

Subharmonic techniques in multiscale analysis: Lecture 2

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Spectrum of ergodic Schrödinger operators

For **self-adjoint operators**

$$(H_x \psi)_n = \psi_{n+1} + \psi_{n-1} + v_n(x) \psi_n, \quad n \in \mathbb{Z}$$

with $v_n(x)$ an “**ergodic potential**”, i.e., $v_n(x) = V(T^n x)$ and $T : X \rightarrow X$ ergodic transformation on a probability space X , and $V : X \rightarrow \mathbb{R}$ measurable. Then there exists fixed compact set $K \subset \mathbb{R}$ with $\text{spec}(H_x) = K$ for a.e. $x \in X$. This follows from ergodic theorem and property of the **spectral resolution** E_x of H_x

$$E_x = S^{-1} \circ E_{T_x} \circ S, \quad S = \text{right shift}$$

In addition, $\text{spec}_{pp}(H_x), \text{spec}_{ac}(H_x), \text{spec}_{sc}(H_x)$ are also deterministic. **Eigenvalues are NOT deterministic**, but their **closure** is.

Anderson localization means precisely that $\text{spec}_{pp}(H_x) = \text{spec}(H_x)$ and eigenfunctions decay exponentially. *Most famous problem in this area: Anderson conjecture in three dimensions for the random case.*

Localization via Oseledec

How to establish AL? Assume positive Lyapunov exponent $\inf_E L(E) > 0$, ω fixed irrational.

By *multiplicative ergodic theorem (Oseledec theorem)*, for every energy and almost every $x \in \mathbb{T}$ there exist directions $\mathbf{v}_x^\pm(E)$ which are **contracting** as $n \rightarrow \pm\infty$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_n(x, E)\mathbf{v}_x^+(E)\| = -L(E) \quad (1)$$

and same for $n \rightarrow -\infty$. If these directions coincide we obtain a globally exponentially decaying solution. If these directions do not coincide, then on one side the solution will grow exponentially, **and thus the energy will not belong to the spectrum.**

Conclusion: The spectrum consists purely of eigenvalues with exponentially decaying eigenfunctions. So why haven't we proved AL?

FALLACY: We need to remove the zero measure sets in x **FOR ALL ENERGIES.** This is not allowed.

Oseledec theorem

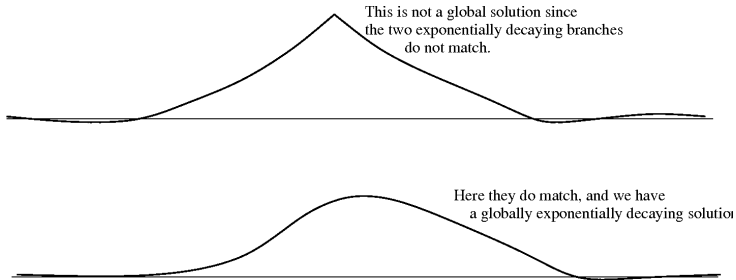


Figure: The solutions of $H_x \psi = E \psi$

Localization theorem by Bourgain-Goldstein

“Localization is a game of resonances” (Phil Anderson). What this means is that we need to make sure that we cannot have infinite tunneling, as this would lead to extended states.

Theorem (Bourgain-Goldstein, Annals, 2000)

Let V be a trigonometric polynomial. Assume that $L(E, \omega) > 0$ for all (ω, E) . Then for almost every $\omega \in \mathbb{T}$ the operator H_0 exhibits AL.

Note:

- Preceded by seminal key result by **S. Jitomirskaya** for Harper operator.
- By Fubini, we also have AL for almost every (ω, x) for H_x .
- In the argument, eliminate zero measure set from Diophantine class precisely to prevent tunneling (eliminate **double resonance**).
- Extends to more than one frequency.
- We can allow analytic potentials by approximation. **Gevrey class**: see **S. Klein** “Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function”, J. Funct. Anal. (2005), no. 2, 255–292; see also “Anderson localization for quasi-periodic Schrödinger operators with multivariable Gevrey potential function”, arXiv:1204.3086.

- In case the **Lyapunov exponent vanishes** on some part of the spectrum the eigenfunctions are expected to be **non-localized but rather oscillating similar to the Floquet eigenfunctions** from the theory of *periodic* Schrödinger equations. For “small” quasi-periodic potentials such eigenfunctions were discovered in the ground-breaking work **E. I. Dinaburg, Ya. G. Sinai**, “*The one dimensional Schrödinger equation with quasiperiodic potential*”, Funkt. Anal. i. Priloz. **9** (1975), 8–21. See also **H. Eliasson’s** work on this topic.

- Recently **A. Avila** made substantial progress towards a “non-perturbative” result if the **Lyapunov exponent vanishes on the spectrum**, similar to B-G theorem in the one frequency case, see **A. Avila**, “*Global theory of one-frequency Schrödinger operators I: stratified analyticity of the Lyapunov exponent and the boundary of nonuniform hyperbolicity*”. Preprint (arXiv:0905.3902), **A. Avila**, “*Global theory of one-frequency Schrödinger operators II: acriticality and finiteness of phase transitions for typical potentials*”, Preprint, 2011, **A. Avila**, “*Almost reducibility and absolute continuity. I*” Preprint (arXiv:1006.0704). In the case of **several frequencies the problem is wide open**, see some further comments below.

