Introduction to Lectures 1-4.

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The periodic problem

Consider the periodic second order ODE

$$-y''(x) + V(x)y(x) = Ey(x), \quad E \in \mathbb{R} \tag{1}$$

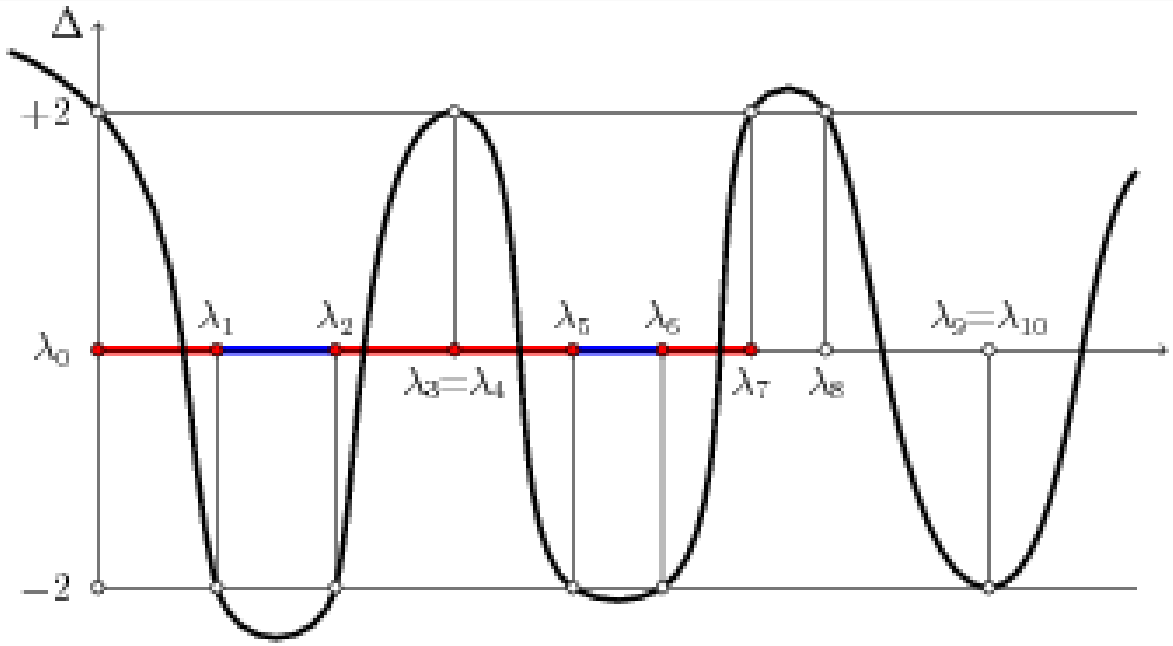
on the line. Assume $V(x+L) = V(x)$ real-valued (may take $L = 1$). Then by Floquet theory any solution of (1) is of the form $y(x) = e^{ik(E)x}a(x,E)$ where $a(x+L,E) = a(x,E)$. This comes from considering the propagator (fundamental matrix) $S(L)$ and its eigenvalues. Since $\det S(L) = 1$, either both eigenvalues lie on the unit circle (and are complex conjugates), or they are real-valued and reciprocal.

What does $\mathrm{spec}(H)$ look like, where

$$(Hy)(x) = -y''(x) + V(x)y(x) \quad ?$$

We need to find those E for which $k(E)$ is real-valued.

The periodic problem: Hill's discriminant Δ



HILL'S DISCRIMINANT

- THE RED INTERVALS ARE THE BANDS OF THE SPECTRUM
- THE BLUE INTERVALS ARE THE GAPS
- THE DOUBLE EIGENVALUES, LIKE $\lambda_3 = \lambda_4$ FALL INTO THE SPECTRUM

Figure: The bands in the spectrum

Here $\Delta = \mathrm{trace}(S(L))$. The red intervals are precisely the ones where the eigenvalues are on the unit circle.

