On discrete Schrödinger operators with stochastic potentials

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Ergodic map $T : \mathbb{T}^d \to \mathbb{T}^d$, potential $V_x(n) = v(T^n x)$, $v : \mathbb{T}^d \to \mathbb{R}$ analytic, nonconstant. Hamiltonian

$$H_x := -\triangle_{\mathbb{Z}} + \lambda V_x.$$

Version of **Anderson's model**: random impurities (potentials) turn conductors (a.c. spectrum) into insulators (p.p. spectrum). Instead of random potentials one considers here potentials given in terms of deterministic dynamics. The "randomness" is given by the parameter $x \in \mathbb{T}^d$.

Basic questions: pure point spectrum for large λ, Anderson localization (exponentially decaying eigenfunctions) dynamical localization: $\sup_t \|(1+|n|)^A e^{itH}\psi_0\|_2 < \infty$, distribution of eigenvalues (density of states), a.c. spectrum for small λ, structure of the spectrum (Cantor set).

<u>Fundamental results by</u>: Dinaburg, Sinai (a.c. spectrum for small λ), Fürstenberg (positivity of the Lyapunov exponent), Goldsheid, Molchanov, Pastur (A.L. for the one-dimensional model) Fröhlich, Spencer ("multiscale analysis" in the random case), Aizenman, Molchanov ("fractional moment method"), Fröhlich, Spencer, Wittwer and Sinai (A.L. for the quasiperiodic model)
Avron, Bellissard, Campanino, Carmona, Delyon, Eliasson, Figotin, Gordon, Jitomirskaya, Klein, Kotani, Kirsch, Last, Martinelli, Oseledec, Pastur, Ruelle, Simon, Simon-Wolff, Souillard, Thouless, Wegner

Eigenvalue equation

$$(H_x\psi)_n = -\psi_{n+1} - \psi_{n-1} + \lambda v(T^n x)\psi_n = E\psi_n$$

Examples of maps T: $Tx = x + \omega$ (mod$\mathbb{Z}^d$), $Tx = 2x$ (mod1), $T(x, y) = (x + y, y + \omega)$ (mod$\mathbb{Z}^2$).

Covariance relation:

$$H_{Tx} = UH_xU^* \text{ with } U\text{=left shift}$$

implies $\Sigma_x :=$ spec(H_x) is constant for a.e. x. Same for $\Sigma_x^{ac}, \Sigma_x^{pp}, \Sigma_x^{sc}$.

Basic objects:

$$\begin{aligned} M_n(x,\lambda,E) &:= \prod_{j=n}^{1} \begin{bmatrix} \lambda v(T^j x) - E & -1 \\ 1 & 0 \end{bmatrix} \\ L_n(\lambda,E) &:= \frac{1}{n}\int_{\mathbb{T}^d} \log \|M_n(x,\lambda,E)\|\,dx \\ L(\lambda,E) &:= \lim_{n\to\infty} L_n(\lambda,E) = \lim \frac{1}{n} \log \|M_n(x,\lambda,E)\| \end{aligned}$$

for a.e. x by Kingman's subadditive ergodic theorem. Ishi-Pastur theorem: $\Sigma_x^{ac} \subset \{E : L(\lambda,E) = 0\}$.

Kotani: meas$\Big[\{E : L(\lambda,E) = 0\} \setminus \Sigma_x^{ac}\Big] = 0$, a.e. x.

Let $Tx = T_\omega x := x + \omega$ (mod1) be the one-dimensional shift. Lyapunov exponent $L(\omega, \lambda, E)$.

Theorem 1 (Bourgain-Goldstein). *Suppose* $\inf_{\omega,E} L(\omega, \lambda, E) > 0$*. Then for a.e.* ω *and a.e.* x *the Hamiltonian* H_x *displays AL (dynamical).*

<u>Note:</u> This **does not** explicitly require large λ. If $v(x) = \cos(2\pi x)$ (almost Mathieu operator), then Theorem 1 applies for $\lambda > 2$ (by Herman's result $L(\omega, \lambda, E) \geq \log(\lambda/2)$ for $\lambda > 2$). Jitomirskaya (ca. 1999) proved this result for the almost Mathieu operator and all **Diophantine** ω and a.e. x. Proof is nonpreturbative, uses transfer matrices.