Strategy of the proof

We know from elementary spectral theory (easy part of Shnol's theorem): For almost every energy E relative to the spectral measure of $H_0 = H(0)$ we have that $H(0)\psi = E\psi$ admits a nonzero solution which grows at most linearly: $|\psi(n)| \lesssim 1 + |n|$.

GOAL: Show that ψ decays exponentially.

For any $\Lambda := [-n, n] \subset \mathbb{Z}$ locally on Λ we have “Poisson formula”

$$(H_\Lambda(0) - E)\psi = \psi_{n+1}\delta_n + \psi_{-n-1}\delta_{-n}$$

Suppose $\psi_0^2 + \psi_1^2 = 1$ and $\psi_{n+1}^2 + \psi_{-n-1}^2 < \gamma^2$. Then

$$\text{dist}(E, \text{spec}(H_\Lambda(0))) \leq \gamma \tag{2}$$

We expect to be able to find “many” intervals $\Lambda = [-n, n]$, in fact along a sufficiently “dense” sequence of $n \rightarrow \infty$ so that (2) holds with $\gamma = \gamma_n \rightarrow 0$ quite rapidly.

Next, we introduce **good** and **bad** Green functions:

Good and bad Green functions

Definition

We say that for any interval $\Lambda \subset \mathbb{Z}$ the Green function

$G_\Lambda(E, \omega) = (H_\Lambda(0, \omega) - E)^{-1}$ is **good** iff

- 1 $\|G_\Lambda(E, \omega)\| < e^{|\Lambda|^{b_1}}$
- 2 $|G_\Lambda(E, \omega)(n, m)| < \exp(-L(E)|n - m| + |\Lambda|^{b_2}) \quad \forall n, m \in \Lambda$

where $0 < b_1, b_2 < 1$. Otherwise, we say the Green function is **bad**.

Clearly, 2 implies 1 by square summation provided $b_1 = b_2$, but we want to keep these two properties separate. *What does LDT have to say about good/bad Green functions?*

Let $\Lambda = [n, n + N]$. We have

$$H_\Lambda(0, \omega) = H_{[0, N]}(n\omega, \omega)$$

We already observed in Lecture 1 that the LDT (for the determinants = entries of M_N) implies that $G_{[0, N]}(E, \theta, \omega)$ is **bad** with probability at most e^{-N^σ} (and parameter $b_1 = b_2 = 1 - \sigma$ determined by the LDT). Depends on Diophantine properties of ω .

Good and bad Green functions

The properties of **good Green functions are intrinsic to Anderson Localization**: Indeed, suppose H_Λ has eigenbasis $\{\psi_j\}_{j \in \mathbb{Z}}$ of exponentially decaying eigenfunctions with eigenvalues E_j on some finite volume Λ . Assume

$$\text{dist}(E, \text{spec}(H_\Lambda)) > \exp(-|\Lambda|^b)$$

Then

$$(H_\Lambda - E)^{-1}(n, m) = \sum_j \frac{\psi_j(n)\psi_j(m)}{E_j - E}$$

satisfies

$$|(H_\Lambda - E)^{-1}(n, m)| \lesssim \exp(-\gamma|n - m| + |\Lambda|^b)$$

Outline of the entire strategy

Basic AL strategy:

- Start with a polynomially growing solution as above with energy E .
- Show that there exists a sufficiently dense sequence $n_j \rightarrow \infty$ so that $G_{[-n_j, n_j]}(0, \omega, E)$ is bad. This uses the “Poisson formula” and averaging of the monodromy matrices over long orbits, which produces uniform closeness to the Lyapunov exponent.
- **Exclude double resonances:** for each n_j we will show that we may eliminate a small set of ω (measure going to zero sufficiently rapidly as $j \rightarrow \infty$) such that **all** Green functions $G_\Lambda(0, \omega, E)$ are good for Λ of length n_j (basically) which are contained in a window $[n_j^C, n_j^{2C}]$. This step is based on **LDT**, **semi-algebraic complexity bound**, and an elementary measure theory argument about small sets in $[0, 1]^2$ that have not too many intersection with horizontal lines.

Outline of strategy

- **Resolvent identity** implies that Green function $G_{[n_j^c, n_j^{2c}]}(0, \omega, E)$ is good. This then of course implies exponential decay of ψ , since we have enough of these intervals to cover the whole line \mathbb{Z} .

This strategy has been implemented in a variety of settings, always on the line \mathbb{Z} :

- 1-dim shift, analytic (Gevrey) potentials. **Nonperturbative** for $1 \leq s < 2$ where $|g^{(k)}(x)| \leq CR^k k^{sk}$, **Perturbative** for $s \geq 2$.
- multi-dimensional shift, same type of potentials. Nonperturbative.
- Skew shift on \mathbb{T}^2 . Perturbative, large disorder depending on ω and not just on the potential. **Not known if $L > 0$ for all nonzero disorders**, but it is conjectured to be so.