The periodic problem

Moreover, the spectrum of H is purely absolutely continuous.
Recall: as a self-adjoint operator H on a suitable domain has a spectral resolution $N(dE)$ so that $H = \int_{\mathbb{R}} E\, N(dE)$ which means that

$$\langle Hf, g\rangle = \int_{\mathbb{R}} E\, \langle N(dE)f, g\rangle \quad \forall\, f, g \in \mathrm{dom}(H)$$

Lebesgue decomposition: $L^2(\mathbb{R}) = L^2(\mathbb{R})_{pp} \oplus L^2(\mathbb{R})_{ac} \oplus L^2(\mathbb{R})_{sc}$ orthogonal decomposition into closed subspaces such that the measure

$$\mu_{f,g}(dE) = \langle N(dE)f, g\rangle \quad \forall\, f, g \in L^2(\mathbb{R})_X$$

is of type X, where $X = pp/ac/sc$.

How does one identify the spectrum as purely absolutely continuous?

Stone formula

$\forall\, f, g \in L^2(\mathbb{R})$ and any test function φ one has

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int \big\langle [(H - (\lambda + i\varepsilon))^{-1} - (H - (\lambda - i\varepsilon))^{-1}] f, g \big\rangle \varphi(\lambda)\, d\lambda$$
$$= \int_{\mathbb{R}} \varphi(\lambda)\, \mu_{f,g}(d\lambda)$$

So there is a clear connection between the spectral measure and the resolvent. For example,

$$\frac{1}{\pi} \operatorname{Im} (H - (\lambda + i0))^{-1}\, d\lambda = N_{ac}(d\lambda) := N(d\lambda) \restriction \mathcal{H}_{ac}$$

This is especially useful on the line, since we can compute the resolvent (i.e., Green function) from a fundamental system of (1) (W=Wronskian)

$$(H - z)^{-1}(x, x') = \frac{y_+(x, z) y_-(x', z)}{W(y_+(\cdot, z), y_-(\cdot, z))} \qquad x > x'$$

where $\operatorname{Im} z > 0$ and $y_\pm(x, z)$ decay as $x \to \pm\infty$, respectively.

Physical relevance

In the early days of quantum mechanics crystals (such as metals and other electric conductors) were modeled by Schrödinger operators with periodic potentials. The wave functions are precisely the Floquet solutions, as noted by Bloch. The fact that these "eigenfunctions" are not localized, or extended is interpreted physically as mobility of the electrons which then translates into electric conductivity. Works in **any dimension**. Mathematically speaking: the spectrum is (absolutely) continuous. Typically, the issue of characterizing singular continuous spectrum is very subtle and many open problems remain (see e.g. Damanik-Killip-Lenz).

In QM it was then an important problem to understand what happens to a crystal (conductor) if random impurities are introduced. The common belief was after Bloch that a.c. spectrum is stable and not destroyed. In **1957 Phil Anderson** surprised many by showing (non-rigorously) that for sufficiently strong random, independent potentials on the whole lattice there is only pure point spectrum and the eigenfunctions decay exponentially.

P. Anderson's work

For this he received the Nobel prize, since experimental confirmation followed. More precisely, consider the random operator

$$H = -\Delta_{\mathbb{Z}^d} + \lambda V \tag{2}$$

on the lattice $\mathbb{Z}^d$ where $V = \{v_n\}_{n\in\mathbb{Z}^d}$ with i.i.d. random variables v_n. E.g., take $v_n = \pm 1$.

- $d = 1$ and any $\lambda \neq 0$: pure point spectrum and exponentially localized eigenfunctions (AL). Fürstenberg's theorem on positive Lyapunov exponents for products of random matrices is the basic mathematical ingredient.
- $d = 2, 3, \ldots$ and LARGE $\lambda \neq 0$: AL as shown by Fröhlich-Spencer in the 1980s.
- Conjecture: $d = 2$ and any $\lambda \neq 0$ have AL.
- Conjecture: $d = 3$ and SMALL $\lambda \neq 0$ have SOME AC SPECTRUM (problem of **extended states**). In other words, one expects a METAL-INSULATOR PHASE TRANSITION depending on the **size of the disorder**.

Dynamical properties

If H exhibits localization, then for any $s \geq 0$ one has

$$\sup_t \|\langle x\rangle^s e^{itH}\psi\|_2 < \infty$$

for any $\psi \in L^2$. In other words, nothing spreads.
Contrast this to the free case: if $\langle x\rangle\psi \in L^2$, then $\|\langle x\rangle e^{it\Delta}\psi\|_2 \simeq \langle t\rangle$. Same for other powers.
On the other hand, we have the following behavior on the continuous spectrum: for all $\psi \in L^2_c$ one has

$$\frac{1}{T}\int_0^T \|\chi_B e^{itH}\psi\|_2^2\, dt \to 0$$

as $T \to \infty$. Here B is any ball. In the free case we have

$$\|e^{it\Delta}f\|_\infty \leq |t|^{-d/2}\|f\|_1$$

This is the standard dispersive estimate (wave packets of different frequencies travel at different speeds).

