

Subharmonic techniques in multiscale analysis: Lecture 1

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Introduction

Consider an operator $H(z)$ depending on a complex parameter $z \in \Omega \subset \mathbb{C}$. For example,

$$(H(z)\psi)_n = \psi_{n+1} + \psi_{n-1} - v_n(z)\psi_n, \quad n \in \mathbb{Z}$$

where $\{v_n(z)\}_{n \in \mathbb{Z}}$ collection of analytic functions such as

$$v_n(z) := F(z \cdot \omega^n), \quad |\omega| = 1,$$

F analytic near $S^1 := \{|z| = 1\}$. Restricting to finite volume $[-N, N]$ with Dirichlet conditions, we obtain a matrix $H_N(z)$. Assume $H_N(z)$ self-adjoint on S^1 .

It is often important to study the **resolvent**

$$G_N(E; z) := (H_N(z) - E)^{-1}$$

on S^1 .

Introduction

By Cramer's rule, we are lead to consider the vanishing of the analytic function

$$f(z) := \det(H_N(z) - E)$$

Basic question: What is the measure of $z \in S^1$ for which $|f(z)|$ is **close to zero**?

We will need some type of *transversality condition* to ensure *nondegeneracy* of $H_N(z) - E$. This is the same as asking when $u(z) = \log |f(z)|$ is *very large and negative*.

The function $u(z)$ is an example of a **subharmonic function**, and classical potential theory provides tools for that purpose.

A subharmonic function is one for which the Laplacian $\Delta u = \mu$ is a non-negative measure. Two important (and elementary) tools:

- **Riesz representation** of subharmonic functions
- **Cartan's estimate** on the size of level sets of subharmonic functions near $-\infty$.

Subharmonic functions 1

Definition

We say $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is **subharmonic** provided

- **upper semi-continuity:**

$$u(z) \geq \limsup_{\zeta \rightarrow z} u(\zeta), \quad \zeta \neq z$$

- **sub mean-value property:** for all $z \in \Omega$ there exists $r_0(z) > 0$

$$u(z) \leq \int_0^1 u(z + re(\theta)) d\theta, \quad \forall r < r_0(z)$$

and $e(\theta) := e^{2\pi i\theta}$.

Examples: $\log |z - z_0|$, $\log |f(z)|$ for any analytic $f(z)$, **logarithmic potential** $\int \log |z - \zeta| \mu(d\zeta)$ for measure $\mu \geq 0$; Fatou gives upper s-c, e.g. for $\mu = \sum 2^{-n} \delta_{2^{-n}}$ the potential at $z = 0$ not continuous, only upper s-c. However: in applications typically have continuity.

Subharmonic functions 2

About **upper-semicontinuous** functions: sub-level sets $\{u < \lambda\}$ are **open**, they attain **supremum on compact sets**.

Some properties of subharmonic functions (s-h) $u(z)$:

- φ increasing, convex. Then $\varphi(u(z))$ is s-h. So $e^u, |f|^\lambda$ are s-h where f analytic, $\lambda > 0$.
- u_α s-h collection, then $\sup_\alpha u_\alpha$ is s-h provided it is upper s-c.
- limit of decreasing or uniformly convergent sequence of s-h functions is s-h
- finite sums of s-h functions are s-h
- **maximum principle**: if u has a **local maximum**, then **constant**. Indeed, if $M = u(z_0)$ local max, then we see that $u = M$ locally near z_0 by sub-mv and upper s-c property. So $\{u = M\}$ open. Closed by upper s-c.
- Averages

$$N(r, z, u) := \int_0^1 u(z + re(\theta)) d\theta, \quad r < \text{dist}(z, \partial\Omega)$$

increasing in r , $\lim_{r \rightarrow 0} N(r, z, u) = u(z)$. **Convex in $\log r$.**

Subharmonic functions 3

Every s-h function is decreasing limit of smooth s-h functions:

$\varphi \geq 0$ radial smooth compactly supported bump-function

$$f_\varepsilon(z) := \int u_\varepsilon(z-w)\varphi_\varepsilon(w) dw, \quad u_\varepsilon := \max(u, -\varepsilon^{-2})$$

$$\varphi_\varepsilon(w) := \varepsilon^{-2}\varphi(w/\varepsilon), \quad 0 < \varepsilon < 1$$

with $2\pi \int_0^\infty s\varphi(s) ds = 1$. Then $u_\varepsilon, f_\varepsilon$ are s-h, f_ε smooth. Further, we have

$$f_\varepsilon(z) = 2\pi \int_0^\infty N(\varepsilon s, z, u_\varepsilon) s\varphi(s) ds \searrow u(z)$$

since

$$u(z) \leq N(\varepsilon s, z, u_\varepsilon) \leq N(\varepsilon s, z, u_{\varepsilon'}) \leq N(\varepsilon' s, z, u_{\varepsilon'})$$

for $\varepsilon < \varepsilon'$. And $N(\varepsilon s, z, u_\varepsilon) \leq N(\varepsilon' s, z, u_\varepsilon) \searrow N(\varepsilon' s, z, u)$ as $\varepsilon \rightarrow 0$ by monotone convergence; then $N(\varepsilon' s, z, u) \searrow u(z)$ as $\varepsilon' \rightarrow 0$. Apply monotone convergence again.