Theorem 2 (Goldstein-S.). *Suppose* ω *is Diophantine (* $\|n\omega\| \geq \frac{C_\omega}{n(\log n)^a}$ *for* $n > 1$*,* $a > 1$*). If* $L(\omega, E) > 0$ *for all* $E_1 < E < E_2$*, then* $L(\omega, \cdot)$ *as well as the IDS* $N(\omega, \cdot)$ *are Hölder continuous in* $E \in [E_1, E_2]$*.*

Note: The integrated density of states (IDS) is the **deterministic** limit

$$N(\omega, E) = \lim_{N\to\infty} \frac{1}{2N+1} \#\{1 \le j \le 2N+1 : E_j^{(N)} < E\}$$

where $E_j^{(N)} = E_j^{(N)}(x, \omega)$ are the eigenvalues of H_x restricted to $[-N, N]$. Thouless' formula relates L, N: $L(\omega, E) = \int \log |E - E'| \, N(\omega, dE')$. Sinai's work on large disorder shows that the IDS can be no better than Hölder-$\frac{1}{2}$ continuous, and Bourgain obtained the exponent $\frac{1}{2}-$ for almost Mathieu and large λ. In [GS] the Hölder exponent depended on the Lyapunov exponent. Bourgain refined [GS] to obtain a Hölder exponent that is uniform in $L(E)$. Bourgain, Jitomirskaya modified the method from [GS] to show that $L(\omega, E)$ is jointly continuous in ω, E at every point (ω_0, E), $\omega_0 \in \mathbb{R} \setminus \mathbb{Q}$ as well as continuous in E for every ω (without assuming that L is positive).

Consider the skew-shift $T_\omega(x,y) = (x+y, y+\omega)$ $(\text{mod}\mathbb{Z}^2)$. <u>Note:</u> Quadratic dependence on n

$$T_\omega^n(x,y) = (x+ny+n(n-1)\omega/2, y+n\omega)\ (\text{mod}\mathbb{Z}^2)$$

conjectured to lead to $L(\omega,\lambda,E) > 0$ for all $\lambda > 0$.

Theorem 3 (BGS). *Fix $\varepsilon > 0$. Then there exist $\lambda_0(\varepsilon)$ and $\Omega(\lambda,\varepsilon) \subset \mathbb{T}^3$ such that* $\text{meas}[\mathbb{T}^3 \backslash \Omega(\lambda,\varepsilon)] < \varepsilon$ *and for all $\lambda > \lambda_0(\varepsilon)$ and $(\omega,x,y) \in \Omega(\lambda,\varepsilon)$, $(H_{\omega,x,y}\psi)_n := -\psi_{n+1} - \psi_{n-1} + \lambda v(T_\omega^n(x,y))\psi_n$ displays A.L.*

This theorem is related to the "quantum kicked rotor" model:

$$i\partial_t\Psi(t,x) = a\partial_x^2\Psi(t,x) + ib\partial_x\Psi(t,x) + V(t,x)\Psi(t,x)$$

where $V(t,x) = \kappa\cos(2\pi x)\sum_{n\in\mathbb{Z}}\delta(t-n)$. Unitary evolution operator $S(t)$, monodromy operator $W = S(1) = U_{a,b}\cdot W_{1,\kappa}$ where $U_{a,b} = e^{i(a\frac{d^2}{dx^2}+b\frac{d}{dx})}$, $(W_{1,\kappa}f)(x) = e^{i\kappa\cos(2\pi x)}f(x) =: \rho(x)f(x)$.

In Fourier modes: $U_{a,b} = e^{-i(4a\pi^2 n^2 + 2\pi bn)}\delta_{mn}$, $W_{1,\kappa}(m,n) = \widehat{\rho}(m-n)$ decays exponentially, and $\widehat{\rho}(0) = 1 + O(\kappa^2)$. Then $H := \frac{1}{2}(W + W^*)$ is of the form $H_{nn} = v(T_\omega^n(0,y))$, $H_{mn} = \phi_{m-n}(T^m x) + \overline{\phi_{n-m}(T^n x)}$ for $m \neq n$, with ϕ exponentially decaying and small (for κ small): H is a long-range version of the skew-shift Hamiltonian. Anderson localization for H holds for small κ and most a, b, thus W has ONB of exponentially decaying eigenfunctions (on the Fourier side).

