Long-term dynamics of nonlinear wave equations

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Wave maps

Let (M, g) be a Riemannian manifold, and $u : \mathbb{R}^{1+d}_{t,x} \to M$ smooth.

Wave maps defined by Lagrangian

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\Big(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2\Big)\, dtdx$$

Critical points $\mathcal{L}'(u, \partial_t u) = 0$ satisfy "manifold-valued wave equation".
$M \subset \mathbb{R}^N$ embedded, this equation is

$$\Box u \perp T_u M \text{ or } \Box u = A(u)(\partial u, \partial u),$$

A being the second fundamental form.

For example, $M = \mathbb{S}^{n-1}$, then

$$\Box u = u(|\partial_t u|^2 - |\nabla u|^2)$$

Note: Nonlinear wave equation, null-form! Harmonic maps are solutions.

Wave maps

Intrinsic formulation: $D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0$, in coordinates

$$-\partial_{tt} u^i + \Delta u^i + \Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0$$

$\eta = (-1, 1, 1, \ldots, 1)$ Minkowski metric

- Similarity with geodesic equation: $u = \gamma \circ \varphi$ is a wave map provided $\Box\varphi = 0$, γ a geodesic.

- Energy conservation: $E(u, \partial_t u) = \int_{\mathbb{R}^d} \left(|\partial_t u|_g^2 + \sum_{j=1}^d |\partial_j u|_g^2 \right) dx$ is conserved in time.

- Cauchy problem:

$$\Box u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

smooth data. Does there exist a smooth local or global-in-time solution?

Local: Yes. Global: depends on the dimension of Minkowski space and the geometry of the target.

Criticality and dimension

If $u(t,x)$ is a wave map, then so is $u(\lambda t, \lambda x)$, $\forall \lambda > 0$.
Data in the Sobolev space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$. For which s is this space invariant under the natural scaling? Answer: $s = \frac{d}{2}$.
Scaling of the energy: $u(t,x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$ same as $\dot{H}^1 \times L^2$.

- Subcritical case: $d = 1$ the natural scaling is associated with less regularity than that of the conserved energy. Expect global existence. Logic: local time of existence only depends on energy of data, which is preserved.
- Critical case: $d = 2$. Energy keeps the balance with the natural scaling of the equation. For $\mathbb{S}^2$ can have finite-time blowup, whereas for $\mathbb{H}^2$ have global existence. Krieger-S.-Tataru 06, Krieger-S. 09, Rodnianski-Raphael 09, Sterbenz-Tataru 09.
- Supercritical case: $d \geq 3$. Poorly understood. Self-similar blowup $Q(r/t)$ for sphere as target, Shatah 80s. Also negatively curved manifolds possible in high dimensions: Cazenave, Shatah, Tahvildar-Zadeh 98.

Wellposedness of Wave Maps

- Energy methods: $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ with $s > \frac{n}{2} + 1$.
- Klainerman-Machedon, Klainerman-Selberg 1990s: $X^{s,b}$ spaces which use symbols of both $\Delta, \Box$, bilinear estimates involving null forms of Christodoulou, Klainerman gives $s > \frac{n}{2}$.
- Push to $s = \frac{n}{2}$ since then local=global. Tataru 1998 introduced *null frame spaces* and achieved small data global regularity, Tao 2000 employed gauge invariance and obtained the desired small energy result, with $d = 2$ the hardest case. Shatah-Struwe 2003: simpler proof, Coulomb gauge in $d \geq 4$
- Large data have dichotomy between blowup/global regularity. For the latter use induction on energy (Bourgain 1990s), for example via Kenig-Merle 2007 concentration compactness (Krieger-S. 2009). Implementation is very complicated: gauge, Tataru/Tao spaces, no linear profile extraction possible (manifold valued functions, no superposition principle).
- Equivariant case (discussed later) is more accessible, Christodoulou, Shatah, Tahvildar-Zadeh, Struwe1990s made fundamental contributions. Many open problems remain for the non-equivariant case (more about this later).
- Next: Concentration-compactness illustrated by a more elementary semi-linear model.