In the **PDE setting** on \mathbb{Z}^2 Bourgain-Goldstein-S, Acta, 2002 use a similar approach, but the technique is different due to absence of transfer matrix formalism. **Again perturbative.**

The AL strategy

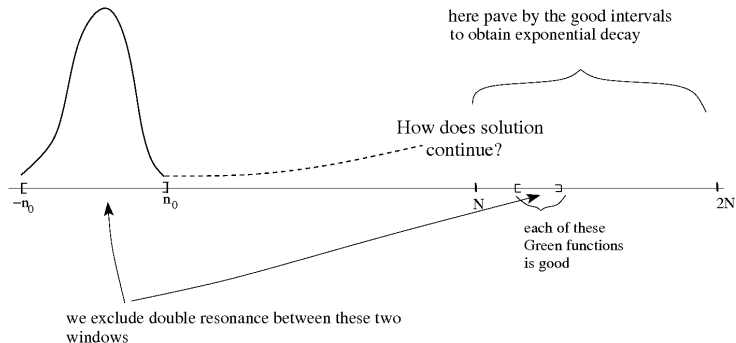


Figure: The three main steps: resonance window about the origin, exclusion of double resonances, paving and resolvent identity

Semi-algebraic sets

Definition

The class of **semi-algebraic subsets** of \mathbb{R}^n is the smallest collection of subsets containing all $\{x \in \mathbb{R}^n : P(x) > 0\}$, where $P \in \mathbb{R}[X_1, \dots, X_n]$, and which is stable under finite intersection, finite union and under taking the complement.

Lemma

The class of semi-algebraic sets is given by expressions of the form

$$\bigcup_{j=1}^J \bigcap_{\ell \in \mathcal{L}_j} \{P_\ell \sigma_{j\ell} 0\} \quad (3)$$

where $P_\ell \in \{P_1, \dots, P_s\} \subset \mathbb{R}[X_1, \dots, X_n]$, and $\mathcal{L}_j \subset \{1, 2, \dots, s\}$, with $\sigma_{j\ell} \in \{<, \leq, >, \geq, =\}$.

s is called **combinatorial complexity**, $\max_\ell \deg P_\ell$ is the **algebraic complexity**.

Properties of semi-algebraic sets

Basic questions: How “complicated” can a semi-algebraic set be? This can be understood **qualitatively** and **quantitatively**. The latter implying bounds in terms of the **number and degrees** of the defining polynomials of, **connected components, Betti numbers** ... Here is an example of a **qualitative theorem**:

Theorem (Seidenberg-Tarski)

The projection of a semi-algebraic set onto any affine subspace is again semi-algebraic in that subspace. I.e., $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ preserves semi-algebraic sets.

The **quantitative version** took longer to develop.

Definition

Let S be semi-algebraic as in (3). Suppose $\deg(P_\ell) \leq d$ for each polynomial. Then we say that $\deg(S) \leq sd$.

A quantitative Seidenberg-Tarski theorem

Note that this definition **does not involve** J . This is surprising at first sight since the total number of words is 2^s .

For our applications crucial to have **polynomial bounds in degree**; we will absorb **polynomial losses** with **exponential gains** from **measure estimates**.

However: It suffices in our applications to work with more elementary **basic s -a sets**, for which $J = 1$ in (3) (thus, they are pure intersections).

Theorem (Basu-Pollack-Roy 1996)

Let \mathcal{S} be semi-algebraic as in (3). Then $\pi(\mathcal{S})$ is as in (3) with

$$\deg(\pi\mathcal{S}) \leq C(\deg \mathcal{S})^C$$

where $C = C(n)$.

In other words, we have **polynomial growth**. A great resource in this area is the 2006 Springer book by these authors: *Algorithms in real algebraic geometry* (pdf on Saugata Basu's website).

Polynomial bounds on the components

Suppose we are given $f, g \in \mathbb{R}[X, Y]$ without common non-constant factor. *How many connected components does the following set have?*

$$\mathcal{S} := \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \geq 0\} \cap \{(x, y) \in \mathbb{R}^2 \mid g(x, y) \geq 0\}$$

It is at most $1 + \#\{f = 0, g = 0\} \leq 1 + \deg(f) \deg(g)$ by Bezout's theorem. In general, we have the following result about semi-algebraic sets.

Theorem

The number of connected components of $\mathcal{S} \subset \mathbb{R}^n$ semi-algebraic is at most $C(n)(\deg(\mathcal{S}))^n$.

Special case of a stronger theorem, see book by Basu et al. For *basic semi-algebraic* sets goes back to Oleinik, Petrovsky (1950s), Thom, Milnor 1964.

Illustration of Bezout

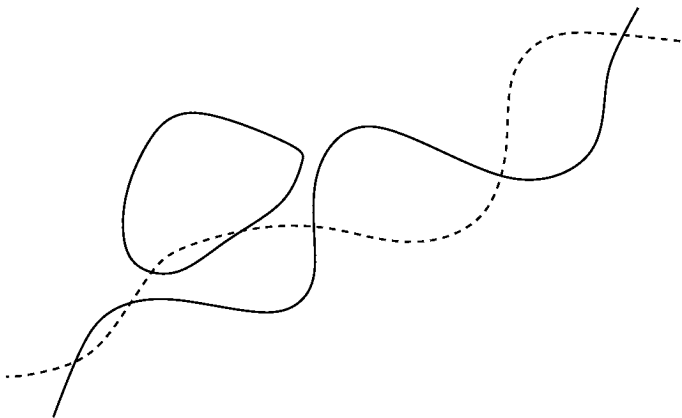


Figure: Applying Bezout's theorem

Tightness of the bounds on the components

Let

$$P_i = L_{i,1}^2 \cdot \dots \cdot L_{i,d}^2 - \varepsilon, \quad 1 \leq i \leq s$$

where $L_{i,j}$ are generic linear polynomials in \mathbb{R}^n , $\varepsilon > 0$ small. The set \mathcal{S} has $\simeq (sd)^k$ connected components.