Subharmonic functions 4

Theorem: $u \in C^2(\Omega)$ s-h if and only if $\Delta u \geq 0$ in Ω

Use Green's formula:

$$\int_D (v\Delta u - u\Delta v) dx = \int_{\partial D} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\sigma$$

Apply to u above and $v(w) = \frac{1}{2\pi} \log \frac{R}{|z-w|}$, $D = D(z, R)$. Then $\Delta v = -\delta_w$ and

$$u(z) + \frac{1}{2\pi} \int_{|z-w| < R} \log \frac{R}{|z-w|} \Delta u(w) m(dw) = N(R, z, u)$$

If $\Delta u(z) < 0$, then get a contradiction for $R > 0$ small.

S-h functions remain s-h under conformal change of variables:

$$\Delta(u \circ F) = |F'|^2 (\Delta u) \circ F$$

Subharmonic functions 5

The distributional Laplacian of any s-h function is a positive Borel measure

First, if u is s-h, then it is in fact a distribution. We need to check that $\int u\varphi$ is finite (thus, not $-\infty$) for any test function φ . By sub-mv property, if $u = -\infty$ on an open set, then it is in fact constant $= -\infty$ in Ω . So we can assume that $u > -\infty$ on a dense set.

But then $\int_D u > -\infty$ for any (small) disk $D \subset \Omega$: consider $N(r, z, u) \geq u(z) > -\infty$ for a suitably chosen point $z \in \Omega$, and integrate in $r_1 \leq r \leq r_2$ so that this annulus contains the disk. So u is a distribution.

Let $u_\varepsilon \searrow u$ with smooth s-h functions u_ε . Then by monotone conv

$$0 \leq \int \varphi \Delta u_\varepsilon = \int u_\varepsilon \Delta \varphi \rightarrow \int u \Delta \varphi = \langle \Delta u | \varphi \rangle$$

for all test functions $\varphi \geq 0$. *Riesz representation* (Rudin RCA) says that $\frac{1}{2\pi} \Delta u = \mu$ is a *positive Borel measure*, called *Riesz mass of u* . We have $\mu(K) < \infty$ for all compact $K \subset \Omega$.

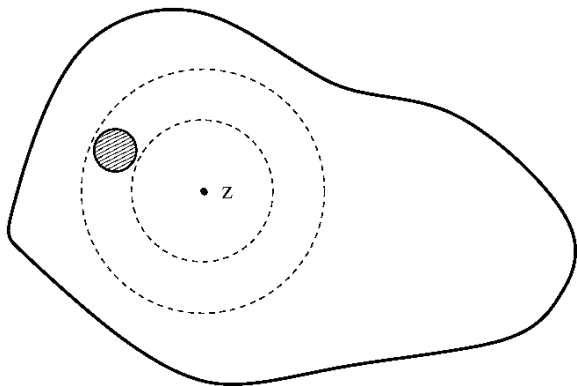


Figure: The geometry in the previous proof

Riesz representation theorem for subharmonic functions

Theorem

Let u be s-h in Ω . Then for any open $G \Subset \Omega$

$$u(z) = \int_G \log |z - \zeta| \mu(d\zeta) + h_G(z) \quad \forall z \in G$$

and h_G harmonic in G .

Let $v_\varepsilon(\zeta) = \frac{1}{2} \log(|z - \zeta|^2 + \varepsilon^2)$. Then $\Delta v_0 = 2\pi\delta_z$, and

$$\langle \mu, \chi v_\varepsilon \rangle = \frac{1}{2\pi} \langle u, v_\varepsilon \Delta \chi + 2\nabla v_\varepsilon \cdot \nabla \chi + \chi \Delta v_\varepsilon \rangle$$

where χ compactly supported in Ω , $\chi = 1$ on G . Limit $\varepsilon \rightarrow 0$

$$\begin{aligned} u(z) &= \int_G \log |z - \zeta| \mu(d\zeta) - \frac{1}{2\pi} \langle u, v \Delta \chi + 2\nabla v \cdot \nabla \chi \rangle \\ &\quad + \int_{\Omega \setminus G} \chi(\zeta) \log |z - \zeta| \mu(d\zeta) \end{aligned}$$

as desired.

Examples of Riesz representation

Let $f(z) = P(z)g(z)$ where f analytic in Ω , polynomial $P(z) = \prod_{j=1}^N (z - z_j)$ with z_j (multiple) zeros, and $g \neq 0$ analytic in Ω . Then

$$\log |f(z)| = \sum_{j=1}^N \log |z - z_j| + \log |g(z)|, \quad \mu = \sum_j \delta_{z_j}$$

Total Riesz mass $\mu(\mathbb{C}) = \deg(P)$. Recall **Jensen's formula**: if $f(z) \neq 0$ then

$$\int_0^1 \log |f(z + re(i\theta))| d\theta - \log |f(z)| = \sum_{j: |z_j - z| < r} \log \frac{r}{|z_j - z|}$$

So we have a zero count:

$$\#\{j : |z - z_j| < r/2\} \leq C \left(\sup_{w \in D(z, r)} \log |f(w)| - \sup_{\zeta \in D(z, r/2)} \log |f(\zeta)| \right)$$

A more quantitative version of Riesz representation

Introducing estimates in the previous argument yields the following more useful quantitative version, see [Lemma 2.2, Goldstein-S., GAFA 2008](#).

Theorem

Let $u : \Omega \rightarrow \mathbb{R}$ be s-h on $\Omega \subset \mathbb{C}$. There exists a positive measure μ on Ω such that for any $\Omega_1 \Subset \Omega$

$$u(z) = \int_{\Omega_1} \log |z - \zeta| \mu(d\zeta) + h(z)$$

where h is harmonic on Ω_1 and μ is unique with this property. Moreover, μ and h satisfy the bounds

$$\mu(\Omega_1) \leq C(\Omega, \Omega_1) (\sup_{\Omega} u - \sup_{\Omega_1} u)$$

$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \leq C(\Omega, \Omega_1, \Omega_2) (\sup_{\Omega} u - \sup_{\Omega_1} u)$$

for any $\Omega_2 \Subset \Omega_1$.

