exterior energy as $t \to \pm\infty$.

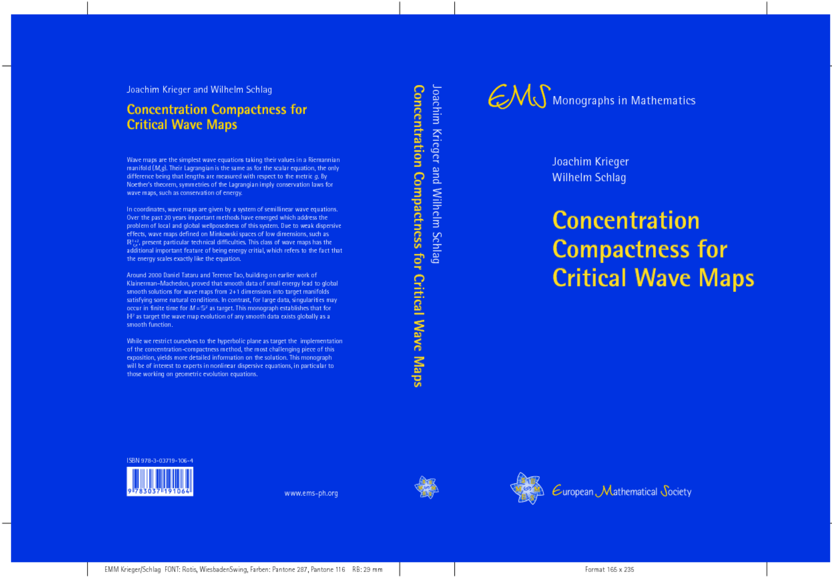
For our $3E(Q, 0)$ theorem we need $d = 4$ result; rather than $d = 2$ due to repulsive $\frac{\psi}{r^2}$-potential coming from $\frac{\sin(2\psi)}{2r^2}$.

Why does $(f, 0)$ result suffice? Because of Christodoulou, Tahvildar-Zadeh, Shatah results from mid 1990s. Showed that at blowup $t = T = 1$ have vanishing kinetic energy

$$\lim_{t\to 1} \frac{1}{1-t} \int_t^1 \int_0^t |\dot{\psi}(t, r)|^2 \, r dr \, dt = 0$$

No result for Yang-Mills since it corresponds to $d = 6$

EMS book with Krieger



Joachim Krieger and Wilhelm Schlag

Concentration Compactness for Critical Wave Maps

Wave maps are the simplest wave equations taking their values in a Riemannian manifold (M,g). Their Lagrangian is the same as for the scalar equation, the only difference being that lengths are measured with respect to the metric g. By Noether's theorem, symmetries of the Lagrangian imply conservation laws for wave maps, such as conservation of energy.

In coordinates, wave maps are given by a system of semillinear wave equations. Over the past 20 years important methods have emerged which address the problem of local and global wellposedness of this system. Due to weak dispersive effects, wave maps defined on Minkowski spaces of low dimensions, such as $\mathbb{R}^{1+2}_{t,x}$, present particular technical difficulties. This class of wave maps has the additional important feature of being energy critial, which refers to the fact that the energy scales exactly like the equation.

Around 2000 Daniel Tataru and Terence Tao, building on earlier work of Klainerman–Machedon, proved that smooth data of small energy lead to global smooth solutions for wave maps from 2+1 dimensions into target manifolds satisfying some natural conditions. In contrast, for large data, singularities may occur in finite time for $M=\mathbb{S}^2$ as target. This monograph establishes that for $\mathbb{H}^2$ as target the wave map evolution of any smooth data exists globally as a smooth function.

While we restrict ourselves to the hyperbolic plane as target the implementation of the concentration-compactness method, the most challenging piece of this exposition, yields more detailed information on the solution. This monograph will be of interest to experts in nonlinear dispersive equations, in particular to those working on geometric evolution equations.

ISBN 978-3-03719-106-4

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Joachim Krieger and Wilhelm Schlag Concentration Compactness for Critical Wave Maps

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ZURICH LECTURES IN ADVANCED MATHEMATICS

Kenji Nakanishi and Wilhelm Schlag

Invariant Manifolds and Dispersive Hamiltonian Evolution Equations

The notion of an invariant manifold arises naturally in the asymptotic stability analysis of stationary or standing wave solutions of unstable dispersive Hamiltonian evolution equations such as the focusing semilinear Klein–Gordon and Schrödinger equations. This is due to the fact that the linearized operators about such special solutions typically exhibit negative eigenvalues (a single one for the ground state), which lead to exponential instability of the linearized flow and allows for ideas from hyperbolic dynamics to enter.

One of the main results proved here for energy subcritical equations is that the center-stable manifold associated with the ground state appears as a hyper-surface which separates a region of finite-time blowup in forward time from one which exhibits global existence and scattering to zero in forward time. Our entire analysis takes place in the energy topology, and the conserved energy can exceed the ground state energy only by a small amount.

This monograph is based on recent research by the authors and the proofs rely on an interplay between the variational structure of the ground states on the one hand, and the nonlinear hyperbolic dynamics near these states on the other hand. A key element in the proof is a virial-type argument excluding almost homoclinic orbits originating near the ground states, and returning to them, possibly after a long excursion.

These lectures are suitable for graduate students and researchers in partial differential equations and mathematical physics. For the cubic Klein–Gordon equation in three dimensions all details are provided, including the derivation of Strichartz estimates for the free equation and the concentration-compactness argument leading to scattering due to Kenig and Merle.

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