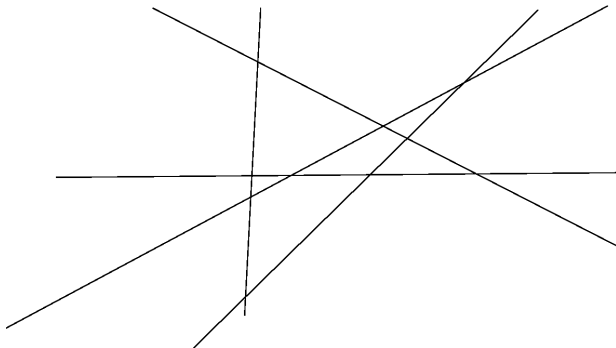


Figure: sd planes in generic position

Relevance to AL

Consider a condition of the form (for the shift dynamics on the circle)

$$n^{-1} \log \|M_n(\omega, \theta, E)\| < L_n(\omega, E) - n^{-\sigma} \quad (4)$$

How to describe this in terms of semi-algebraic sets? First, we replace matrix norm on the left-hand side by the Hilbert-Schmidt norm (or work with det on the left):

$$f_1^2 + f_2^2 + f_3^2 + f_4^2 < \exp(2n(L_n(\omega, E) - n^{-\sigma})) \quad (5)$$

where the f_j are determinants: $f_1 = \det(H_{[1,n]}(\omega, \theta) - E)$ etc. Second, let $V = V(\theta)$ be a real-valued trigonometric polynomial of degree ν . Then we can write $\det(H_\Lambda(\omega, \theta) - E)$ as polynomial in $e(\theta), e(-\theta)$ of degree at most $|\Lambda|\nu$. So **relative to θ** we have a semi-algebraic description of (4) in terms of a polynomial in $(x_1, x_2) \in \mathbb{R}^2$ of degree $O(n)$ intersected with $\{x_1^2 + x_2^2 = 1\}$. So this was easy. More relevant to us, however, is a semi-algebraic description of (4) **in all three variables (ω, θ, E)** .

Relevance to AL

A small issue here is that ω, E sit in the Lyapunov exponent (average over θ of the left-hand side) on the right-hand side. To overcome it, we may resort to averaging over a long shift orbit.

Lemma

Suppose $\|k\omega\| > c_0|k|^{-A}$, $\forall 0 < |k| < n^{2A}$. Then for all $J > n^{2A}$ we have

$$\frac{1}{J} \sum_{j=1}^J \log \|M_n(\omega, \theta + j\omega, E)\| = L_n(\omega, E) + O(n^{-1}) \quad (6)$$

uniformly in θ, E (the latter in bounded set).

The $O(\cdot)$ depends on the c_0 in the Diophantine condition. Note that the latter is in a finite range of k . The lemma is again proved by Fourier representation, but we need **more regularity** than just subharmonicity as in the LDT: use derivative bound in θ , which is C^n . So truncate Fourier series at exponentially large modes.

Relevance to AL

Using this, we see that (4), or more precisely (5) **together with** the Diophantine condition in the Lemma, is semi-algebraic in all three variables (ω, θ, E) of degree $O(n^C)$.

Proof of lemma: Let $u(\theta) = n^{-1} \log \|M_n(\omega, \theta, E)\|$. Then using $\|u'\|_\infty \leq C^n$ we conclude that

$$\sum_{|k| > K} |\hat{u}(k)| \leq K^{-\frac{1}{2}} \|u'\|_2 \leq 2^{-n}, \quad K = (2C)^{2n}$$

Thus, using also $|\hat{u}(k)| \lesssim |k|^{-1}$, we have

$$\begin{aligned} \left| \frac{1}{J} \sum_{j=1}^J u(\theta + j\omega) - L_n \right| &\lesssim \sum_{k=1}^K |\hat{u}(k)| (1 + J\|k\omega\|)^{-1} + \sum_{|k| > K} |\hat{u}(k)| \\ &\lesssim J^{-1/A} \log K + 2^{-n} \lesssim n^{-1} \end{aligned}$$

The implicit constant depends on c_0, V, E . □

The first step in AL

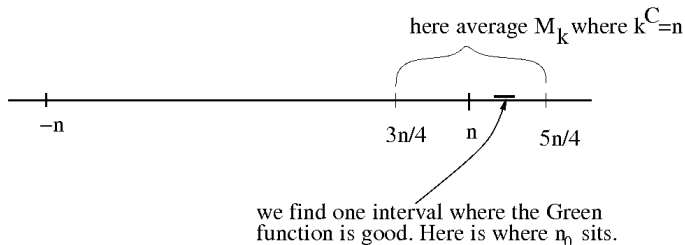


Figure: Finding the resonance window $[-n_0, n_0]$

Using lemma, we average $k^{-1} \log \|M_k(\omega, j\omega, E)\|$,
 $3n/4 \leq j \leq 5n/4$. Together with uniform upper bound this produces some (or many) intervals $[a, b]$, $b - a \in \{k, k - 1, k - 2\}$ for which Green function is good. Use Poisson formula to say that $|\psi(n_0)| \leq \exp(-n_0^\delta)$. Thus, we obtain the desired **“almost eigenvalue”** $\text{dist}(E, \text{spec}(H_{[-n_0, n_0]}(\omega, 0))) \leq \exp(-n_0^\delta)$.