A more quantitative version of Riesz representation

The control of the Riesz mass here follows from the analogue of Jensen's formula:

$$\int_0^1 u(z + re(\theta)) d\theta - u(z) = \int_0^r \frac{\mu(D(z, t))}{t} dt$$

Controlling deviations from above allows to bound the Riesz mass.

In Lecture 3 we will address the question about the **size and complexity** of the set where a subharmonic function can be very **large and negative**. In fact, the fundamental *Cartan estimate* states that the measure of the level set $\{u < -\lambda\}$ is less than $\exp(-\lambda/\mu(\mathbb{C}))$ (up to constants) where $\lambda > 0$.

An application to large deviation estimates

Let V be analytic, real-valued on \mathbb{T}^d , and $Tx := x + \omega$ **irrational rotation**. Consider the Schrödinger equation on \mathbb{Z}

$$(H_x \psi)(n) = -\psi(n+1) - \psi(n-1) + V(T^n x)\psi(n) = E\psi(n) \quad (1)$$

Rewrite as a system (**linear cocycle**):

$$\begin{bmatrix} \psi(n+1) \\ \psi(n) \end{bmatrix} = A(T^n x, E) \begin{bmatrix} \psi(n) \\ \psi(n-1) \end{bmatrix},$$
$$A(x, E) = \begin{bmatrix} V(x) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

Propagator $M_n(x, E) := A(T^n x, E) \dots A(Tx, E)$. **Lyapunov exp.:**

$$L_n(E) := \frac{1}{n} \int_{\mathbb{T}^d} \log \|M_n(x, E)\| dx$$

Subadditivity: $L_n(E) \rightarrow L(E)$ exists. Since $\det A = 1$, one has $L(E) \geq 0$. **Pointwise (Furstenberg-Kesten):** for a.e. x

$$L(E) = \lim_{n \rightarrow \infty} n^{-1} \log \|M_n(x, E)\|$$

LDT for $d = 1$

We now establish a **quantitative version of this convergence** under a **Diophantine condition**: $\|n\omega\| > n^{-1}(\log n)^{-2}$ for all $n \geq n_0(\omega)$.
A.e. $\omega \in \mathbb{T}$ satisfies such a condition.

Theorem

Let $\omega \in \mathbb{T}$ satisfy a Diophantine condition. Then there exists $\sigma > 0$ such that

$$|\{x \in \mathbb{T} : |n^{-1} \log \|M_n(x, E)\| - L_n(E)| > n^{-\frac{1}{4}}\}| < e^{-n^\sigma} \quad (\text{LDT})$$

for all $n \geq n_0(E, V, \omega)$.

On some strip around the real line

$$u(z) := n^{-1} \log \|M_n(z, E)\|$$

subharmonic, ≥ 0 , and 1-periodic, and of size $\lesssim 1$. By RRT,

$$u(x) = \int \log |e(x) - \zeta| \mu(d\zeta) + h(x) \quad \forall x \in \mathbb{R} \quad (2)$$

Plot of log of norm for almost Mathieu

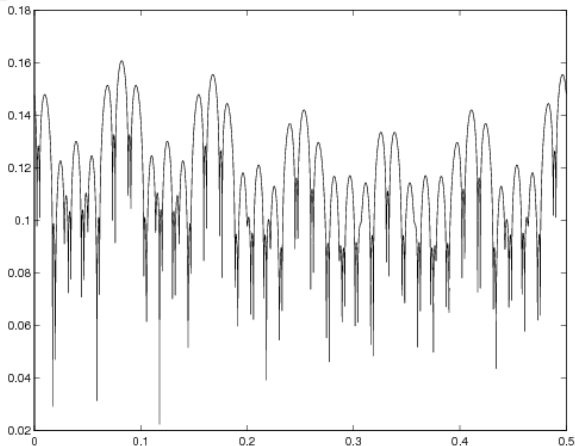


Figure: The graph of $\frac{1}{100} \log \|M_{100}(x)\|$, $\omega = \sqrt{2}$, $\lambda = 2.2$

The almost Mathieu (or Harper) operator is

$$(H_x \psi)(n) = -\psi(n+1) - \psi(n-1) + \lambda \cos(2\pi(x + n\omega))\psi(n)$$