Theorem 4 (B). *For small* κ *and* (a,b) *outside a set of small measure one has: Let* $\Psi(t,x) = (S(t)\Psi(0,\cdot))(x)$ *be a solution of the kicked rotor equation. If* $\Psi(0,\cdot)$ *is a smooth function on* $\mathbb{T}$*, then* $\Psi(t)$ *is an almost-periodic* $H^1(\mathbb{T})$*-valued function and* $\sup_t \|\Psi(t)\|_{H^1(\mathbb{T})} < \infty$*.*

In recent work Bourgain, Jitomirskaya have obtained a version of Theorem 4 that applies to a.e. choice of the parameters (a, b). Theorem 3 is proved by means of the transfer matrix formalism, whereas Theorem 4 cannot be obtained this way (because of long-range interactions). One uses instead an approach that originated in the proof of the following theorem.

Theorem 5 (BGS). *Let* $v : \mathbb{T}^2 \to \mathbb{R}$ *be analytic, nonconstant on any vertical and horizontal line segment. Let* $(H_{xy}\psi)_{nm} = -\triangle_{\mathbb{Z}^2}\psi_{nm} + \lambda v(x + n\omega_1, y + m\omega_2)\psi_{nm}$ *for* $(n, m) \in \mathbb{Z}^2$*. The for all* $\varepsilon > 0$, $\lambda > \lambda_0(\varepsilon)$ *the operator* H_{xy} *displays Anderson localization for all* $(x, y, \omega_1, \omega_2) \in \mathbb{T}^4$ *up to a set of measure at most* ε*.*

<u>Methods:</u> Basic step is to control

$$\mathrm{meas}\Big[x \in \mathbb{T}^d \ : \ \mathrm{dist}(H_x^\Lambda, E) < e^{-\rho}\Big]$$

where $\Lambda = [-N, N]$ and ρ large ($\gg (\log N)^A$). Same as controlling $\Big\|(H_x^\Lambda - (E + i0))^{-1}\Big\| = \|G_\Lambda(x, E)\|$. One has

$$G_\Lambda(x, E)(n, m) = \frac{f_{[-N,n-1]}(x, E) f_{[m+1,N]}(x, E)}{f_{[-N,N]}(x, E)}$$

where $f_{[p,q]}(x, E) = \det(H_x^{[p,q]} - E)$. Moreover,

$$M_n(x, E) = \begin{bmatrix} f_{[1,n]}(x, E) & -f_{[2,n]}(x, E) \\ f_{[1,n-1]}(x, E) & -f_{[3,n]}(x, E) \end{bmatrix}.$$

"Quantitative Kingman" theorem: For large N

$$\mathrm{meas}\Big[x \ : \ \Big|\log \Big\|M_N(x, E)\| - N\, L_N(E)\Big| > N^\sigma\Big] < e^{-N^\tau}$$

for some $0 < \sigma, \tau < 1$ (large deviation theorems or LDT). Commutative model case:

$$u(x) = \sum_{n=1}^{N} \log |e(x + n\omega) - 1| = \int \log |z - \zeta|\, \mu(d\zeta)$$

where $z = e(x) = e^{2\pi i x}$, $\mu = \sum_{n=1}^N \delta_{e(-n\omega)}$.

Koksma's inequality: $\mathcal{S} := \{x_n\}_{n=1}^N \subset \mathbb{T}$

$$\Big| \sum_{n=1}^N f(x_n) - \int_0^1 f(x)\,dx \Big| \le C\, D_N(\mathcal{S}) \|f\|_{BV}$$

where $D_N(\mathcal{S}) = \sup_{J\subset\mathbb{T}} \Big| \#\{n : x_n \in J\} - N|J| \Big|$.

Problem: $\log|e(x)-1| \notin BV$. More direct approach: let $f_0(x) = -\frac{1}{2} - x$ for $-\frac{1}{2} < x < 0$ and $f_0(x) = \frac{1}{2} - x$ for $0 < x < \frac{1}{2}$. Then

$$\begin{aligned} u(x) &= \mathcal{H}\Big(\sum_{n=1}^N f_0(\cdot + n\omega)\Big)(x) \\ \|u\|_{BMO} &\le C \Big\| \sum_{n=1}^N f_0(\cdot + n\omega) \Big\|_\infty \sim D_N(\{n\omega\}_{n=1}^N) \end{aligned}$$

Since for a.e. ω and N large $D_N(\{n\omega\}_{n=1}^N) < N^\varepsilon$, the John-Nirenberg inequality implies

$$\text{meas}\Big[x : |u(x) - \langle u\rangle| > N^\sigma\Big] \le \exp\Big[- N^{\sigma-\varepsilon}\Big]$$

Transition to noncommutative case by means of:

- subharmonicity
- avalanche principle (only if $L(E) > 0$)

Since v is analytic, $u(x) = \log \|M_N(x, E)\|$ extends to a neighborhood of the unit circle as a subharmonic function. Riesz representation:

$$u(z) = \int \log |z - \zeta| \, \mu(d\zeta) + h(z)$$

$\mu \geq 0$, h harmonic. Here $\|\mu\| \sim N$, but there is a lot of cancellation in the integral to ensure, for example, that $\|u\|_{BMO} \leq (\log N)^A$.

