Blowup for hyperbolic equations

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Consider Cauchy problem in $\mathbb{R}^n$ with $\Box = \partial_{tt} - \Delta$,

$$\Box u = N(u), \qquad \Box u = N(u, \nabla u) \qquad \text{(NLW)}$$

data $(u, \partial_t u)(0) = (u_0, u_1) \in H^s \times H^{s-1}$. Basic question of solvability and uniqueness:

- Locally well posed (LWP) in $H^s \times H^{s-1}$?
- If so, solutions global (GWP) or finite time blow-up ?

(LWP) in $H^s \times H^{s-1}$ ***if*** $\forall\ (u_0, u_1) \in H^s \times H^{s-1}$ ***there is*** $T > 0$ ***and a neighborhood*** U ***of*** (u_0, u_1) ***so that for all data in*** U $\exists!$ ***solution*** $(u, u_t) \in C((-T, T); H^s) \times C((-T, T); H^{s-1})$***, depending continuously on data. Also, higher regularity preserved.***

Easy: (LWP) for large s via energy methods, (GWP) for small data

Recast (**NLW**) as

$$u = S(u_0, u_1) + \Box^{-1} N(u)$$

Find fixed point for small times in Banach spaces X, Y with

$$S : H^s \times H^{s-1} \to X, \quad \Box^{-1} : Y \to X, \quad N : X \to Y$$

Energy method means

$$X = \{u \in L^\infty(H^s),\ \partial u \in L^\infty(H^{s-1})\}, \qquad Y = L^1(H^{s-1})$$

Boundedness of $S, \Box^{-1}$ trivial:

$$u(t) = \cos(\sqrt{-\Delta}t)u_0 + \frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}} u_1 + \int_0^t \frac{\sin(\sqrt{-\Delta}(t-t'))}{\sqrt{-\Delta}} f(t')\, dt'$$

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|f(t')\|_{H^{s-1}}\, dt'$$

Obstruction from N: $s \geq s_0 + 1$ where s_0 is the scaling exponent: $s_0 = \frac{n}{2} - \alpha$, **(NLW)** invariant under $\mathcal{D}_\lambda : u \to \lambda^\alpha u(\lambda x, \lambda t),\ \lambda > 0$. Point: norm of $\dot{H}^{\frac{n}{2}-\alpha}$ unchanged under $\mathcal{D}_\lambda$.

- $s < s_0$ Small data, small time equivalent to large data, large time result. Expect *local ill-posedness below scaling.*
- $s = s_0$ Small data, small time equivalent to small data, large time result. Same for large data *provided time of existence only depends on size of data.*
- $s > s_0$ Small data, large time equivalent to large data, small time result. $T_{\max} \gtrsim \|(u_0, u_1)\|_{H^s \times H^{s-1}}^{s_0 - s}$

Other symmetries besides scaling also serve as obstructions to (LWP): Lorentz invariance leads to concentration along light-rays.

(GWP): Suppose energy $E : H^{s_c} \times H^{s_c - 1} \to \mathbb{R}$ preserved under the flow of (NLW). Require that $E(u, u_t) \sim \|(u, u_t)\|_{s_c}^2$.

- subcritical case: $s_c > s_0$. (**LWP**) at $s \geq s_c$ implies (**GWP**) at $s \geq s_c$ since norm $\| \cdot \|_{s_c}$ does not grow. *Energy penalizes point-singularity formation via scaling.*

- critical case: $s_c = s_0$. (**LPW**) *equivalent* to (**GWP**) for small data. Large data: *Energy neutral to point-singularity formation via scaling*, exclude *concentration of energy in the tip of characteristic cone* (Morawetz identity).

- supercritical case: $s_c < s_0$. *Energy rewards point-singularity formation via scaling.* No global large data results known, even for Schwartz data.

Consider (**NLW**) with $N(u) = -|u|^{p-1}u$ and energy

$$E(u) = \int \frac{1}{2}|\partial u|^2 + \frac{1}{p+1}|u|^{p+1}\,dx$$

Then $s_c = 1, s_0 = \frac{n}{2} - \frac{2}{p-1}$. If n=3, p=5 the problem is **energy critical**, and (GWP) known (Grillakis, 90). **Open problem**: for example, prove that $p = 7$ (**NLW**)

$$\partial_{tt}u - \Delta u + u^7 = 0$$

admits *global smooth solutions for all smooth data*.