A nonlinear defocusing Klein-Gordon equation

Consider in $\mathbb{R}^{1+3}_{t,x}$

$$\Box u + u + u^3 = 0, \quad (u(0), \dot{u}(0)) = (f, g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3)$$

Conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{4}|u|^4 \right) dx$$

With $S(t)$ the linear propagator of $\Box + 1$ we have

$$\vec{u}(t) = (u, \dot{u})(t) = S(t)(f, g) - \int_0^t S(t-s)(0, u^3(s))\, ds$$

whence by a simple energy estimate, $I = (0, T)$

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u^3\|_{L^1(I;L^2)} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u\|^3_{L^3(I;L^6)}$$
$$\lesssim \|(f,g)\|_{\mathcal{H}} + T\|\vec{u}\|^3_{L^\infty(I;\mathcal{H})}$$

Contraction for small T implies local wellposedness for $\mathcal{H}$ data.

Defocusing NLKG3

T depends only on $\mathcal{H}$-size of data. From energy conservation we obtain global existence by time-stepping.

Scattering (as in linear theory): $\|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \to 0$ as $t \to \infty$ where $\Box v + v = 0$ energy solution.

$$\vec{v}(0) := \vec{u}(0) - \int_0^\infty S(-s)(0, u^3)(s)\, ds \text{ provided } \|u^3\|_{L_t^1 L_x^2} < \infty$$

Strichartz estimate uniformly in intervals I

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} + \|u\|_{L^3(I;L^6)} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u\|_{L^3(I;L^6)}^3$$

Small data scattering: $\|\vec{u}\|_{L^3(I;L^6)} \lesssim \|(f,g)\|_{\mathcal{H}} \ll 1$ for all I. So $I = \mathbb{R}$ as desired.

Large data scattering valid; induction on energy, concentration compactness (Bourgain, Bahouri-Gerard, Kenig-Merle).

Scattering blueprint

Let $\vec{u}$ be nonlinear solution with data $(u_0, u_1) \in \mathcal{H}$. Forward scattering set

$$\mathcal{S}_+ = \{(u_0, u_1) \in \mathcal{H} \mid \vec{u}(t) \text{ exists globally, scatters as } t \to +\infty\}$$

We claim that $\mathcal{S}_+ = \mathcal{H}$. This is proved via the following outline:

- (Small data result): $\|(u_0, u_1)\|_{\mathcal{H}} < \varepsilon$ implies $(u_0, u_1) \in \mathcal{S}_+$

- (Concentration Compactness): If scattering fails, i.e., if $\mathcal{S}_+ \neq \mathcal{H}$, then construct $\vec{u}_*$ of minimal energy $E_* > 0$ for which $\|u_*\|_{L_t^3 L_x^6} = \infty$. There exists $x(t)$ so that the trajectory

$$K_+ = \{\vec{u}_*(\cdot - x(t), t) \mid t \geq 0\}$$

  is pre-compact in $\mathcal{H}$.

- (Rigidity Argument): If a forward global evolution $\vec{u}$ has the property that K_+ pre-compact in $\mathcal{H}$, then $u \equiv 0$.

Kenig-Merle 2006, Bahouri-Gérard decomposition 1998; Merle-Vega.

Bahouri-Gérard: symmetries vs. dispersion

Let $\{u_n\}_{n=1}^{\infty}$ free Klein-Gordon solutions in $\mathbb{R}^3$ s.t.

$$\sup_n \|\vec{u}_n\|_{L_t^\infty \mathcal{H}} < \infty$$

$\exists$ free solutions v^j bounded in $\mathcal{H}$, and $(t_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}^3$ s.t.

$$u_n(t,x) = \sum_{1 \le j < J} v^j(t + t_n^j, x + x_n^j) + w_n^J(t,x)$$

satisfies $\forall\, j < J$, $\vec{w}_n^J(-t_n^j, -x_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$, and

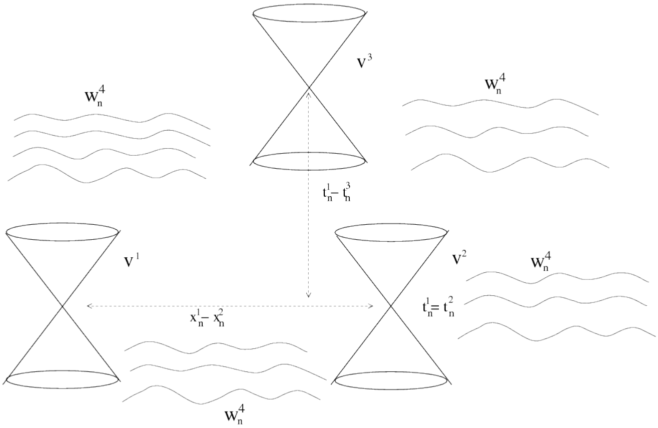
- $\lim_{n\to\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty \;\forall\, j \neq k$
- dispersive errors w_n^k vanish asymptotically:

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\|w_n^J\right\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R}\times\mathbb{R}^3)} = 0 \quad \forall\, 2 < p < 6$$

- orthogonality of the energy:

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1 \le j < J} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{w}_n^J\|_{\mathcal{H}}^2 + o(1)$$

Profiles and Strichartz sea



We can extract further profiles from the Strichartz sea if w_n^4 does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n\to\infty} \|w_n^4\|_{L_t^\infty L_x^p(\mathbb{R}^3)} > 0$.

Lorentz transformations

$$\begin{bmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

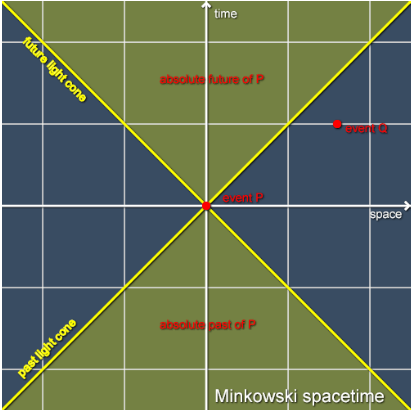


Figure: Causal structure of space-time

Further remarks on Bahouri-Gérard

- Noncompact symmetry groups: space-time translations and Lorentz transforms.

  Compact symmetry groups: Rotations
  Lorentz transforms do not appear in the profiles: Energy bound compactifies them.

- Dispersive error w_n^J is not an energy error!

- In the radial case only need time translations

The focusing NLKG equation

The focusing NLKG

$$\Box u + u = \partial_{tt} u - \Delta u + u = u^3$$

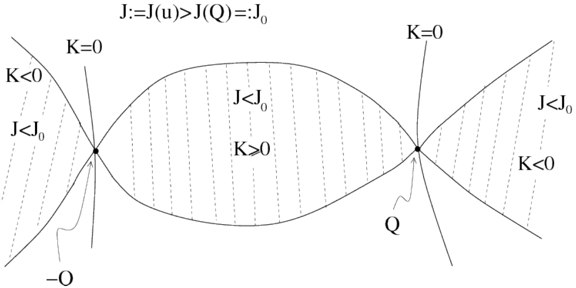
has indefinite conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4 \right) dx$$

- Local wellposendness for $H^1 \times L^2(\mathbb{R}^3)$ data
- Small data: global existence and scattering
- Finite time blowup $u(t) = \sqrt{2}(T-t)^{-1}(1 + o(1))$ as $t \to T-$
  Cutoff to a cone using finite propagation speed to obtain finite energy solution.
- stationary solutions $-\Delta\varphi + \varphi = \varphi^3$, ground state $Q(r) > 0$

Payne-Sattinger theory; saddle structure of energy near Q

Criterion: finite-time blowup/global existence?
Yes, provided the energy is less than the ground state energy Payne-Sattinger 1975.



$$J(\varphi) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\right) dx$$

$$K(\varphi) = \int_{\mathbb{R}^3} \left(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\right) dx$$

Uniqueness of Q is the foundation!

Payne-Sattinger theory

$j_\varphi(\lambda) := J(e^\lambda \varphi)$, $\varphi \neq 0$ fixed.

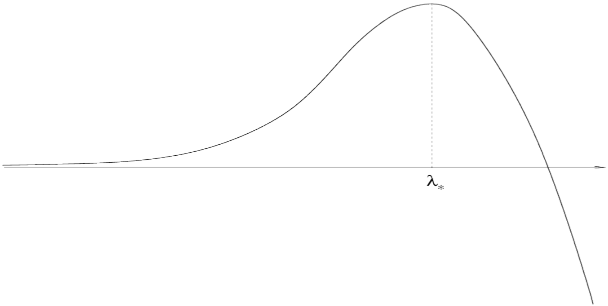


Figure: Payne-Sattinger well

Normalize so that $\lambda_* = 0$. Then $\partial_\lambda j_\varphi(\lambda)\big|_{\lambda=\lambda_*} = K(\varphi) = 0$.