Structure of u as a sum of shifts of another function comes from the Avalanche Principle (Goldstein-S.):

Let $\{A_n\}_{n=1}^N \in S\ell(2, \mathbb{R})$. *Suppose* $\|A_n\| > \mu > N$, $\|A_{n+1}A_n\| \leq \sqrt{\mu}\, \|A_{n+1}\| \, \|A_n\|$ *for all* n. *Then*

$$\left| \log \|M_N\| + \sum_{n=2}^{N-1} \log \|A_n\| - \sum_{n=1}^{N-1} \log \|A_{n+1}A_n\| \right| \leq C\frac{N}{\mu}$$

where $M_N = A_N \cdots A_1$.

- Positivity of the Lyapunov exponent for large disorder
- Inductive prood of large deviation theorems

Typical application: Write

$$M_{nN}(x,E) = \prod_{j=N-1}^{0} M_n(T^{jn}x,E), \qquad n \sim (\log N)^A.$$

Suppose

$$\text{meas}\Big[x \,:\, \Big|\log\Big\|M_n(x,E)\| - n\,L_n(E)\Big| > n^\sigma\Big] < e^{-n^\sigma}$$

and the same for $2n$, where $A\sigma \geq 100$. Furthermore, suppose $L_n(E) \geq L_{2n}(E) \geq 1$ for all E, and $L_n(E) - L_{2n}(E) \leq 1/100$. Then up to a set of measure $< CNe^{-n^\sigma} \sim N^{-99}$, and for n large,

$$\begin{aligned}
&\|M_n(T^{jn}x)\| > e^{n-n^\sigma} > e^{n/2} =: \mu \\
&\|M_n(T^{(j+1)n}x)M_n(T^{jn}x)\| > e^{2nL_{2n}(E)-n^\sigma} \\
&> e^{2nL_n(E)-2n^\sigma} > \mu^{-\frac{1}{2}}\|M_n(T^{(j+1)n}x)\|\,\|M_n(T^{jn}x)\|.
\end{aligned}$$

avalanche principle: Up to a set of meaure $< N^{-99}$,

$$\Big|\log\|M_{nN}(x,E)\| + \sum_{j=1}^{N-2}\log\|M_n(T^{jn}x,E)\| - \sum_{j=0}^{N-2}\log\|M_{2n}(T^{jn}x,E)\|\Big| < CNe^{-n/2}$$

Typically, the sums are **uniformly** (in x) close to their means (averages over **long orbits**). Conclusion:

$$u(x) := \frac{1}{Nn}\log\|M_{nN}(x,E)\| = u_0(x) + u_1(x)$$

where $\|u_0 - \langle u_0\rangle\|_\infty \leq CN^{-1+\varepsilon} =: \varepsilon_0$, $\|u_1\|_1 \leq C\,N^{-90} =: \varepsilon_1$. Provided the Riesz mass of the subharmonic extension of $u(x)$ is $< N^{20}$, the **splitting lemma** yields $\|u\|_{BMO} \leq C\Big(\varepsilon_0 + \sqrt{N^{20}\varepsilon_1}\Big) \leq C\,N^{-1+\varepsilon}$.

Motivation: Given N points $z_j = e(x_j)$ in $\mathbb{T}$, suppose that $P(z) = \prod_{j=1}^N (z - z_j)$ satisfies $\sup_{|z|=1}|P(z)| < e^\tau$. Then $D_N(\{x_j\}_{j=1}^N) \leq C\sqrt{N\tau}$.