Expect finite time blow-up in many cases:

$$\partial_{tt}u - \Delta u - |u|^{p-1}u = 0$$

Solution $u(t,x) = c_p \cdot (T-t)^{-\frac{2}{p-1}}$, truncate to light cone.

Merle-Zaag: if $p \leq 3$, then blow-up is self-similar with this rate!

The Wave Maps Equation

$\phi : \mathbb{R}^{n+1} \to M$, (M, g) Riemann manifold with Lagrangian

$$L(\phi) = \frac{1}{2} \int_{\mathbb{R}^{n+1}} -|\partial_t \phi|_g^2 + |\nabla \phi|_g^2 \, dxdt$$

and associated Euler-Lagrange equation

$$\mathbf{D}^\alpha \partial_\alpha \phi = 0, \quad \Box \phi^i = \Gamma^i_{jk} \partial^\alpha \phi^j \partial_\alpha \phi^k, \quad \Box \phi \perp T_\phi M$$

E.g. $\phi = \gamma \circ u$, $D\dot{\gamma} = 0$, $\Box u = 0$. Conserved energy

$$E(\phi) = \frac{1}{2} \int |\partial_t \phi|_g^2 + |\nabla \phi|_g^2 \, dx$$

Cauchy Problem $\phi(0) = \phi_0 \in M$, $\partial_t \phi(0) = \phi_1 \in T_{\phi_0} M$. (WM) scaling $\phi(t, x) \mapsto \phi(\lambda t, \lambda x)$. So $s_0 = \frac{n}{2}$: (**LWP**) for $s > \frac{n}{2}$ (Machedon, Klainerman, Selberg), (**GWP**) for $s = \frac{n}{2}$, small data, "reasonable" M (Tataru, Tao, Krieger), and (**LIP**) for $s < \frac{n}{2}$ (d'Ancona, Georgiev). Large data (**GWP**): compare $s_0 = \frac{n}{2}$, $s_c = 1$

Expect (GWP) for $n = 1$ since $s_c > s_0$ (known, even down to scaling critical), blow-up for $n \geq 3$ (Shatah): there exist self-similar solutions $u(t,x) = v(x/t)$ with $u = v$, $\partial_t u = -x \cdot \nabla v$ at $t = 1$.
Rewrite the *Lagrangian in self-similar coordinates* $\tau = \sqrt{t^2 - |x|^2}$, $\xi = \frac{x}{t} = \rho\omega$, $\omega \in S^{n-1}$. Then v solves the elliptic PDE

$$-v_{\rho\rho} - \Big(\frac{n-1}{\rho} + \frac{(n-3)\rho}{1-\rho^2}\Big) v_\rho + \frac{1}{\rho^2(1-\rho^2)} \Delta_\omega v \perp T_v M$$

which is an harmonic map equation on the unit ball of $\mathbb{R}^n$ with **hyperbolic metric**

$$\frac{d\rho^2}{(1-\rho^2)^2} + \frac{\rho^2}{1-\rho^2} d\omega^2$$

For $n \geq 3$ non-constant solutions exist (equivariant harmonic maps) which can be smoothly continued beyond $\rho = 1$ with target manifolds given in terms of surfaces of revolution (spheres). Fails for $n = 2$, since the only solutions are $v = const$.

Critical case: $n=2$, $s_c = s_0 = 1$. We'll take target M a *surface of revolution* with metric $ds^2 = d\rho^2 + g^2(\rho)d\theta^2$, $\theta \in S^1$, $g \in C^\infty(\mathbb{R})$, $g(0)=0, g'(0)=1$ and also assume this: g odd and either

$$g(\rho) > 0 \ \ \forall\, \rho > 0, \qquad \int_0^\infty |g(\rho)|\, d\rho = \infty \tag{1}$$

or if M is compact, that g has first zero $\rho_1 > 0$, $g'(\rho_1) = -1$ and g periodic with period $2\rho_1$. Consider **equivariant wave maps** $\phi(t,x) : \mathbb{R}^{1+2} \to M$: if (r,ϕ) polar coordinates on $\mathbb{R}^2$ then

$$\rho = u(t,r), \qquad \theta = \phi$$

with equation $\partial_{\mathbf{tt}}\mathbf{u} - \partial_{\mathbf{rr}}\mathbf{u} - \frac{\mathbf{1}}{\mathbf{r}}\partial_{\mathbf{r}}\mathbf{u} + \frac{\mathbf{g(u)g'(u)}}{\mathbf{r^2}} = \mathbf{0}$. Smooth data with finite energy have local smooth solutions which cannot be continued beyond $t = T_*$ iff *energy concentrates*: $\exists\, \varepsilon_0 = \varepsilon_0(M) > 0$ so that

$$\liminf_{t \to T_*} \frac{1}{2}\int_0^{T_*-t} |\partial u|^2 + \frac{g^2(u)}{r^2}\, dr > \varepsilon_0$$