Plot of log of norm

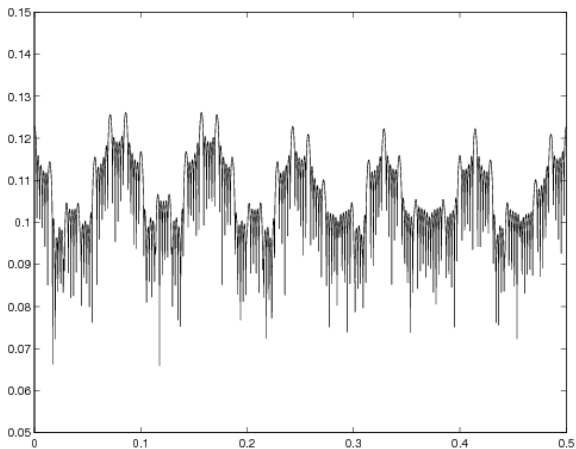


Figure: The graph of $\frac{1}{200} \log \|M_{200}(x)\|$

Plot of log of norm

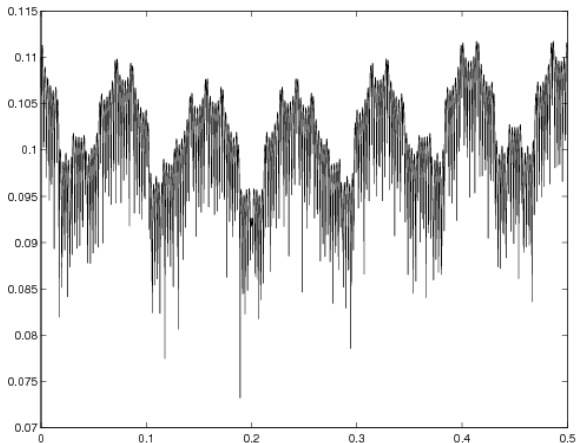


Figure: The graph of $\frac{1}{400} \log \|M_{400}(x)\|$

Proof of LDT

$\mu \geq 0$ and finite on some nbhd of $\{|z| = 1\}$ and h harmonic and 1-periodic on a strip around the real line, and $|h| \lesssim 1$ there. From this alone cannot derive (LDT).

Examples: $P_1(z) = (z - 1)^n$, $P_2(z) = z^n - 1$, generate s-h functions

$$u_1(z) = n^{-1} \log |P_1(z)| = \log |z - 1|, \quad u_2(z) := n^{-1} \log |z^n - 1|$$

of Riesz mass 1, average = 0 over S^1 . **Deviation estimates:**

$$|\{x \in \mathbb{T} : |u_1(e(x))| > n^{-\frac{1}{2}}\}| \sim 1$$

$$|\{x \in \mathbb{T} : |u_2(e(x))| > n^{-\frac{1}{2}}\}| \sim e^{-\sqrt{n}}$$

We want to be closer to the second example, obviously. **Note:** u_2 is $1/n$ periodic, and u is almost invariant under translation by ω :

$$\sup_{x \in \mathbb{T}} |u(x + k\omega) - u(x)| \leq C \frac{k}{n} \quad (3)$$

from the matrix product structure of M_n .

Proof of LDT

Indeed:

$$\begin{aligned}u(x + k\omega) - u(x) &= \frac{1}{n} \log \frac{\|M_n(x + k\omega, E)\|}{\|M_n(x, E)\|} \\ \|M_n(x + k\omega, E)\| &\leq \|A(T^{n+k}x, E) \dots A(T^{n+1}x, E)\| \cdot \\ &\quad \cdot \|M_n(x, E)\| \|A(Tx, E)^{-1} \dots A(T^kx, E)^{-1}\| \\ &\leq C^k \|M_n(x, E)\|\end{aligned}$$

In order to prove the LDT, we need to combine (2) and (3). Use Fourier series, with $K = K(n) \gg n$

$$u(x) = L_n(E) + \sum_{0 < |\nu| \leq K} \hat{u}(\nu) e(\nu x) + \sum_{|\nu| > K} \hat{u}(\nu) e(\nu x) \quad (4)$$

From (2)

$$\hat{u}(\nu) = \int \int_0^1 \log |e(\theta) - \zeta| e(-\nu\theta) d\theta \mu(d\zeta) + \int_0^1 h(\theta) e(-\nu\theta) d\theta$$

Now $\hat{h}(\nu) = O(\nu^{-N})$, whereas the logarithmic integral satisfies

$$\int_0^1 \log |e(\theta) - \zeta| e(-\nu\theta) d\theta = O(\nu^{-1})$$

Since $\mu(\mathbb{C}) \lesssim 1$, we have $|\hat{u}(\nu)| \lesssim |\nu|^{-1}$. Thus,

$$\|u_K\|_2 \lesssim K^{-\frac{1}{2}}, \quad u_K(x) := \sum_{|\nu| > K} \hat{u}(\nu) e(\nu x) \quad (5)$$

We need to use the **properties of the translation by ω** . Average (4) using almost invariance (3):

$$\begin{aligned} u(x) &= \frac{1}{m} \sum_{j=1}^m u(x + j\omega) + O\left(\frac{m}{n}\right) \\ &= L_n(E) + \sum_{0 < |\nu| \leq K} \hat{u}(\nu) \frac{1}{m} \sum_{j=1}^m e(\nu(x + j\omega)) \\ &\quad + \frac{1}{m} \sum_{j=1}^m u_K(x + j\omega) + O\left(\frac{m}{n}\right) \end{aligned} \quad (6)$$

Now

$$\sup_x \left| \frac{1}{m} \sum_{j=1}^m e(\nu(x + j\omega)) \right| \lesssim \min(1, m^{-1} \|\nu\omega\|^{-1}) \quad (7)$$

Inserting this into (6) yields by **Diophantine condition**

$$\begin{aligned} & \sup_x \left| \sum_{0 < |\nu| \leq K} \hat{u}(\nu) \frac{1}{m} \sum_{j=1}^m e(\nu(x + j\omega)) \right| \\ & \lesssim \sum_{0 < |\nu| \leq K} |\nu|^{-1} \min(1, m^{-1} \|\nu\omega\|^{-1}) \\ & \lesssim m^{-1} (\log K)^3 \lesssim n^{-\frac{1}{3} + 3\sigma} \end{aligned} \quad (8)$$

where we set $m = n^{\frac{1}{3}}$, $K = \exp(2n^\sigma)$. To pass to last line, divide into cases $\|\nu\omega\| \leq m^{-1}$, $2^k m^{-1} \leq \|\nu\omega\| \leq 2^{k+1} m^{-1}$.