P. Anderson's work

Needless to say, all of this is about random potentials in INFINITE VOLUME. Finitely many "impurities" have **no effect** on the **essential spectrum** by Weyl's criterion (on resolvent-compact perturbations of self-adjoint operators).

Anderson's quote: Localization is a *GAME OF RESONANCES*.

A **resonance** here means that on any given *finite volume* $\Lambda \subset \mathbb{Z}^d$ such as a *large cube*, and any given energy E the spectrum of the restricted operator $H \restriction \Lambda$ comes "very close" to E.
Of course, one needs to make this quantitative, but the basic idea is to control (or rule out) the presence of long chains of such resonant cubes. Indeed, if they are present, then the associated eigenfunction will have equal mass on each of these cubes, and therefore be extended.
Not surprisingly, it is easier to **exclude** this type of *tunneling* than to show that it occurs and leads to infinitely extended states.

The game of resonances

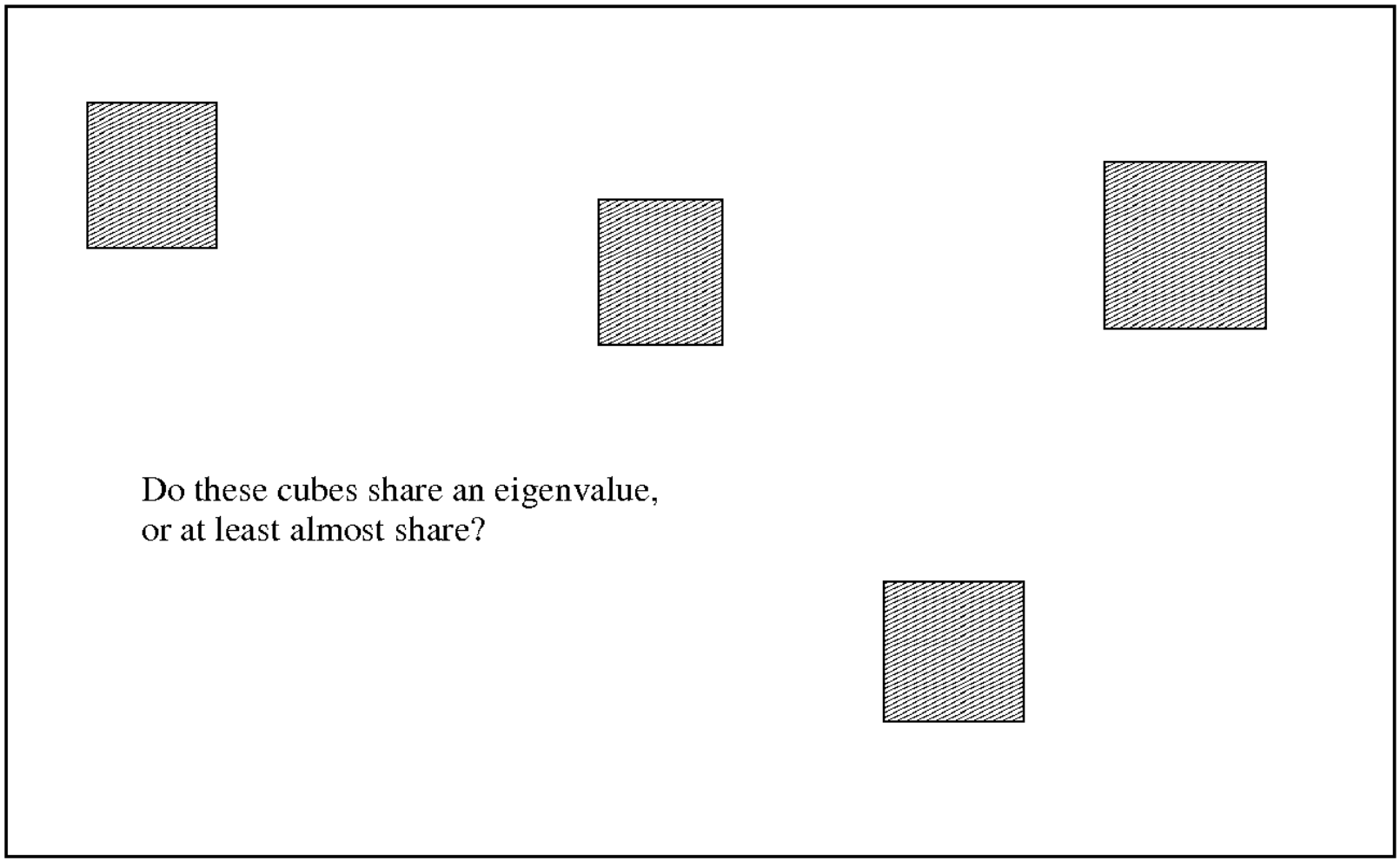


Figure: Resonant cubes

The game of resonances

If we have a **periodic structure** then there are clearly *infinitely extended periodic chains* of cubes which have **identical spectrum**. So this is what is captured — without any reference to the finite volume analysis — by *Floquet-Bloch solutions* in infinite volume. In the random case, we can exploit that disjoint cubes are independent and thus the probability of having an eigenvalue **close to** E in a given cube (which is called "Wegner estimate") is SQUARED. This essentially leads to the same rapid convergence as for a **Newton scheme** in the AL proof at large disorder.

But what if we have *strong dependence* between the values of the potential at different sites, such as in **quasi-crystals**? To be more specific, suppose $v_n(x) = F(T^n x)$ where $T : X \to X$ is an ergodic transformation on some measure space $X \ni x$ and $F : X \to \mathbb{R}$. Here $n \in \mathbb{Z}$, in higher dimension can consider the analogue $v_n(x) = F(T_1^{n_1} \circ T_2^{n_2} \circ \ldots \circ T_d^{n_d} x)$.