Struwe's bubbling off theorem: Let ϕ be smooth (EQWM) blowing up at t_0. Then $\exists\, r_j \to 0+, t_j \to t_0-$ s.t.

$$\phi_j(t,x) := \phi(t_j + r_j t, r_j x) \to \Phi_\infty(t,x), \qquad r_j = o(t_0 - t_j)$$

strongly in $H^1_{\mathrm{loc}}(]-1,1[\times\mathbb{R}^2)$, where Φ_∞ a nonconstant, time-independent solution generating nonconstant, smooth, (EQHM) $\Psi : S^2 \to M \quad (\Delta_{S^2}\Psi \perp T_\Psi M)$.

If M is non-compact as in (1), then harmonic spheres are known not to exist: (GWP) for such targets in the *equivariant case*. However: for $M = S^2$ harmonic spheres do exist: $z^\ell, \bar{z}^\ell : \mathbb{C} \to \mathbb{C}$, $\ell \geq 1$, via stereographic projection $u(r) = 2\arctan(r^\ell), \theta = \pm\ell\phi$.

Main result of this talk for wave maps is to show that blow-up does occur for S^2! In the *non-equivariant case* also expect dichotomy between (GWP) for large data and blow-up depending on the **geometry of the target manifold** (S^2 vs. $\mathbb{H}^2$).

Theorem (K-S-T, 2006): Let $\nu > \frac{1}{2}$, $t_0 > 0$, $\lambda(t) = t^{-1-\nu}$, N large. $\exists$ (WM) $\phi(t, r, \phi) = (u(r, t), \phi) : [0, t_0] \times \mathbb{R}^2 \to S^2$

$$u(r,t) = 2\arctan(\lambda(t)r) + u^e(r,t) + \varepsilon(r,t), \qquad 0 \le r \le t$$

$$\mathcal{E}_{\mathrm{loc}}(u^e)(t) \lesssim (t\lambda(t))^{-2}\,|\log t|^2, \quad \mathcal{E}_{\mathrm{loc}}(\varepsilon)(t) \lesssim t^N \text{ as } t \to 0$$

$$u^e \in C^{\nu+1/2-}(\{t_0 > t > 0,\ |x| \le t\}), \quad \varepsilon \in t^N H^{1+\nu-}_{\mathrm{loc}}(\mathbb{R}^2)$$

Also, $u(0,t) = 0$ for all $0 < t < t_0$. The solution $u(r,t)$ extends as an $H^{1+\nu-}$ solution to all of $\mathbb{R}^2$ and the energy of u concentrates in the cuspidal region $0 \le r \lesssim \frac{1}{\lambda(t)}$ leading to blow-up at $r = t = 0$.

Remarks: i) Expect $\nu > 0$, improve on nonlinear part of proof; blow-up *non-generic*, study *conditional stability*

ii) Rodnianski-Sterbenz 06 independently obtained blow up; bulk term $Q(\lambda(t)r)$ with Q (**HM**) degree $\ell \ge 4$, obtained *stable* rate $\lambda(t) \gtrsim t^{-1}\sqrt{-\log t}$ (prior work Bizon-Ovchinnikov-Sigal 04, Cote)

Related result for energy critical **(SLWE)** in $n = 3$:

$$\partial_{tt} u - \Delta u - u^5 = 0, \quad (t, x) \in \mathbb{R}^{1+3}$$

Stationary solutions $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$.

Theorem (K-S-T, 2006): Let $\nu > \frac{1}{2}$, $\delta > 0$, $\lambda(t) = t^{-1-\nu}$. $\exists$ energy solution u of (SLWE) blowing up at $r = t = 0$ and $\forall\ |x| = r \le t < t_0$ small,

$$u(x, t) = \lambda^{\frac{1}{2}}(t) W(\lambda(t) r) + \eta(x, t)$$

where $\mathcal{E}_{\mathrm{loc}}(\eta(\cdot, t)) \to 0$ as $t \to 0$ and outside the cone

$$\int_{[|x| \ge t]} \left[|\nabla u(x, t)|^2 + |u_t(x, t)|^2 + |u(x, t)|^6 \right] dx < \delta$$

for small $t > 0$. In particular, the energy of these blow-up solutions can be chosen arbitrarily close to $\mathcal{E}(W, 0)$, i.e., the energy of the stationary solution.