Step 2: double resonances

We now want to show that we can completely cover $[N, 2N]$ by **good Green functions** $G_\Lambda(\omega, 0, E)$. It will turn out that reasonable choices of scales are $N = n_0^C$, $|\Lambda| \simeq n_0^{\delta/2}$. This requires **removing a small set of ω** .

Lemma

Let $n \gg 1$, and define $S_n \subset \mathbb{T}^2$ set of all (ω, θ) so that

(i) $\|k\omega\| > c_0|k|^{-A}$, $\forall 0 < |k| < n^{2A}$

(ii) for some $n_0 \simeq n$, $E \in I$ (some bounded interval)

$$\text{dist}(E, \text{spec}(H_{[-n_0, n_0]}(\omega, 0))) \leq \exp(-n_0^\delta)$$

(iii) for some $k \simeq n_0^{\delta/2}$

$$k^{-1} \log \|M_k(\omega, \theta, E)\| < L_k(\omega, E) - k^{-\sigma} \quad (\star)$$

Then $|S_n| < e^{-\frac{1}{2}k^\sigma}$ and every horizontal section of S_n is covered by at most n^B intervals.

Step 2 continued

For the proof, note that by (ii) $|E - E_j| < \exp(-n_0^\delta)$ for some eval E_j of $H_{[-n_0, n_0]}(\omega, 0)$. Then we may replace E in (iii) with E_j just by differentiation. Indeed, we pay $|E - E_j|C^k$ for this which is $\exp(O(n_0^{\delta/2})) \exp(-n_0^\delta) \ll k^{-\sigma}$ since n_0 large. Now apply LDT to say that probability in θ for (iii) to happen at E_j is at most $\exp(-k^\sigma)$. We need to take the union of $O(n^C)$ such sets, which can be absorbed.

For the complexity bound: we already know that (iii) is semi-algebraic in (ω, θ, E) of a degree at most a power of n . Similar for (ii): write the condition as

$$\|(H_{[-n_0, n_0]}(\omega, 0) - E)^{-1}\| > \exp(n_0^\delta)$$

Express the inverse by Cramer's rule, recast as condition in determinants. So semi-algebraic again to degree some power of n . Now project out E . □

Removing double resonances

We need to **remove the set**

$$\{\omega \in \mathbb{T} : (\omega, l\omega) \in S_n \pmod{\mathbb{Z}^2} \text{ for some } N \leq l \leq 2N\}$$

l gives position of small interval to the right of $[-n_0, n_0]$.

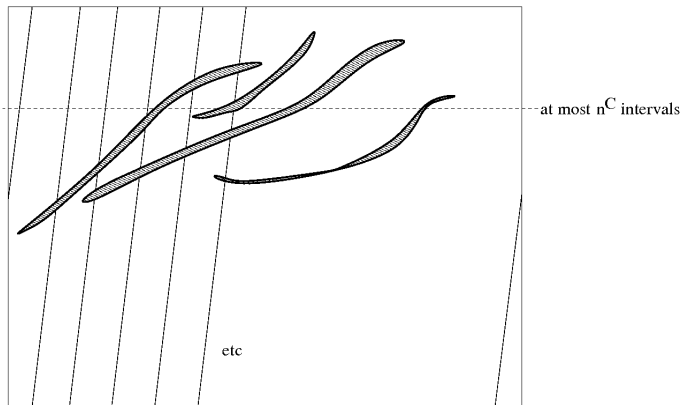


Figure: Eliminating “bad” ω

The lemma on steep lines

Lemma

Let $S \subset \mathbb{T}^2$ be a measurable set with the following properties:

- For each $\theta \in \mathbb{T}$ the horizontal section S_θ is covered by at most M intervals.
- $|S| < N^{-3}$, where $N > M$

Then

$$\begin{aligned} & |\{\omega \in \mathbb{T} : (\omega, \ell\omega) \in S \pmod{\mathbb{Z}^2} \text{ for some } N \leq \ell \leq 2N\}| \\ & \lesssim N^{\frac{3}{2}} |S|^{\frac{1}{2}} + MN^{-1} \end{aligned}$$

We apply this to S_n with $M = n^B$, $N = n^{2B}$, $|S| < \exp(-n^\epsilon)$. So we eliminate a set \mathcal{B}_n of bad ω of measure $|\mathcal{B}_n| \lesssim n^{-2}$, say. This is summable, and we can apply Borel-Cantelli to conclude that we just need to remove a measure zero set.

Final step: resolvent identity

What he have so far is enough to conclude that $\text{spec}(H(\omega, 0))$ consists entirely of eigenfunctions with decay like $|\psi(n)| \lesssim \exp(-|n|^\epsilon)$. For this note that the converse of (\star) in (iii) of the definition of the set S_n together with uniform upper bounds implies that **one of the entries of $M_k(\omega, \ell\omega, E)$** is good, and since these are determinants, we conclude that the **Green function on the associated interval is good**. Then one applies “Poisson formula” on that interval, together with the at most polynomial growth of ψ . But to obtain *exponential decay* need to do one more thing, namely pave the interval $[N, 2N]$ with all the good intervals we have so far (see picture on AL strategy above).