Sharp LDT

Again from (6) we conclude that for σ small and n large,

$$\begin{aligned} & \left| \left\{ x \in \mathbb{T} : |u(x) - L_n(E)| > n^{-\frac{1}{4}} \right\} \right| \\ & \leq \left| \left\{ x \in \mathbb{T} : \left| \frac{1}{m} \sum_{j=1}^m u_K(x + j\omega) \right| > \frac{1}{2} n^{-\frac{1}{4}} \right\} \right| \lesssim n^{\frac{1}{2}} K^{-1} \lesssim e^{-n^\sigma} \end{aligned}$$

which is the **LDT**. Under our **strong Diophantine condition** one can prove a sharper estimate, see **Goldstein-S., Annals Math, 2001**.

Theorem (Sharper LDT)

For any $\delta > 0$, any n , and $u = n^{-1} \log \|M_n\|$ as above,

$$\left| \left\{ x \in \mathbb{T} : |u(x) - L_n(E)| > \delta \right\} \right| \leq \exp\left(-c\delta^2 n + C(\log n)^A\right).$$

The constants c, C only depend on the size of E , the potential, and ω . A is absolute.

Notice the δ, δ^2 dependence. **Can we do better? Yes if $L > 0$.**

Further remarks on LDT

Under a **weaker Diophantine condition** $\|n\omega\| > n^{-a}$, $a > 1$, the same proof gives this **weak LDT**

$$|\{x \in \mathbb{T} : |u(x) - L_n(E)| > n^{-\sigma}\}| < e^{-n^\sigma}$$

where $\sigma > 0$.

- What is the **LDT good for**? When do we **need the sharp LDT**?
- How to prove the LDT on **higher-dimensional tori** with the shift dynamics? Or for other underlying dynamics, such as the **skew shift** $T(x, y) = (x + \omega, x + y)$? Note: T^n gives $n^2\omega$. Is there a proof that does not use the Fourier transform?

Main applications are (i) **Anderson Localization for (1)** assuming $L > 0$ (LDT + **elimination of energies via exclusion of DOUBLE RESONANCES**) Bourgain-Goldstein, Annals, 2000. (ii) **Regularity properties of the integrated density of states, distribution and separation of eigenvalues of (1) on finite volume, gaps in the spectrum in infinite volume** Goldstein-S., Annals, 2001, GAFA 2008, Annals 2009.

Avalanche Principle – induction on scales

We know from subadditivity that $L_n(E) \rightarrow L(E)$. What can we say about the **rate**? In other words, **can we control** $L_n(E) - L_{2n}(E)$? It turns out we can, assuming $L(E) > 0$. We need a tool that **produces exponential growth in the norm of a long product of matrices**. This need not be the case: $AA^{-1}AA^{-1}AA^{-1}AA^{-1} \dots$

Proposition (Goldstein-S)

Let $A_1, \dots, A_n \in SL(2, \mathbb{R})$ so that

$$\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n \quad (9)$$

$$\max_{1 \leq j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|] < \frac{1}{2} \log \mu \quad (10)$$

Then

$$\left| \log \|A_n \cdot \dots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C \frac{n}{\mu} \quad (11)$$

Proof of Avalanche Principle

Fix $K \in SL(2, \mathbb{R})$. Polar decomposition: $K = RP$, R rotation, $P > 0$. Eigenvectors of $P = \sqrt{K^*K}$ are \mathbf{u}_K^+ , \mathbf{u}_K^- , respectively. One has $K\mathbf{u}_K^+ = \|K\|\mathbf{v}_K^+$, $K\mathbf{u}_K^- = \|K\|^{-1}\mathbf{v}_K^-$ with unit vectors \mathbf{v}_K^+ , \mathbf{v}_K^- . Given $K, M \in SL(2, \mathbb{R})$, let $b^{(+,+)}(K, M) = \mathbf{v}_K^+ \cdot \mathbf{u}_M^+$, similar for $b^{(+,-)}$, $b^{(-,+)}$, $b^{(-,-)}$. We have

$$\begin{aligned} & \|K\| \|M\| |b^{(+,+)}(K, M)| - \|K\| \|M\|^{-1} \leq \|MK\| \\ & \leq \|K\| \|M\| |b^{(+,+)}(K, M)| + \|K\|^{-1} \|M\| + \|K\| \|M\|^{-1}. \end{aligned}$$

In particular,

$$\begin{aligned} & \frac{\|A_{j+1}A_j\|}{\|A_{j+1}\| \|A_j\|} - \frac{1}{\|A_j\|^2} \leq |b^{(+,+)}(A_j, A_{j+1})| \\ & \leq \frac{\|A_{j+1}A_j\|}{\|A_{j+1}\| \|A_j\|} + \frac{1}{\|A_j\|^2} + \frac{1}{\|A_{j+1}\|^2}. \end{aligned}$$