Background on (SLWE): $s_0 = s_c = 1$, $u(t,x) \to \lambda^{\frac{1}{2}} u(\lambda t, \lambda x)$ leaves both $\dot{H}^1(\mathbb{R}^3)$ and (**SLWE**) invariant. (**LWP**) in energy space $\dot{H}^1 \times L^2(\mathbb{R}^3)$, solution cannot be continued beyond $T_* < \infty$ iff $\|u\|_{L^8([0,T_*)\times\mathbb{R}^3)} = \infty$. Critical points of Lagrangian

$$L(u) = \int_{\mathbb{R}^{1+3}} \frac{1}{2}(-|\partial_t u|^2 + |\nabla u|^2) - \frac{1}{6}|u|^6 \, dtdx$$

and conserved energy

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) - \frac{1}{6}|u|^6 \, dx$$

The critical Sobolev imbedding $\dot{H}^1 \to L^6(\mathbb{R}^3)$ has extremizers $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$. Conformal group generates all extremizers. Associated EL-equation is $-\Delta W - W^5 = 0$, hence stationary solutions of (**SLWE**). Perturb around W: Seek $u = W + \psi$ with ψ *small*. Then $(-\Delta - 5W^4(r))\psi(t,r) = N(W,\psi) = O(|\psi|^2)$ has solutions, with $H = -\Delta - 5W^4(r)$,

$$\psi(t,x) = \cos(\sqrt{H}t)\psi_0 + \frac{\sin(\sqrt{H}t)}{\sqrt{H}}\psi_1 + \int_0^t \frac{\sin(\sqrt{H}(t-t'))}{\sqrt{H}} N(t')\,dt'$$

Spectrum of H restricted to $L^2_{\rm rad}$: since $H(\partial_\lambda[\lambda^{\frac12}W(\lambda r)]) = 0$ and the zero-mode has a unique positive zero: zero is second eigenvalue (actually: resonance), bottom eigenvalue negative, $Hg_0 = -k_0^2 g_0$,

$$\cos(\sqrt{H}t)g_0 = \cosh(k_0 t)g_0, \quad \frac{\sin(\sqrt{H}t)}{\sqrt{H}}g_0 = \frac{\sinh(k_0 t)}{k_0}g_0.$$

Thus, need to project onto $g_0^\perp$ to have linear stability. Nonlinearly, one has a **local center stable mf theorem for radial data (KS 05): In $B_\delta(0) \subset \langle x\rangle^{-\sigma}(H^3_{\rm rad} \times H^2_{\rm rad}) \,\exists$ co-dim one Lipschitz graph $\mathcal{G}$ parameterized by tangent plane $\langle k_0 f_1 + f_2, g_0\rangle = 0$ s.t. data on $(W,0) + \mathcal{G}$ have global solutions $\lambda_\infty^{\frac12} W(\lambda_\infty x) + R$ where R disperses and scatters to a free energy wave and $|\lambda_\infty - 1| \lesssim \delta$.**

Expect $\mathcal{G}$ to divide $(W,0)+B_\delta(0)$ into **blow-up/scattering** halves. Karageorgis-Strauss 06 proved that there is blow-up above the tangent plane of $\mathcal{G}$ (for $|u|^5$). Graph of energy at $(W,0)$ is a saddle surface:

$$\mathcal{E}(W+f_1,f_2)=\mathcal{E}(W,0)+\langle D\mathcal{E}(W,0),(f_1,f_2)\rangle+$$
$$+\frac{1}{2}\langle D^2\mathcal{E}(W,0)(f_1,f_2),(f_1,f_2)\rangle+\ldots$$

Euler-Lagrange equation for W is equivalent to $D\mathcal{E}(W,0)=0$, whereas the second variation is an indefinite quadratic form

$$\langle D^2\mathcal{E}(W,0)(f_1,f_2),(f_1,f_2)\rangle=-k_0^2\xi_1^2+\xi_2^2+\langle Hf_1^\perp,f_1^\perp\rangle+\langle f_2^\perp,f_2^\perp\rangle$$

with $\xi_1=\langle f_1,g_0\rangle,\xi_2=\langle f_2,g_0\rangle$.