"Trap" the solution in the well on the left-hand side: need $E < \inf\{j_\varphi(0) \mid K(\varphi) = 0, \varphi \neq 0\} = J(Q)$ (lowest mountain pass). Expect global existence in that case.

Above the ground state energy

Theorem (Nakanishi-S. 2010)

Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\mathrm{rad}}$. *In* $t \geq 0$ *for NLKG:*

1. *finite time blowup*
2. *global existence and scattering to* 0
3. *global existence and scattering to* Q: $u(t) = Q + v(t) + o_{H^1}(1)$ *as* $t \to \infty$, *and* $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 *combinations of this trichotomy allowed as* $t \to \pm\infty$.

- Applies to $\dim = 3$, $|u|^{p-1}u$, $7/3 < p < 5$, or $\dim = 1$, $p > 5$.
- Third alternative forms the **center stable manifold** associated with $(\pm Q, 0)$. Linearized operator $L_+ = -\Delta + 1 - 3Q^2$ has spectrum $\{-k^2\} \cup [1, \infty)$ on $L^2_{rad}(\mathbb{R}^3)$. Gap $[0, 1)$ difficult to verify, Costin-Huang-S., 2011.
- $\exists$ 1-dim. stable, unstable manifolds at $(\pm Q, 0)$. Stable manifolds: Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko 2009

The invariant manifolds

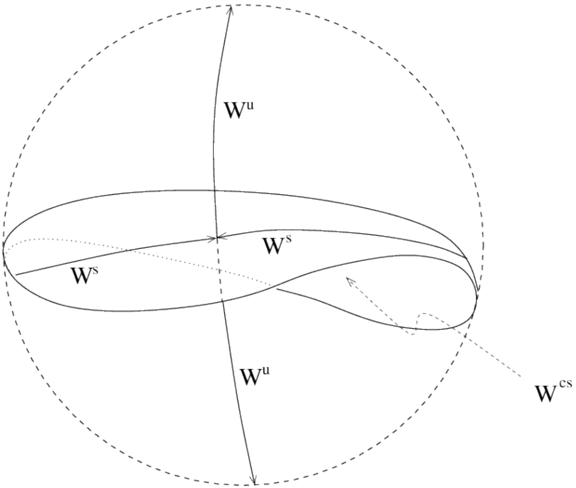


Figure: Stable, unstable, center-stable manifolds

Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$

- $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$
- $L_+ \rho = -k^2 \rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: L_+ has no eigenvalues in $(0, 1]$, no threshold resonance (delicate!) Use Kenji Yajima's L^p-boundedness for wave operators.

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}$$

Then $\mathrm{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ simple evals. Formally:

$X_s = P_1 L^2$, $X_u = P_{-1} L^2$, X_c is the rest.

Spectrum of matrix Hamiltonian

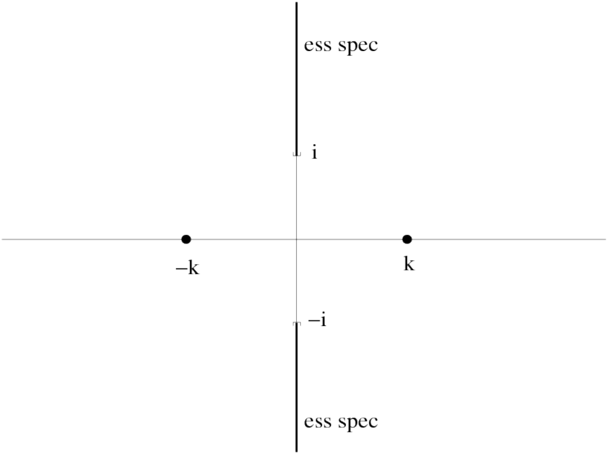


Figure: Spectrum of nonselfadjoint linear operator in phase space

Numerical 2-dim section through $\partial\mathcal{S}_+$ (with R. Donninger)