Deterministic potentials

Such potentials go by the name of *deterministic potentials*, and all the "randomness" sits in the variable $x \in X$. To play the game of resonances, we clearly have to face the issue of **recurrence** of the dynamical system. But we need much more (such as a quantitative ergodic theorem) since we must precisely control the **small divisors** which arise in the resolvent $(H \restriction \Lambda - z)^{-1}$.

In this generality only have "soft" results such as constancy of spectrum etc. Any result which requires dealing with small divisors can only be done with very specific dynamics such as the shift=rotation and very limited potentials (trigonometric polynomials such as cos, or analytic functions, Gevrey class also studied). **In many ways, our understanding is very poor**. For example, our methods give **weaker conclusions** for higher-dimensional shifts, although they should exhibit "more randomness" and thus the results should be closer to the random case – and not further as the current techniques would suggest.

Spectrum of ergodic Schrödinger operators

Consider the self-adjoint operators

$$(H_x\psi)_n = \psi_{n+1} + \psi_{n-1} + v_n(x)\psi_n, \quad n \in \mathbb{Z}$$

with $v_n(x)$ an "ergodic potential", i.e., $v_n(x) = V(T^n x)$ and $T : X \to X$ ergodic transformation on a probability space X, and $V : X \to \mathbb{R}$ measurable. Then there exists *fixed* compact set $K \subset \mathbb{R}$ with $\mathrm{spec}(H_x) = K$ for a.e. $x \in X$. This follows from ergodic theorem and property of the spectral resolution N_x of H_x

$$N_x = S^{-1} \circ N_{Tx} \circ S, \quad S = \text{ right translation}$$

In addition, $\mathrm{spec}_{pp}(H_x), \mathrm{spec}_{ac}(H_x), \mathrm{spec}_{sc}(H_x)$ are *also deterministic*. Eigenvalues are **NOT** deterministic, but their **closure** is.

Very delicate: Structure of the spectrum such as Cantor set (dense collection of open gaps), versus no gaps.