Binary structure in the proof of the Avalanche Principle

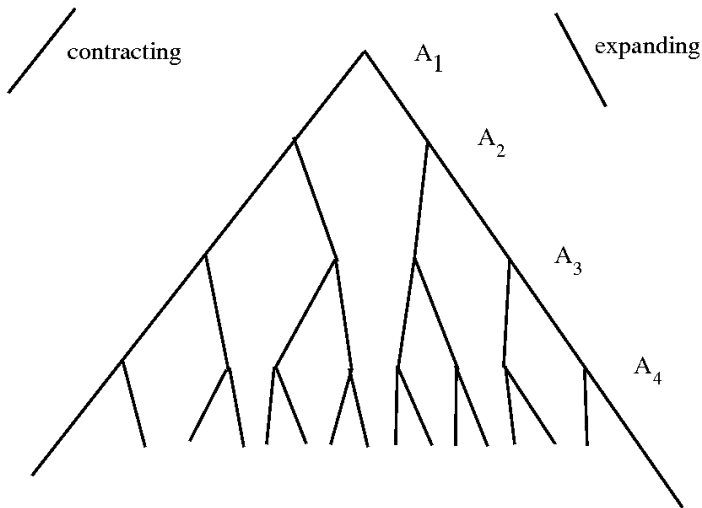


Figure: The expanding, contracting structure in the proof

Proof of Avalanche Principle

In view of our assumptions therefore

$$1 - \frac{\sqrt{\mu}}{\mu^2} \leq |b^{(+,+)}(A_j, A_{j+1})| \frac{\|A_{j+1}\| \|A_j\|}{\|A_{j+1} A_j\|} \leq 1 + \frac{2\sqrt{\mu}}{\mu^2} \quad (12)$$

which implies $|b^{(+,+)}(A_j, A_{j+1})| \geq \frac{1}{\sqrt{\mu}}(1 - \mu^{-\frac{3}{2}}) \geq \frac{1}{2}\mu^{-\frac{1}{2}}$ if $n \geq 2$, say. One checks easily by induction that for any vector \mathbf{u}

$$A_n \cdot \dots \cdot A_1 \mathbf{u}$$

$$= \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \|A_n\|^{\varepsilon_n} \prod_{j=1}^{n-1} \|A_j\|^{\varepsilon_j} b^{(\varepsilon_j, \varepsilon_{j+1})}(A_j, A_{j+1}) (\mathbf{u}_{A_1}^{\varepsilon_1} \cdot \mathbf{u}) \mathbf{v}_{A_n}^{\varepsilon_n}$$

Hence

$$\|A_n \cdot \dots \cdot A_1 \mathbf{u}\| = \|A_n\| \prod_{j=1}^{n-1} \|A_j\| |b^{(+,+)}(A_j, A_{j+1})| \|\mathbf{u}_{A_1}^+ \cdot \mathbf{u}\| [1 + R_n(\mathbf{u})]$$

$$\begin{aligned}
 |R_n(\mathbf{u})| &\leq \sum_{\substack{\varepsilon_1, \dots, \varepsilon_n = \pm 1 \\ \min_j \varepsilon_j = -1}} \prod_{j=1}^n \|A_j\|^{\varepsilon_j - 1} \prod_{k=1}^{n-1} \left| \frac{b^{(\varepsilon_k, \varepsilon_{k+1})}(A_k, A_{k+1})}{b^{(+,+)}(A_k, A_{k+1})} \right| \\
 &\leq \sum_{\ell=1}^n \binom{n}{\ell} \mu^{-2\ell} (2\sqrt{\mu})^{2\ell} = \sum_{\ell=1}^n \binom{n}{\ell} (4/\mu)^\ell = \left(1 + \frac{4}{\mu}\right)^n - 1 < 4e^4 \frac{n}{\mu}
 \end{aligned}$$

So we have shown with $b^{(+,+)}(A_j, A_{j+1}) =: b_j^{(+,+)}$,

$$\left| \log \|A_n \cdot \dots \cdot A_1\| - \sum_{j=1}^n \log \|A_j\| - \sum_{j=1}^{n-1} \log |b_j^{(+,+)}| \right| < C \frac{n}{\mu}$$

(11) follows from this and the sum of (12):

$$\left| \sum_{j=1}^{n-1} \left[\log |b_j^{(+,+)}| - \log \|A_{j+1} A_j\| + \log \|A_j\| + \log \|A_{j+1}\| \right] \right| \leq C \mu^{-\frac{3}{2}} n$$

Rate of convergence for Lyapunov exponents

Assume $L(E) > \gamma > 0$. Let $N = nk$ and apply AP to

$$M_N(x, E) = \prod_{j=0}^{n-1} A_j, \quad A_j := M_k(T^{jk}x, E)$$

Let $k \sim C(\log N)^{\frac{1}{\sigma}}$, and apply (weak) LDT. Then on $\mathbb{T} \setminus \mathcal{B}$ with $|\mathcal{B}| < N^{-10}$, conditions (9), (10) hold (using that $L_k(E) - L_{2k}(E) \ll 1$ for k large):

$$\begin{aligned} |\log \|A_j\| - kL_k(E)| &< k^\sigma \\ |\log \|A_{j+1}A_j\| - 2kL_{2k}(E)| &< k^\sigma \end{aligned}$$

whence averaging over x yields

$$\begin{aligned} |NL_N(E) + (n-2)kL_k(E) - 2k(n-1)L_{2k}(E)| &< N^{-9} \\ |L_N(E) + L_k(E) - 2L_{2k}(E)| &< \frac{k}{N} \end{aligned} \tag{13}$$

Repeat the same analysis with twice as many matrices, i.e., with $2N$ instead of N .