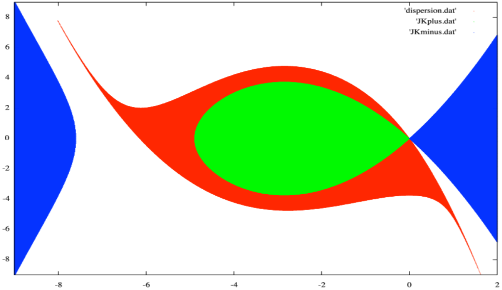
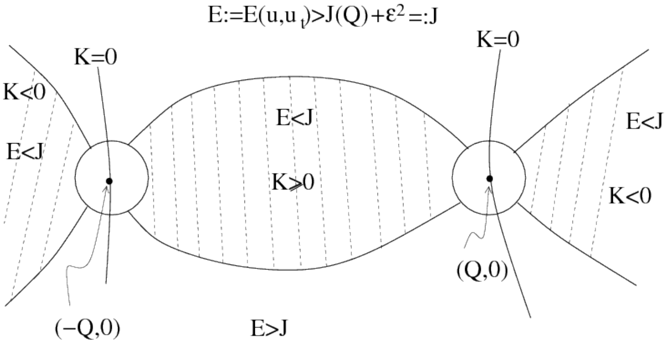


Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, (A, B) vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_+$, BLUE: $\mathcal{PS}_-$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)

Variational structure above $E(Q, 0)$



- Solution can pass through the balls. Energy is no obstruction anymore as in the Payne-Sattinger case.
- Key to description of the dynamics: One-pass (no return) theorem. The trajectory can make only one pass through the balls.
- Point: Stabilization of the sign of $K(u(t))$.

One-pass theorem (non-perturbative)

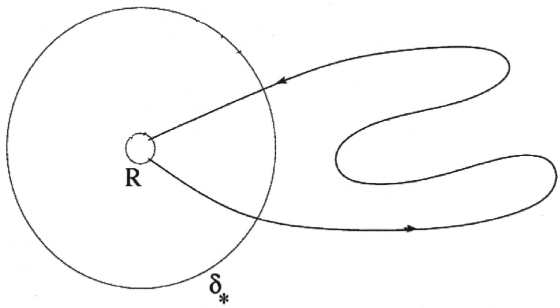


Figure: Possible returning trajectories

Such trajectories are excluded by means of an indirect argument using a variant of the virial argument that was essential to the rigidity step of concentration compactness.

One-pass theorem

Crucial no-return property: Trajectory does **not return to balls around** $(\pm Q, 0)$. Suppose it did; Use *virial identity*

$$\partial_t \langle w\dot{u} \,|\, Au \rangle = -\int_{\mathbb{R}^3} (|\nabla u|^2 - \frac{3}{4}|u|^4)\, dx + \text{error}, \quad A = \frac{1}{2}(x\nabla + \nabla x)$$

where $w = w(t,x)$ is a space-time cutoff that lives on a rhombus, and the “error” is controlled by the external energy.

Finite propagation speed $\Rightarrow$ error controlled by free energy outside large balls at times T_1, T_2.
Integrating between T_1, T_2 gives contradiction; the **bulk** of the integral of $K_2(u(t))$ here comes from exponential ejection mechanism near $(\pm Q, 0)$.

Non-perturbative argument.

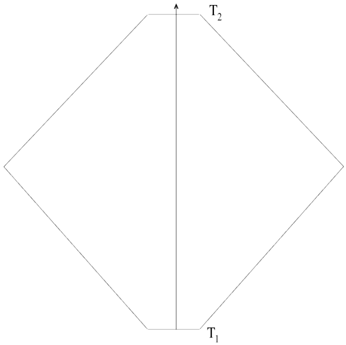


Figure: Space-time cutoff for the virial identity

Open problem

Complete description of possible long-term dynamics: Given focusing NLKG3 in $\mathbb{R}^3$ with radial energy data, show that the solution either

- blows up in finite time
- exists globally, scatters to one of the stationary solutions $-\Delta\varphi + \varphi = \varphi^3$ (including 0)

Moreover, describe dynamics, center-stable manifolds associated with φ.

Evidence: With dissipation given by $-\alpha\partial_t u$ term, result holds (Burq-Raugel-S.).

Critical equation: $\Box u = u^5$ in $\mathbb{R}^3$, Duyckaerts-Kenig-Merle proved analogous result with rescaled ground-state profiles $\sqrt{\lambda}W(\lambda x)$, $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$.

Obstruction: Exterior energy estimates in DKM scheme fail in the KG case due to speed of propagation < 1.