Compare this to the definition of the tangent plane to $\mathcal{G}$ as well as the recent work of Kenig-Merle 06 on the regime of energy below $\mathcal{E}(W,0)$. Their work implies that for $\mathcal{E}(W+f_1,f_2)<\mathcal{E}(W,0)$ the center stable manifold is $\|\nabla f_1\|_2=\|\nabla W\|_2$.

Strategy of proof for (WM): We seek a solution of

$$-\partial_{tt}u + u_{rr} + \frac{u_r}{r} - \frac{\sin(2u)}{2r^2} = 0$$

of the form $u(t,r) = Q(\lambda(t)r) + v(t,r)$ inside the light-cone $r \leq t$, with $\mathcal{E}_{\text{loc}}(v(t,\cdot)) \to 0$ as $t \to 0+$. By Struwe's result, $t\lambda(t) \to \infty$. Applying ∂_{tt} to the bulk-term generates

$$\partial_{tt}Q(\lambda(t)r) = \ddot{\lambda}(t)rQ'(\lambda(t)r) + \dot{\lambda}^2(t)r^2Q''(\lambda(t)r)$$

Note: $rQ'(r) = \frac{r}{1+r^2} \notin L^2(\mathbb{R}^2)$. To cancel at $r = \infty$ need ODE

$$\lambda(t)\ddot{\lambda}(t) - 2\dot{\lambda}(t)^2 = 0 \Leftrightarrow \lambda(t) = (c_1 t + c_2)^{-1}$$

Not allowed by Struwe's theorem!

Way out: Replace $Q(\lambda(t)r)$ by $Q(\lambda(t)r)\chi(t, r/t)$ (a cut-off to light-cone) with $\chi(t,\cdot) = 1 + o_{H^1}(1)$ as $t \to 0+$.

Ignoring χ_t, setting $a = \frac{r}{t}, \lambda(t) = t^{-1-\nu}$ yields ODE in a with principal Sturm-Liouville term

$$(1-a)^{\nu+\frac{1}{2}}\partial_a[(1-a)^{-\nu+\frac{1}{2}}\partial_a]\chi(a) = \ldots$$

Fundamental system $1, (1-a)^{\nu+\frac{1}{2}}$, gives energy solutions as long as $\nu > 0$.

Actual renormalization scheme: $R = \lambda(t)r$, $a = r/t$, $u_0(R) = Q(R)$. Seek $u_k = u_{k-1} + v_k$ approximate solutions of (WME) in light-cone $r \leq t$, improve with k. **Iterative scheme** of errors and corrections

$$e_k = (-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r)u_k - \frac{\sin(2u_k)}{2r^2}$$

$$v_{2k+1} = \big(\partial_{rr} + \frac{1}{r}\partial_r - \frac{\cos(2u_0)}{r^2}\big)e_{2k}$$

$$v_{2k} = \big(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r - \frac{1}{r^2}\big)e_{2k-1}$$

Then $e_{2k} = -N_{2k}(v_{2k})$, $e_{2k+1} = -\partial_t^2 v_{2k+1} - N_{2k+1}(v_{2k+1})$ with

$$\begin{aligned} N_{2k+1}(v) &= \frac{\cos(2u_0) - \cos(2u_{2k})}{r^2} v + \frac{\sin(2u_{2k})}{2r^2}(1 - \cos(2v)) \\ &\quad + \frac{\cos(2u_{2k})}{2r^2}(2v - \sin(2v)) \\ N_{2k}(v) &= \frac{1 - \cos(2u_{2k-1})}{r^2} v + \frac{\sin(2u_{2k-1})}{2r^2}(1 - \cos(2v)) \\ &\quad + \frac{\cos(2u_{2k-1})}{2r^2}(2v - \sin(2v)) \end{aligned}$$

Upshot: Get errors that decay like $t^m \; \forall \; m$. Indeed,

$$u_{2k-1}(r,t) = Q(\lambda(t)r) + \frac{c_k}{(t\lambda)^2} R\log(1+R^2) + O\left(\frac{(\log(1+R^2))^2}{R(t\lambda)^2}\right)$$

$$e_{2k-1} = O\left(\frac{R(\log(2+R))^{2k-1}}{t^2(t\lambda)^{2k}}\right)$$

where correction is $C^{\nu+\frac{1}{2}-}$ at $r = t$.