Proof: $u(x) = \log|P(e(x))| = \mathcal{H}\Big(\sum_{j=1}^N f_0(\cdot - x_j)\Big) =: \mathcal{H}(F)$. Let K_N be a smooth bump function $K_N(\theta) = K(N\theta)$, $K \geq 0, \int K = 1, \mathrm{supp} K \subset [-.01, .01]$. Then

$$(K_N * f_0)\Big(\cdot - \frac{C}{N}\Big) - \frac{C}{N} < f_0(\cdot) < (K_N * f_0)\Big(\cdot + \frac{C}{N}\Big) + \frac{C}{N}$$

Using $\int_{\mathbb{T}} u = 0$, $\|u\|_1 \le C\tau$, $F = \mathcal{H}^{-1}u = -\mathcal{H}u$,

$$\begin{aligned} \|F\|_\infty &\le 2\|\mathcal{H}^{-1}(u * K_M)\|_\infty + \frac{CN}{M} \\ &\le C\sqrt{M}\|\mathcal{H}^{-1}(u * K_M)\|_2 + \frac{CN}{M} \le C\,M\|u\|_1 + \frac{CN}{M} \\ &\le C\,M\tau + \frac{CN}{M}. \end{aligned}$$

Setting $M = \sqrt{N/\tau}$ gives $\|F\|_\infty \le C\sqrt{N\tau}$ (and thus also) $\|u\|_{BMO} \le C\sqrt{N\tau}$. Finally, $\|F\|_\infty \sim D_N(\{x_j\}_{j=1}^N)$.

Note that this argument shows that if $u(z) = \int \log|z-\zeta|\,d\mu(\zeta)$ with $\mathrm{supp}(\mu) \subset \mathbb{T}$, $\sup_{\mathbb{T}} u \le \langle u\rangle + \tau$ (here $\langle u\rangle = 0$ and thus $\|u\|_1 \le \tau$), then $\|u\|_{BMO} \le C\sqrt{\|\mu\|\,\tau}$.

A small variation of this argument gives the following splitting lemma: *Let u be subharmonic on a neighborhood of $\mathbb{T}$ with Riesz mass N. Suppose $u = u_0 + u_1$ on $\mathbb{T}$ with $\|u_0\|_{L^\infty(\mathbb{T})} = \varepsilon_0$ and $\|u_1\|_{L^1(\mathbb{T})} = \varepsilon_1$. Then $\|u\|_{BMO} \le C(\varepsilon_0 + \sqrt{N\varepsilon_1})$.*

Review of the transfer matrix formalism:

large deviation theorems for $u(x) = \log \|M_N(x,E)\|$ (as well as for the entries $\log |f_N(x,E)|$: Wegner estimate!) obtained via (a) subharmonicity (b) almost invariance $u(x) \approx u(Tx)$. The latter can be either $\sup_x |u(x) - u(Tx)| \leq C$ (shift) or avalanche principle (skew-shift). Difference between these cases: Riesz mass of u in the former is N, in the latter N^2 (or N^C for higher-dim. versions of the skew-shift). The AP requires positive exponents, and the first step (but only that one) is perturbative (large disorder - or computer assisted?). The AP gives positive Lyapunov exponents for large disorders. Shift: the LDT does **not** require positive exponents and is non-perturbative. For the doubling map $Tx = 2x$ (mod1) the Riesz mass grows $\sim e^N$; use LDT for sums of martingale differences.

Localization: LDT only controls the probability of a single resonance at a fixed energy E, i.e.,

$$\text{meas}\Big[x \in \mathbb{T} \,:\, \text{dist}(E, H_{x,\omega}^{[-N,N]}) < e^{-N^\sigma}\Big].$$

For AL one needs to control the probability of double resonances, i.e., show $\sum_j \delta_{2^j} < \infty$ where

$$\delta_N := \text{meas}\Big[\omega \in DC_{A,c} \,:\, \text{dist}(E, \text{spec}(H_{0,\omega}^{[-n,n]})) < e^{-n^\sigma},$$
$$G_{[k,k+N]}(0,\omega,E) \text{ is bad for } \textbf{some } E \text{ and } |k| \sim N^C\Big].$$

Here $n \sim (\log N)^A$, and $G_{[k,k+N]}(x,\omega,E)$ is **good** if both $\|G_{[k,k+N]}(x,\omega,E)\| < e^{N^\sigma}$ and $|G_{[k,k+N]}(x,\omega,E)|(j,\ell) < e^{-\gamma N}$ if $k \le j,\ell \le N+k$, $|j-\ell| \sim N$. Also, $DC_{A,c} \subset \mathbb{T}$ denotes the set of ω with $\|n\omega\| \ge \frac{c}{|n|^A}$ for all $n \ne 0$. One has

$$\delta_N \le \sum_{E \in \text{spec}(H_{0,\omega}^{[-n,n]})} \text{meas}\Big[\omega \in DC_{A,c} \,:\, \log \|M_N(k\omega,\omega,E)\|$$
$$< N L_N(\omega,E) - N^\sigma, \text{ for some } |k| \sim N^C\Big]$$