Resolvent identity: A, B Hermitian matrices, then for $\text{Im } z \neq 0$

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1},$$

Apply this to $\Lambda' \subset \Lambda \subset \mathbb{Z}$. With $\chi_{\partial\Lambda'}$ the hopping operator

$$(H_\Lambda - E)^{-1}R_{\Lambda'} = (H_{\Lambda'} - E)^{-1}R_{\Lambda'} + (H_\Lambda - E)^{-1}\chi_{\partial\Lambda'}(H_{\Lambda'} - E)^{-1}R_{\Lambda'}$$

Here $R_{\Lambda'}$ is the restriction to Λ' .

Applying the resolvent identity

To be specific, let $n \in \Lambda = [N, 2N]$ and put short interval $\Lambda' \subset \Lambda$ around this point. Then (suppressing the argument $(\omega, 0)$)

$$\begin{aligned}(H_\Lambda - E)^{-1}(n, m) &= (H_{\Lambda'} - E)^{-1}(n, m)\chi_{\Lambda'}(m) \\ &\quad + (H_\Lambda - E)^{-1}(m, b+1)(H_{\Lambda'} - E)^{-1}(n, b) \\ &\quad + (H_\Lambda - E)^{-1}(m, a-1)(H_{\Lambda'} - E)^{-1}(n, a)\end{aligned}\tag{7}$$

where $\Lambda' = [a, b]$. If, say, a coincides with the left endpoint of Λ , then the last term is not there. Now the Green function on each Λ' is good. So this means that with some small $0 < b < 1$

$$\|(H_\Lambda - E)^{-1}\| \lesssim \exp(N^b)$$

Moreover, by **iterating** (7) we obtain exponential decay of the Green function on Λ as desired.

The resolvent expansion

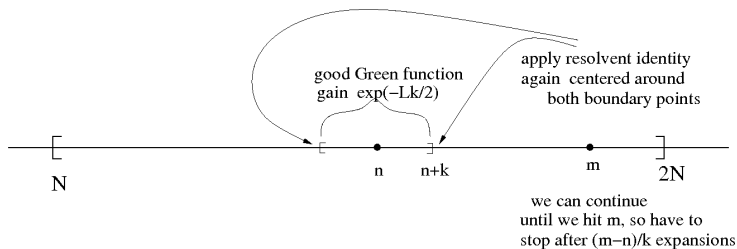


Figure: Applying the resolvent identity iteratively

So we gain $(\exp(-Lk/2))^{|m-n|/k} = \exp(-L|m-n|/2)$ decay.

The proof of the lemma with the steep lines

We first write

$$\begin{aligned} & |\{\omega \in \mathbb{T} : (\omega, l\omega) \in S \pmod{\mathbb{Z}^2} \text{ for some } N \leq \ell \leq 2N\}| \\ & \leq \sum_{\ell \simeq N} \int_{\mathbb{T}} \chi_S(\omega, l\omega) d\omega = \\ & = \sum_{\ell \simeq N} \sum_{j=0}^{\ell-1} \frac{1}{\ell} \int_{\frac{j}{\ell}}^{\frac{j+1}{\ell}} \chi_S((j+\theta)/\ell, \theta) d\theta \end{aligned} \quad (8)$$

Let S_θ denote the horizontal section. Then $|S_\theta| > \gamma$ happens at most with probability $|S|\gamma^{-1}$. So the contribution to (8) is at most

$$\sum_{\ell \simeq N} \sum_{j=0}^{\ell-1} \frac{1}{\ell} \int_{\frac{j}{\ell}}^{\frac{j+1}{\ell}} \chi_{[|S_\theta| > \gamma]} d\theta \lesssim \gamma^{-1} |S|. \quad (9)$$

Next, $|\{\theta \in \mathbb{T} : \inf_{1 \leq k \leq 2N} \|k\theta\| \leq 4N^2\gamma\}| \lesssim N^3\gamma$. So it remains to consider to contribution to (8) of all other θ . We assume $N^3\gamma \ll 1$.

The proof of the lemma with the steep lines

By assumption, $S_\theta \subset \bigcup_{m=1}^M J_m$, covering by intervals, $|J_m| < \gamma$. If

$$\frac{j+\theta}{\ell}, \frac{j'+\theta}{\ell'} \in J_m, \quad (j, \ell) \neq (j', \ell')$$

then we have $|(\ell - \ell')\theta + (j\ell' - j'\ell)| < 4N^2\gamma$. If $\ell = \ell'$ then get the contradiction $1 < 4N^2\gamma$. Otherwise, $\|(\ell - \ell')\theta\| < 4N^2\gamma$ which is the case treated before.

Conclusion: At most **one pair** (j, ℓ) contributes to (8) for each interval J_m , resulting in a bound of M/N .