Perturbative analysis: Exact (**WME**) solution $u = u_{2k-1} + \varepsilon$.
Solve in coordinates $R = \lambda(t)r$, $d\tau = \lambda(t)dt$, $\tau = \int_t^1 \lambda(s)\, ds + \nu^{-1}$.
Set $\tilde{\varepsilon} = R^{\frac{1}{2}}\varepsilon(t(\tau), \lambda^{-1}R)$. **Main PDE**:

$$\Big(-(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R)^2 + \frac{1}{4}(\frac{\lambda_\tau}{\lambda})^2 + \frac{1}{2}\partial_\tau(\frac{\lambda_\tau}{\lambda})\Big)\tilde{\epsilon} - \mathcal{L}\tilde{\epsilon}$$
$$= \lambda^{-2}R^{\frac{1}{2}}\left(N_{2k-1}(R^{-\frac{1}{2}}\tilde{\epsilon}) + e_{2k-1}\right)$$
$$\mathcal{L} := -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$$

Note: e_{2k-1} decays like τ^{-N}.

Two key issues: Appearance of generator of dilations $R\partial_R$ and the **strongly singular Schrödinger operator** $\mathcal{L}$ on $L^2(0,\infty)$ as part of the driving linear hyperbolic operator.

- Spectral, scattering theory, Fourier transform $\mathcal{F}$ of $\mathcal{L}$
- $\mathcal{F}$ hits (PDE), solve transport equation for $\mathcal{F}\tilde{\epsilon}$, $R\partial_R \to \xi\partial_\xi$

Spectral theory: Our Sturm-Liouville operator is **singular** at both $R = 0$ and $R = \infty$. Consider first **regular** case at $R = 0$: $\tilde{\mathcal{L}} = -\frac{d^2}{dR^2} + V(R)$ with $V(R) \in L^1(0, \infty)$ is **self-adjoint** subject to *Dirichlet BC at zero*, say. For $\mathrm{Im} z > 0$ $\exists$ Weyl-Titchmarsh solution $\psi(R, z) \in L^2(0, \infty)$ - unique up to scalar multiples (limit point case at $R = \infty$). Set $\psi(0, z) = 1$, and write

$$\psi(R, z) = \theta(R, z) + m(z)\phi(R, z)$$

where $\mathcal{L}\theta(\cdot, z) = z\theta(\cdot, z), \mathcal{L}\phi(\cdot, z) = z\phi(\cdot, z)$ and

$$\theta(0, z) = \phi'(0, z) = 1, \qquad \theta'(0, z) = \phi(0, z) = 0$$

Then $m(z)$ *Herglotz*. **Fourier inversion** (where $\chi_{(0,\infty)}(\mathcal{L})f = f$)

$$\hat{f}(\xi) := \int_0^\infty f(R)\phi(R, \xi)\, dR, \qquad f(R) = \int_0^\infty \hat{f}(\xi)\phi(R, \xi)\rho(\xi)\, d\xi$$

with **spectral measure** $\rho(d\xi) := \pi^{-1}\mathrm{Im} m(\xi + i0) d\xi$. **Free case**: $V = 0, \psi = e^{iR\xi^{\frac{1}{2}}}$, $\theta = \cos(R\xi^{\frac{1}{2}})$, $\phi = \xi^{-\frac{1}{2}} \sin(R\xi^{\frac{1}{2}})$, $m(\xi) = i\xi^{\frac{1}{2}}$

Singular case: $\mathcal{L}_0 = -\frac{d^2}{dR^2} + \frac{3}{4R^2}$, $\mathcal{L}_0 R^{\frac{3}{2}} = \mathcal{L}_0 R^{-\frac{1}{2}} = 0$, self-adjoint without BC at $R = 0$, *limit point case*. For $\mathcal{L} = \mathcal{L}_0 - \frac{8}{(1+R^2)^2}$ have $\mathcal{L}\phi_0 = \mathcal{L}\theta_0 = 0$, $W(\theta_0, \phi_0) = 1$, where

$$\phi_0(R) := \frac{R^{\frac{3}{2}}}{1+R^2}, \qquad \theta_0(R) := \frac{1 - 4R^2 \log R - R^4}{2R^{\frac{1}{2}}(1+R^2)}$$

$\forall\, \mathrm{Im} z > 0\ \exists$ **fund. systems** $\phi(R,z), \theta(R,z)$, and $\psi(R,z), \overline{\psi(R,z)}$ of $\mathcal{L}f = zf$. Former entire in z, $W(\theta(\cdot,z), \phi(\cdot,a)) = 1$, and

$$\phi(R,z) = \phi_0(R) + R^{-\frac{1}{2}} \sum_{j=1}^{\infty} (R^2 z)^j \phi_j(R)$$