The set on the right-hand side is the projection onto the ω-axis of the intersections $\Omega_N := \bigcup_{|k|\sim N^C} \mathcal{S}_N \cap \ell_k$ of the lines $\ell_k := \{(\omega, k\omega) \,:\, \omega \in \mathbb{T}\}$ with

$$\begin{aligned}\mathcal{S}_N \;:=\; &\Big\{(\omega, x) \in DC_{A,c} \times \mathbb{T} \,:\, \log \|M_N(x, \omega, E)\| \\ &\quad < NL_N(\omega, E) - N^\sigma\Big\}.\end{aligned}$$

The measure of $\mathcal{S}_N$ is very small by the LDT, but this by itself does not preclude $\Omega_N = \mathbb{T}$. We need to know that the intersections of $\mathcal{S}_N$ with any horizontal line consist of a small number (say N^C many) connected components. This property is given by the Milnor-Thom bound of d^C on the number of connected components of semi-algebraic sets of degree d. From this complexity bound and the LDT bound get $\delta_N \le Cn\,\mathrm{meas}[\Omega_N] < N^{-\tau}$ for some $\tau > 0$. Localization then follows by standard methods (resolvent identity plus the polynomial Shnol-Simon bound on generalized eigenfunctions).

Lemma 1 (BG). *Let* $\mathcal{S} \subset \mathbb{T}^2$ *be such that* $\{\omega \in \mathbb{T} : (\omega, x) \in \mathcal{S}\}$ *consists of* M *intervals for every* $x \in \mathbb{T}$*. Then*

$$\text{meas}\Big[\omega \in \mathbb{T} \;:\; (\omega, k\omega) \in \mathcal{S} \quad \textit{for some } |k| \sim N\Big]$$
$$\leq C\, N^C (\text{meas}[\mathcal{S}])^{\frac{1}{2}} + C\, M N^{-1}.$$

Concluding remarks:

- Localization can be obtained by this method for shifts of any dimension, as well as the skew-shift. In the latter case the main difficulty is the LDT (only known for large disorders).
- The long-range case as well the Laplacian on $\mathbb{Z}^2$ cannot be treated by the transfer matrix formalism, so no LDT or AP available. Here need to develop estimates on the probability that a given Green's function is bad (in the spirit of Fröhlich-Spencer's multiscale method).
- Hölder regularity of the IDS requires a sharp LDT

$$\mathrm{meas}\Big[x \in \mathbb{T} : |\log \|M_N(x,E) - NL_N(E)| > \delta N\Big] < e^{-c_\delta N}.$$

This LDT is known, but **only** for the shift on $\mathbb{T}$. Averaging the avalanche principle over x by means of this LDT yields

$|L_N(E)+L_n(E)-2L_{2n}(E)| < N^{-1+\varepsilon}$ where $n \sim \log N$.

Thus $|L(E)-L_N(E)| \le CN^{-1+\varepsilon}$ and

$$|L_N(E)-L_N(E')| \le CN^{-1+\varepsilon} + e^{C_2 n}|E-E'|.$$

Thus $|L(E)-L(E')| \le C|E-E'|^{\alpha}$, as desired.

Open problems:

- Hölder regularity of IDS for shifts on $\mathbb{T}^d$ with $d \ge 2$.

Does the IDS become more regular as the number of frequencies increases? Currently, the results deteriorate with the number of frequencies. Related question: Is there a LDT of the form

$$\text{meas}\Big[x \in \mathbb{T} : |\log \|M_N(x,E) - NL_N(E)| > \delta N\Big] < e^{-c_\delta N}$$

for shifts in higher dimensions?

- Determine the Hölder exponent for the IDS. More precisely, can one get Hölder $\frac{1}{2}-$ provided the potential has only non-degenerate critical points? This is known (Bourgain) for the almost Mathieu model and large disorders.
- Prove a version of Theorem 5 on $\mathbb{Z}^d$, $d \geq 3$.
- Positivity of the Lyapunov exponent for small disorders for the skew-shift.