Total estimate:

$$\gamma^{-1}|S| + N^3\gamma + M/N \lesssim N^{\frac{3}{2}}|S|^{\frac{1}{2}} + M/N$$

as claimed. □

Further remarks on AL

This scheme has been applied in various contexts, still based on **transfer matrix formalism**. In each case we need to resolve two main issues:

- Obtain **large deviation measure estimates**. Ideally, this is done **nonperturbatively**, i.e., **free of any assumption on large potentials, or even assuming positive Lyapunov exponents**. But in more difficult situations, such as for the **skew shift**, or for general Gevrey class potentials, we need an inductive scheme, based on **avalanche principle**. In Lecture 3 we will discuss other analytical techniques (**Cartan estimates**) which are helpful for LDTs.
- **Elimination of double resonances** based on *semi-algebraic complexity estimates* as above. This also goes by the name of *elimination of the energy* (note: LDT is for a **fixed energy**). This is often delicate, requiring ad hoc arguments. For example, for **two or more frequencies** the lemma on lines does not directly apply. More semi-algebraic machinery is needed, e.g. **Gromov-Yomdin triangulation/parametrization theorem**.

But what do we do if we consider **quasi-periodic operators on higher-dimensional lattices**? For example

$$(H_{x_1, x_2} \psi)(n_1, n_2) := -(\Delta_{\mathbb{Z}^2} \psi)(n_1, n_2) + \lambda V(x_1 + n_1 \omega_1, x_2 + n_2 \omega_2) \psi(n_1, n_2) \quad (10)$$

where $V : \mathbb{T}^2 \rightarrow \mathbb{R}$ is a non-constant trigonometric polynomial *on any line*, and $\underline{\omega} := (\omega_1, \omega_2)$ generic in some sense. We **lose the notion of a Lyapunov exponent**, so we instead assume that λ is large (we will need to remove a set of $\underline{\omega}$ which is small but of positive measure depending on the size of this λ).

Higher-dimensional case

Our proof of localization for (10) is in the spirit of KAM.

- To approach AL for (10) we work **directly** with the resolvent identity. We also need to allow for **many resonance sites**, but of course this is delicate since we do not want an entire **chain** of resonant cubes at the smaller scale. So in our first work on this problem (Bourgain-Goldstein-S, Acta 2002) we obtained that the number of such bad cubes is **sub-linear**.
- Semi-algebraic/arithmetical arguments used to eliminate ω s.t. number of resonant sites is too large for **fixed energy**.
- To apply the resolvent identity, the issue is really to obtain an **exponential measure estimate at the next larger scale at fixed energy**. For this we develop a so-called **Cartan estimate for matrix valued functions**. Here it is essential to keep the number of resonant sites sub-linear, too, so that we can pack all **resonance sites into a black box** which then determine the size of the Riesz mass of the subharmonic functions along each coordinate direction. Will become clearer in Lecture 3.

Positivity of the Lyapunov exponent

We now return to the 1-dim equation, and introduce M. Herman's subharmonicity method for proving positivity of $L(\omega, E)$ for Harper:

$$(H_\lambda(x)\psi)_n = \psi_{n+1} + \psi_{n-1} + 2\lambda \cos(2\pi(n\omega + x))\psi_n$$

We proceed as follows, with $z = e(x)$, $w = e(\omega)$:

$$\begin{aligned} \int_0^1 \log \|M_N(e(x), E)\| dx &= \int_0^1 \log \left\| \prod_{j=N}^1 \begin{pmatrix} \lambda(zw^j + z^{-1}\bar{w}^j) - E & -1 \\ 1 & 0 \end{pmatrix} \right\| dx \\ &= \int_0^1 \log \left\| \prod_{j=N}^1 \begin{pmatrix} \lambda(z^2w^j + \bar{w}^j) - zE & -z \\ z & 0 \end{pmatrix} \right\| dx \\ &\geq \log \left\| \prod_{j=N}^1 \begin{pmatrix} \lambda\bar{w}^j & 0 \\ 0 & 0 \end{pmatrix} \right\| = N \log |\lambda| \end{aligned}$$

So $L(\omega, E) \geq \log |\lambda| > 0$ if $|\lambda| > 1$.

Positivity of the Lyapunov exponent

This is **uniform** in E, ω . Known to be **optimal**: for $|\lambda| \leq 1$ we have $L(\omega, E) = 0$ on the spectrum. For $|\lambda| > 1$ we have $L(\omega, E) = \log |\lambda| > 0$ on the spectrum.

This method applies to other trigonometric polynomials, but the bound is in terms of the **coefficient of the largest power**.

For **analytic potentials** we have alternative arguments, of course not as precise (Sorets-Spencer). But they give positivity for large disorders **nonperturbatively** – which means depending on the potential, but not the rotation number ω .

Duarte-Klein 2012 introduced a new method for establishing lower bounds for Lyapunov exponents nonperturbatively, also for higher-dimensional co-cycles. It is a variant of Sorets-Spencer/Bourgain approach which uses **harmonic measure**.

Positivity of the Lyapunov exponent

The Duarte-Klein method hinges on the following analytical fact: a radial function is sub-harmonic if and only if it is convex in $\log r$.

This means that for all $r_1 < r < r_2$ we have

$$u(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} u(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} u(r_2) \quad (11)$$

This is not so surprising if we recall that radial harmonic functions are exactly affine in $\log r$.

If $u(z)$ is any subharmonic function, then the means

$$\zeta \mapsto \int_0^1 u(z_0 + \zeta e(x)) dx$$

form a radial sub-harmonic function and therefore these means satisfy (11). The logic is now that we obtain a lower bound for $r_1 = 1$ if we have one for $1 < r$ and an upper bound for $r_2 > r$. The latter are gotten by exploiting the benefits of complexification in x .