Applying the AP to the propagators M_N

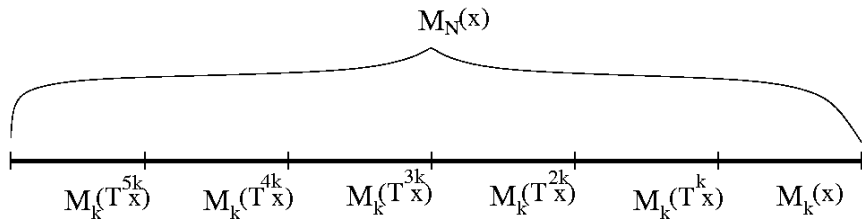


Figure: Writing M_N as product of shifted shorter M_k

Rates of convergence

Hence

$$|L_{2N}(E) + L_k(E) - 2L_{2k}(E)| < \frac{k}{N}$$

Subtracting from the previous estimate we obtain

$$0 \leq L_N(E) - L_{2N}(E) < \frac{(\log N)^C}{N} \quad (14)$$

whence $L_N(E) - L(E) \lesssim (\log N)^C N^{-1}$ (works for **any** Diophantine condition). By a more careful rendition of the same argument we see that in fact

$$L_N(E) - L(E) \lesssim N^{-1}$$

and that this is optimal **unless the convergence takes place exponentially fast** due

$$|L(E) + L_n(E) - 2L_{2n}(E)| < e^{-n^\sigma}$$

Moreover, convergence is **uniform** on any compact interval on which $L > 0$.

Regularity of Lyapunov exponent and IDS assuming $L > 0$

Combine (14) with (13), this time using the **sharper LDT**:

$$|L(E) + L_k(E) - 2L_{2k}(E)| < \exp(-ck)$$

and so taking difference for E, E' and differentiating on small scale $L_k(E)$ one obtains:


$$\begin{aligned} |L(E) - L(E')| &\leq |L_k(E) - L_k(E')| + 2|L_{2k}(E) - L_{2k}(E')| + e^{-ck} \\ &\leq e^{Ck}|E - E'| + e^{-ck} \end{aligned}$$

Theorem

Assume sharp Diophantine condition, and $L(E) > \gamma > 0$ for $E \in I = [E_0, E_1]$. Then L is Hölder continuous on I .

The Hölder exponent depends on γ , but in fact it can be shown by a refinement of this argument from G-S 2001 **that it does not**.

Under a weaker Diophantine condition we have

$\exp(-|\log|E - E'||^b)$ continuity with $b < 1$. **However: You, Zhang 2013** refined LDT and showed **Hölder for all Diophantine**. 

Integrated Density of States

$E_{\Lambda,j}(x)$, $j = 1, \dots, b - a + 1 = |\Lambda|$ are the eigenvalues of (1) restricted to the interval $\Lambda = [a, b]$ with Dirichlet BC $\varphi(a-1) = \varphi(b+1) = 0$. Consider

$$N_{\Lambda}(E, x) = \frac{1}{|\Lambda|} \sum_j \chi_{(-\infty, E)}(E_{\Lambda,j}(x)).$$

The limit (in the weak sense of measures)

$$\lim_{a \rightarrow -\infty, b \rightarrow +\infty} N_{\Lambda}(\cdot, x) = N(\cdot)$$

exists for a.e. x and is **deterministic**. $N(\cdot)$ is the IDS. Connection with Lyapunov exponent given by the **Thouless formula**

$$L(E) = \int \log |E - E'| N(dE'). \quad (15)$$

In other words, L is the **Hilbert transform** of N . Hence we have a corollary of Hölder regularity theorem: the **IDS is also Hölder continuous**.

Controlling the resolvent

Connection with the resolvent:

$$(H_N(x) - E)^{-1}(m, n) = \frac{f_{[1,m]}(x, E) f_{[n+1, N]}(x, E)}{f_{[1, N]}(x, E)}$$

We have a LDT for the determinants:

Proposition

For some small constant $\tau > 0$

$$|\{x \in \mathbb{T} : |N^{-1} \log |f_{[1, N]}(x, E)| - L_N(E)| > N^{-\tau}\}| \leq e^{-N^\tau}.$$

By $|L_N(E) - L(E)| \lesssim N^{-1}$ we may replace $L_N(E)$ with $L(E)$.

Conclusion: if $L(E) > 0$ then

$$|(H_N(x) - E)^{-1}(m, n)| \lesssim \exp(-L(E)(m - n) + N^{1-\tau})$$

up to set of measure $\lesssim \exp(-N^\tau)$.

Uniform upper bounds

We do **not** need LDT for the **numerator**: subharmonic functions can only fall significantly **below** the average but not **rise much above it**. Proof of LDT, use a **regularized form** of (2):

$$u(x) \leq \int \log(|e(x) - \zeta| + \delta) \mu(d\zeta) + h(x) \quad \forall x \in \mathbb{R}$$

with $\delta = N^{-1}$. Work with $\frac{1}{m} \sum_{j=1}^m u(T^j x)$ as in the proof of LDT. Harmonic part is harmless, and we have better decay of the Fourier transform

$$\left| \int_0^1 \log(|e(x) - \zeta| + \delta) e(-\nu x) dx \right| \lesssim \min(|\nu|^{-1}, \delta^{-1} |\nu|^{-2})$$

This gives

$$\sup_x N^{-1} \log \|M_N(x, E)\| \leq L(E) + N^{-\frac{1}{2}+}$$

Assuming $L(E) > 0$ we may further improve this to (G-S 2008)

$$\sup_x N^{-1} \log \|M_N(x, E)\| \leq L(E) + N^{-1} (\log N)^A$$

Higher-dimensional matrices

Let $A : \mathbb{T} \rightarrow GL(d, K)$ be continuous, co-cycle

$$\mathbb{T} \times K^d \in (x, v) \mapsto (x + \omega, A(x)v) \quad (17)$$

where ω is (strongly) Diophantine. $(\mathcal{E}, \mathfrak{d})$ compact metric space.