Basic gap formation resulting from a double resonance, Sinai's work

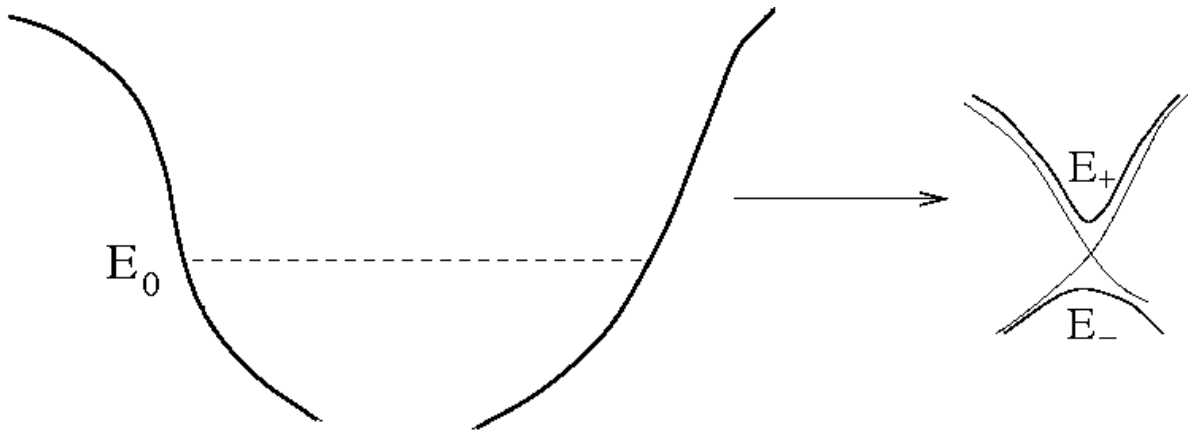


Figure: Crossing of graphs of eigenvalues creates a gap

$$\det \begin{pmatrix} \lambda_1(x) - E & \varepsilon \\ \varepsilon & \lambda_2(x) - E \end{pmatrix} = 0, \quad \lambda_1(x_0) = \lambda_2(x_0) = E_0$$
$$E_{\pm}(x) = \frac{1}{2}(\lambda_1(x) + \lambda_2(x)) \pm \sqrt{(\lambda_1(x) - \lambda_2(x))^2 + 4\varepsilon^2} \tag{3}$$

This is a reflection of the fact that for the Dirichlet problem eigenvalues are simple. On the level of eigenfunctions the following is going on:

Basic mechanism behind gap formation

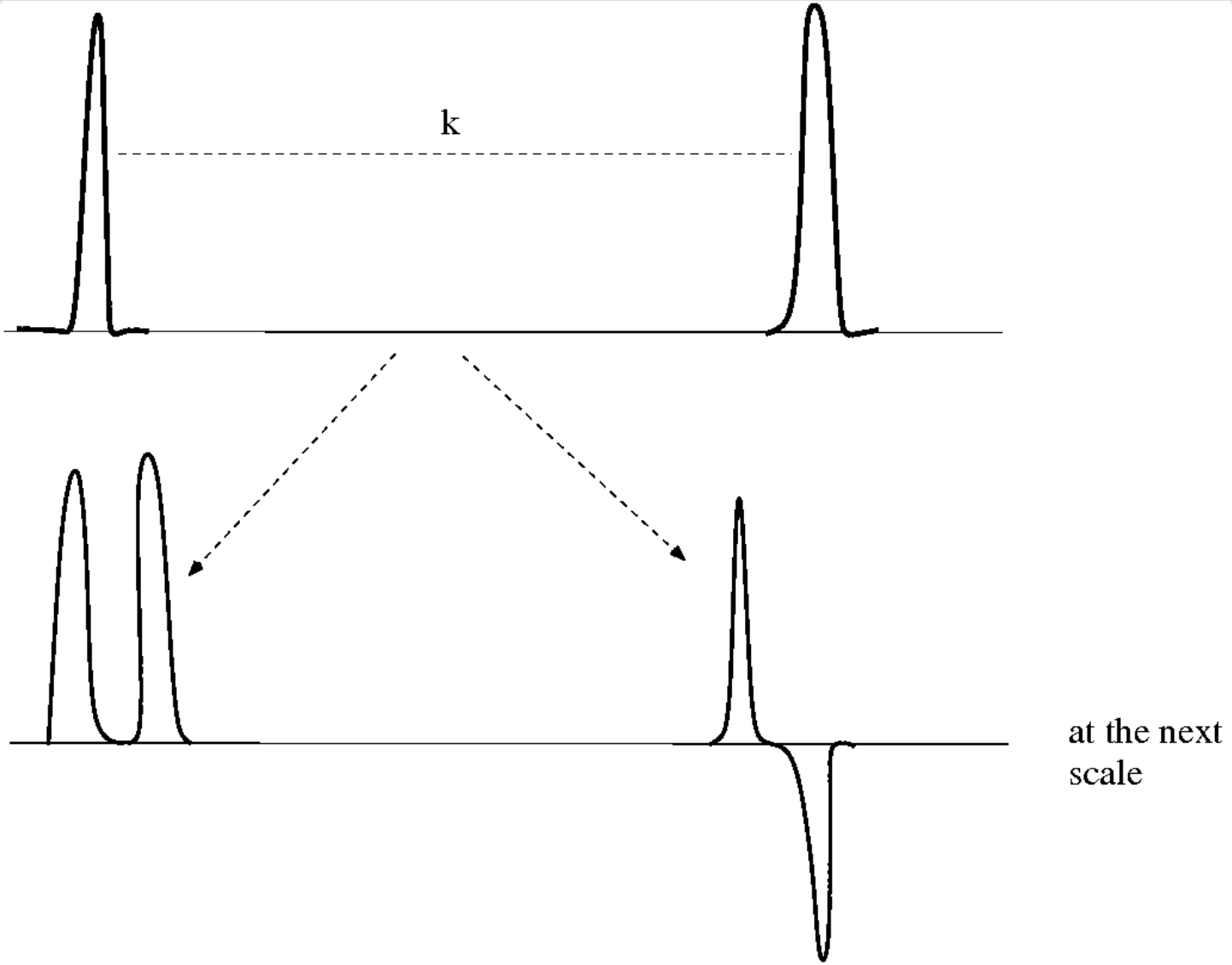


Figure: Crossing of graphs of eigenvalues create two peaks

Organization of lectures

The goal of these lectures is to present a body of techniques based on two main ingredients

- Estimates for subharmonic functions (Cartan estimate); requires analytic potentials. Basic analytical ingredient which controls the small divisors. The dynamics enters here.
- Bounds for semi-algebraic sets. These require polynomial potentials or such that can be (exponentially) well approximated by polynomials. *Goal:* control the "complexity" (connected components) of "bad sets" in the "random parameter" $x \in X$ to recapture independence of events that both x, $T^n x$ are "bad" with **LARGE** $n \in \mathbb{Z}$. The specific type of dynamics enters here, treat on a case-by-case basis. This second step prevents chains of resonant cubes.
- Vary the dynamics analytically in a parameter (=frequency/rotation number); *eliminate bad frequencies*.

Overview

Much of our presentation is for the basic shift=rotation on the circle. Parameter equals rotation number and we eliminate a thin set of bad rotation numbers in the process of controlling small divisors. We will also make reference to shifts on higher-dimensional tori, as well as skew shifts. The final Lecture 4 is devoted to operators on higher-dimensional lattices.

- **Lecture 1:** subharmonic functions, Riesz representation, transfer matrices, Lyapunov exponent, large deviation estimates, avalanche principle.
- **Lecture 2:** Localization, semi-algebraic sets, elimination of double resonances, positivity of the Lyapunov exponent
- **Lecture 3:** Cartan estimates, derivation of LDT from Cartan, BMO and subharmonic functions, matrix-valued Cartan, splitting lemma
- **Lecture 4:** higher-dimensional lattices, resolvent expansions, applying the matrix-valued Cartan estimate, dealing with large collections of resonant boxes.

Overview

Many open problems remain, some very basic. Of course, the random case stands out with the problem of extended states. Appears very difficult. For "deterministic potentials", higher-dimensional shifts are very poorly understood (gaps in the spectrum?). Endless variations of dynamical systems possible, combine different types of dynamics – for example, mixing with not mixing: $F(x_1 + n\omega, 2^n x_2)$

A basic reference for these lectures is Bourgain's book *"Green's function estimates for lattice Schrödinger operators and applications"*. We do not cover all of it by any means, but go beyond it in some ways (the higher-dimensional lattices, for example, or the systematic developments of Cartan estimates). A major omission are the KAM applications at the end of Bourgain's book aiming at the construction of (quasi)periodic solutions to various Hamiltonian wave equations.

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