Aubry duality

Consider Harper's (or almost Mathieu) as above. Assume $|\lambda| > 1$ so that we have AL. Let ψ be an exponentially localized eigenfunction, $\mathcal{H}_\lambda(0)\psi = E\psi$. Define **analytic function** by means of Fourier transform

$$F(\theta) = \sum_n \psi_n e(n\theta)$$

and set $\varphi_k = F(x + k\omega) \forall k \in \mathbb{Z}$. Then

$$\begin{aligned}\varphi_{k+1} + \varphi_{k-1} &= \sum_n \psi_n e(n(x + k\omega)) 2 \cos(2\pi n\omega) \\ &= E\varphi_k - \frac{1}{\lambda} \sum_n (\psi_{n+1} + \psi_{n-1}) e(n(x + k\omega)) \\ &= E\varphi_k - \frac{2}{\lambda} \cos(2\pi(x + k\omega))\varphi_k\end{aligned}$$

Conclusion: $\mathcal{H}_{\frac{1}{\lambda}}(x)\varphi = E\varphi$. Since $\varphi \in \ell^\infty$ we see that $\mathcal{H}_{\frac{1}{\lambda}}(x)$ has **purely continuous spectrum** for $|\lambda| > 1$.

Aubry duality

Suppose we apply the Fourier transform to a more general operator

$$(\mathcal{H}_\lambda(x)\psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda v(x + n\omega)\psi_n \quad (12)$$

with v analytic real-valued on the circle. Assume we are in regime of AL: $\mathcal{H}_\lambda(0)\psi = E\psi$. Dualizing yields a **long-range** operator

$$(\tilde{\mathcal{H}}_\lambda(x)\varphi)_k = \frac{1}{\lambda} \sum_\ell \hat{v}(\ell)\varphi_{k+\ell} + 2 \cos(2\pi(k\omega + x))\varphi_k = E\varphi_k \quad (13)$$

Laplacian replaced by **Töplitz operator**.

LOGIC: If we prove AL for (13) and λ **small**, then we have shown that (12) has continuous spectrum for **small disorder**. Also note that for **multi-frequency shift dynamics** Aubry duality establishes a correspondence between an operator on the integer line \mathbb{Z} with one on a **higher-dimensional lattice \mathbb{Z}^d** .

AL for long-range difference operators

The long-range operators *do not fall into the frame work of the transfer matrix* formalism developed here. Other approach needed to establish Anderson localization. In Lectures 3 and 4 we will describe in detail a method based on **resolvent expansions/matrix-valued Cartan theorem**. These give **exponential measure estimates** for the event that a Green function at fixed energy is bad (or more precisely: **resonant**). But these methods are **inductive in the scale** and therefore **perturbative** requiring the removal of sets of positive but small measure in frequency and phase.

- In **contrast** to this, one has this nonperturbative result.

Theorem (Bourgain-Jitomirskaya 2002)

For $|\lambda| < \lambda_0(v)$ any Diophantine ω , a.e. x the operator (12) has absolutely continuous spectrum.

Proof is based on a careful analysis of the determinant in the denominator of the Green function (random walk expansion). The strongest results in this setting were obtained in a remarkable paper by **Avila, Jitomirskaya**, “*Almost localization and almost reducibility*”, Journal EMS 12 (2010), 93–131. Their approach goes via conjugation of the transfer-matrix cocycle with fixed E to a constant co-cycle (this is so called reducibility). As a striking application they establish the optimal $1/2$ -Hölder continuity of the Lyapunov exponent $L(E)$.

Summary of Lecture 2

- Anderson localization is a **game of resonances**.
- LDT is for **fixed energy**. Tells us that with very high probability (depending on scales) any **given Green function is good**.
- For AL we need to handle **double resonances**. *Eliminate the energy*.
- In the random case, any two such resonances are *independent events*. So we get to square the probability - energy frozen from first box.
- In the case of deterministic dynamics, we bring some degree of independence (relative to the parameter of the dynamics, such as the frequency ω) back into the picture by means of two ingredients (i) control the **complexity in the parameter** via semi-algebraic considerations (ii) **separation of boxes ensures sufficient decoupling** (this is why we want **steep lines** in the lemma).
- Analyticity of the potential (or something “close” – Gevrey)

Summary of Lecture 2

- AL for **long-range difference operators** can also be obtained, but **not** by the transfer matrix formalism. One method which we will encounter later is based on the matrix-valued Cartan theorem and the resolvent expansion. The crux there is to control the number of **bad or resonant** sites at the smaller scale, which however is not a problem for difference operators on the line \mathbb{Z} . It becomes much more delicate, though, for higher-dimensional lattices \mathbb{Z}^d .
- Aubry duality is basically conjugation by the Fourier transform and allows one to switch between small and large disorders. For the Harper operator this is especially clean, as the class of these operators is self-dual.
- Using Aubry duality we may establish (absolute) continuity of the spectrum for small disorder.
- Important topic we did not develop here: Floquet theory for small disorders and quasi-periodic potentials. Closely related to KAM. **Dinaburg-Sinai, Eliasson, Krikorian, Avila etc.**

See Lecture 1 and the Introduction.