Theorem (S., 2012)

$A : \mathbb{T} \times \mathcal{E} \rightarrow GL(d, K)$ continuous ($K = \mathbb{R}, \mathbb{C}$), analytic
 $x \mapsto A(x, E)$ uniformly in $E \in \mathcal{E}$. Suppose $E \mapsto A(x, E)$ Hölder
continuous, uniformly in $x \in \mathbb{T}$. Lyapunov exponents satisfy the
gap condition

$$\lambda_j(E) - \lambda_{j+1}(E) > \kappa > 0 \quad \forall E \in \mathcal{E}, \forall 1 \leq j < d \quad (18)$$

Then all $\lambda_j(E)$ are Hölder continuous as a function of $E \in \mathcal{E}$. If
(18) holds at some point $E_0 \in \mathcal{E}$, then each $\lambda_j(E)$ is Hölder
continuous locally around E_0 . In other words, if all exponents are
distinct at E_0 , then they are all Hölder continuous locally
around E_0 , and therefore also remain distinct near E_0 .

Avalanche Principle for $d \times d$ matrices

Lemma

$\{A_j\}_{j=1}^n \subset GL(d, K)$ satisfy: for each $1 \leq j \leq n \exists$ a **1-dimensional subspace** $\mathcal{S}_j \subset K^d$ s.t.

$$|A_j v| = \|A_j\| |v| \quad \forall v \in \mathcal{S}_j$$

$$|A_j w| \leq \alpha_j |w| \quad \forall w \in \mathcal{S}_j^\perp$$

$$\|A_j\| \geq \alpha_j \mu, \quad \mu \geq 16n^2$$

In addition, assume

$$\|A_{j+1}\| \|A_j\| < \mu^{\frac{1}{4}} \|A_{j+1} A_j\| \quad \forall 1 \leq j < n.$$

Then

$$\left| \log \|A_n \cdot \dots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1} A_j\| \right| < C \frac{n}{\sqrt{\mu}}$$

One-dimensionality condition in the AP

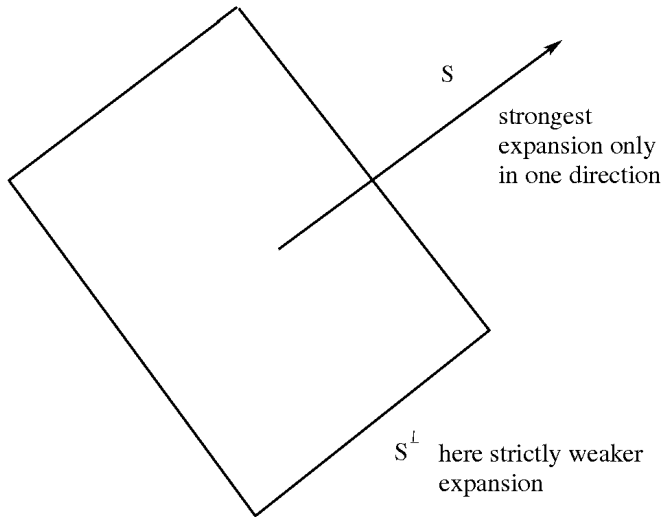


Figure: The unique expanding direction condition

Remarks on higher dimensions

The 2×2 AP is a special case, “line condition” holds automatically. Gap condition is the same as $L(E) > 0$, since the two Lyapunov exponents are $\{L(E), -L(E)\}$.

Oseledec multiplicative ergodic theorem guarantees a filtration of \mathbb{R}^d associated with the sequence of Lyapunov exponents.

Higher-dimensional cocycles arise in the transfer-matrix formalism of higher-order difference equations. Duarte, Klein 2013: very general investigation of avalanche principle, LDT etc. for such cocycles. In particular, they also investigate the situation where the Lyapunov gaps collapse and obtain a version of the AP in that case. They also announce that they will come out with a paper in which they give sufficient conditions for different types of Lyapunov spectrum.

For the 1-dimensional Schrödinger operator we can change the underlying dynamics: instead of \mathbb{T} we consider a potential on \mathbb{T}^d with the multi-dim shift, or the skew shift.

Or we switch from a 1-dim operator to a higher-dimensional one, and we lose the entire transfer matrix formalism.

Summary of Lecture 1

- **Subharmonic functions** arise naturally in the context of Schrödinger operators with potentials determined by evaluating analytic functions along an orbit of an *ergodic deterministic transformation*. Examples: $\log \|M_n\|$, $\log |\det H_\Lambda|$. Also relevant to higher-dimensional lattices \mathbb{Z}^d where we do not have a transfer matrix formalism; then s-h in each coordinate direction (pluri-s-h).
- Modulo harmonic functions, subharmonic ones are **logarithmic potentials of positive Borel measures** (Riesz representation). The latter are the key to analytic control, and the derivation of large deviation estimates.
- But size of Riesz measure alone too crude to imply a large deviation estimate. We require also some *structural information* on the measure, such as near invariance of the function itself under the dynamics. Easier to implement structure directly for the subhamonic function itself rather than for its Riesz measure.

Summary of Lecture 1

- **Avalanche principle** in combination with LDT allows for *induction on scales* arguments. Essential for nonperturbative approach assuming only positivity of the Lyapunov exponent. Gives rates of convergence for these exponents. Can also be used to **derive** perturbatively (need big potentials for the initial step) large deviation estimates, such as for the skew shift. See Lecture 3.
- In the PDE setting (higher-dimensional lattices \mathbb{Z}^d), or for long-range operators, cannot use AP, instead rely on resolvent identity. This will again be **perturbative**. We control the Green functions directly in this way. A key analytical ingredient in this approach are the **Matrix-valued Cartan estimates**, see Lecture 3.
- All of the analysis in this lecture was for **fixed energy**. In the following lecture, we will encounter the so-called *elimination of the energy* which is needed for **Anderson Localization**.

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