Fourier inversion: as above with "Weyl-Titchmarsh" function

$$m(z) = \frac{W(\theta(.,z), \psi(.,z))}{W(\psi(.,z), \phi(.,z))}$$

Proved by Gesztesy-Zinchenko 05

For $\mathcal{L}_0 = -\frac{d^2}{dR^2} + \frac{n^2-1/4}{R^2}$ this amounts to **Bessel functions**:

$$\phi(R;z) = \frac{\pi}{2} C^{-1} z^{-n/2} R^{1/2} J_n(z^{1/2}R)$$

$$\theta(R;z) = C\, z^{n/2} R^{1/2} [-Y_n(z^{1/2}R) + \pi^{-1} \log(z) J_n(z^{1/2}R)]$$

$$\begin{aligned} \psi(R;z) &= C\, z^{n/2} R^{1/2} [-Y_n(z^{1/2}R) + i J_n(z^{1/2}R)] \\ &= C\, z^{n/2} R^{1/2} i H_n^{(1)}(z^{1/2}R) \\ &= \theta(R;z) + m(z)\phi(R;z) \end{aligned}$$

$$m(z) = C^2 \frac{2}{\pi} z^n [i - \pi^{-1} \log(z)], \quad z \in \mathbb{C} \setminus \mathbb{R}^+$$

For our $\mathcal{L} = -\frac{d^2}{dR^2} + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$ obtain

$$\rho(\xi) \asymp \frac{\chi_{[\xi \le 1]}}{\xi \log^2 \xi} + \xi \chi_{[\xi \ge 1]}$$

Singularity at $\xi = 0$ due to zero energy resonance.

Transference identity: With $\mathcal{F}$ the Fourier transform of $\mathcal{L}$,

$$\mathcal{F}\big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\big) = \big(\partial_\tau + \frac{\lambda_\tau}{\lambda}(-2\xi\partial_\xi + \mathcal{K})\big)\mathcal{F}$$

where $\mathcal{K} = -\big(\frac{3}{2} + \frac{\eta\rho'(\eta)}{\rho}\big)\delta(\xi-\eta) + \frac{\rho(\xi)}{\xi-\eta}F(\xi,\eta)$, and

$$|F(\xi,\eta)| \lesssim \langle \xi+\eta\rangle^{-\frac{3}{2}}(1+|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}|)^{-N}$$

Point: $(R\partial_R - 2\xi\partial_\xi)e^{iR\xi^{\frac{1}{2}}} = 0$ and $\mathcal{K}$ is the error when applied to $\phi(R,\xi)$ rather than $e^{iR\xi^{\frac{1}{2}}}$. **Boundedness property** $\forall\, \alpha$

$$\mathcal{K}_0(\xi,\eta) := \frac{\rho(\xi)}{\xi-\eta}F(\xi,\eta) \;:\; L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho$$

$$[\mathcal{K}_0, \xi\partial_\xi] \;:\; L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho$$

with $\|f\|^2_{L^{2,\alpha}_\rho} = \int_0^\infty |f(\xi)|^2 \langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi$.

Final iteration: Apply Fourier transform $\mathcal{F}$ to main (**PDE**)

$$-\Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)^2 x - \xi x = 2\beta\mathcal{K}\Big(\partial_\tau - 2\beta\xi\partial_\xi\Big)x + \beta^2(\mathcal{K}^2 + 2[\xi\partial_\xi, \mathcal{K}])x$$
$$- \Big(\frac{\beta^2}{4} + \frac{\dot{\beta}}{2}\Big)x + \lambda^{-2}\mathcal{F}R^{\frac{1}{2}}(N_{2k-1}(R^{-\frac{1}{2}}\mathcal{F}^{-1}x) + e_{2k-1})$$

where $\beta = \frac{\lambda_\tau}{\lambda}$. We solve this with zero terminal conditions. I.e., let $H(\tau,\sigma)$ be the backward fundamental solution for the operator

$$\Big(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\Big)^2 + \xi$$

and by $H(\tau,\sigma)$ its kernel, $x(\tau) = \int_\tau^\infty H(\tau,\sigma)f(\sigma)\,d\sigma$. ***Then*** $\forall\, \alpha \geq 0\ \exists C = C(\alpha)$ ***large so that uniformly in*** $\sigma \geq \tau$

$$\|H(\tau,\sigma)\|_{L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho} \lesssim \tau\Big(\frac{\sigma}{\tau}\Big)^C$$
$$\Big\|\Big(\partial_\tau - \frac{\lambda_\tau}{\lambda}2\xi\partial_\xi\Big)H(\tau,\sigma)\Big\|_{L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho} \lesssim \Big(\frac{\sigma}{\tau}\Big)^C$$

Main linear estimate, small Lipschitz constant: Define

$$\|f\|_{L^{\infty,N}L^{2,\alpha}_\rho} := \sup_{\tau\geq 1}\tau^N\|f(\tau)\|_{L^{2,\alpha}_\rho}$$

Given $\alpha \geq 0$ let N be large enough. Then

$$\|Hb\|_{L^{\infty,N-2}L^{2,\alpha+1/2}_\rho}+\left\|\left(\partial_\tau-2\beta\xi\partial_\xi\right)Hb\right\|_{L^{\infty,N-1}L^{2,\alpha}_\rho} \lesssim \frac{1}{N}\|b\|_{L^{\infty,N}L^{2,\alpha}_\rho}$$

To close the loop, it remains to show this:

Assume that N is large enough and $\frac{\nu}{2}+\frac{3}{4} > \alpha > \frac{1}{4}$. Then the map

$$x \to \lambda^{-2}\mathcal{F}R^{\frac{1}{2}}(N_{2k-1}(R^{-\frac{1}{2}}\mathcal{F}^{-1}x))$$

is locally Lipschitz from $L^{\infty,N-2}L^{2,\alpha+1/2}_\rho$ to $L^{\infty,N}L^{2,\alpha}_\rho$.

In order to lower $\nu > \frac{1}{2}$ to $\nu > 0$, say, would need to improve on this nonlinear estimate. Need $\frac{\nu}{2} > \alpha > \frac{1}{4}$ due to regularity of e_{2k-1} from **renormalization step**.

The semilinear case Similar scheme, but some major differences:

- No "cuspidal result" as Struwe's theorem expected for (**SLWE**); in fact, blow-up could occur on complicated set. Currently little intuition why $\lambda(t) = t^{-1-\nu}$ rates appear. Role of criticality? Compare work of Merle-Zaag for (**SLWE**) and Merle-Raphael, G. Perelman for L^2 critical (**NLS**).

- **Spectrum** of linearized operator $-\Delta - 5W^4$ (restricted to L^2_{rad}) is $\{-k_0^2\} \cup [0, \infty)$ with a *zero energy resonance*. Need to kill the *exponentially growing mode*; heuristically (!?), tie eval $-k_0^2$ to stable, **self-similar** blow-up rate $\lambda(t) = t^{-1}$ (with corrections), whereas the resonance (by itself) leads to all the *slow rates* that we observed. Since for (**WM**), $\mathcal{L}_Q$ = linearized operator has *purely continuous spectrum* provided $\deg(Q) = 1$, perhaps no generic rate for (**WM**) relative to the ground state (HM). **Note:** If $\deg(Q) > 1$, then $\mathcal{L}_Q$ has zero eigenvalue!! Rodnianski-Sterbenz 06 obtain stable rate $\lambda(t) \gtrsim t^{-1}\sqrt{-\log t}$.

- In (**WM**) the scaling is $Q(\lambda r)$ whereas for (**SLWE**) it is $\lambda^{\frac{1}{2}} W(\lambda r)$. Difference in the nonlinearity: $\frac{\sin(2u)}{2r^2}$ versus u^5. The former comes from $\Box\phi = \phi(|\nabla\phi|^2 - |\phi_t|^2)$, the latter has no gradient; $\lambda^{\frac{1}{2}}$ in front of W produces same scaling as r^{-2}.
- The role of the zero energy resonance in the formation of blow-up not understood. It appears in both (**WM**) and (**SLWE**). In $4+1$-dim. energy critical **Yang-Mills**, however, the linearized operator has a zero energy eigenvalue which appears to destroy the renormalization procedure:

$$\partial_{tt} u - \Delta_{\mathbb{R}^2} u - \frac{2}{r^2} u(1-u^2) = 0$$

$$Q(r) = \frac{1-r^2}{1+r^2}, \qquad rQ'(r) = -4\frac{r^2}{(1+r^2)^2}$$

$$\mathcal{L} = -\frac{d^2}{dR^2} + \frac{15}{4R^2} - \frac{24}{(1+R^2)^2}, \qquad \mathcal{L} R^{-\frac{3}{2}} = \mathcal{L} R^{\frac{5}{2}} = 0$$

Perhaps *slow blow-up* does not occur here !?