Center-stable manifold, u^5 critical equation

Nakanishi-S theorem applies to nonradial NLKG, NLS, different subcritical powers and dimensions. Critical equations exhibit similar, yet qualitatively essentially different phenomena due to scaling symmetry.

$$\ddot{u} - \Delta u = |u|^{2^*-2}u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \quad 2^* = \frac{2d}{d-2},$$

Static Aubin, Talenti solutions

$$W_\lambda = T_\lambda W, \quad W(x) = \left[1 + \frac{|x|^2}{d(d-2)}\right]^{1-\frac{d}{2}},$$

T_λ is $\dot{H}^1$ preserving dilation

$$T_\lambda \varphi = \lambda^{d/2-1}\varphi(\lambda x)$$

Positive radial solutions $-\Delta W - |W|^{2^*-2}W = 0$. Functionals:

$$J(\varphi) := \int_{\mathbb{R}^d}\left[\frac{1}{2}|\nabla\varphi|^2 - \frac{1}{2^*}|\varphi|^{2^*}\right]dx, \quad K(\varphi) := \int_{\mathbb{R}^d}[|\nabla\varphi|^2 - |\varphi|^{2^*}]\,dx$$

Critical wave equation

Radial $\dot{H}^1 \times L^2$, $E(\vec{\varphi}) < J(W) + \varepsilon^2$, outside soliton tube

$$\mathcal{S} = \{\pm \vec{W}_\lambda \mid \lambda > 0\} + O(\varepsilon)$$

There exists four open disjoint sets which lead to all combinations of FTB/GE and scattering to 0 as $t \to \pm I$.

- Krieger-Nakanishi-S. 2013: complete description of all solutions with energy $E(\vec{\varphi}) < J(W) + \varepsilon^2$. Type-I conjecture!
- center-stable manifold exists in $\dot{H}^1 \times L^2$, contains all W_λ, solutions with $\lambda \to 0, \infty$ (but Krieger-S. 05 showed that in a stronger non-invariant topology exists codim-1 manifold with global solutions, $\lambda(t) \to \lambda_* \in (0, \infty)$).
- Inside the soliton tube there exist blowup solutions, as found by Krieger-S.-Tataru 06. Then Duyckaerts-Kenig-Merle 09 showed that all type II blowup are of the KST form, as long as energy below $2J(Q)$. So trapping by the soliton tube cannot mean scattering to $\{W_\lambda\}$ as it did in the subcritical case.

Equivariant wave maps

$u : \mathbb{R}^{1+2}_{t,x} \to \mathbb{S}^2$ satisfies WM equation

$$\Box u \perp T_u\mathbb{S}^2 \Leftrightarrow \Box u = u(|\partial_t u|^2 - |\nabla u|^2)$$

as well as equivariance assumption $u \circ R = R \circ u$ for all $R \in SO(2)$

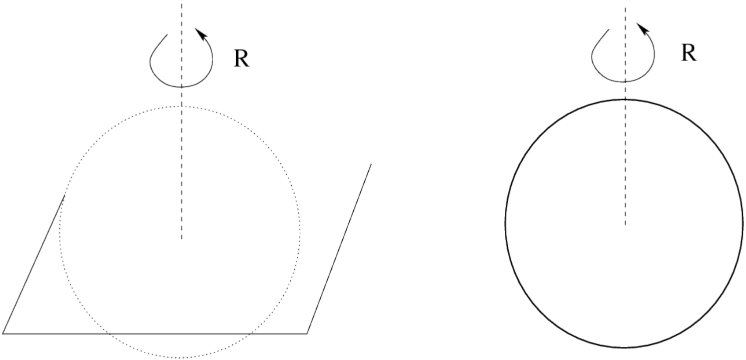


Figure: Equivariance and Riemann sphere

Equivariant wave maps

$u(t, r, \phi) = (\psi(t, r), \phi)$, spherical coordinates, ψ angle from north pole satisfies

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi, \psi_t)(0) = (\psi_0, \psi_1)$$

- Conserved energy

$$E(\psi, \psi_t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\,dr$$

- $\psi(t, \infty) = n\pi, n \in \mathbb{Z}$, homotopy class = degree = n
- stationary solutions $=$ harmonic maps $= 0, \pm Q(r/\lambda)$, where $Q(r) = 2\arctan r$. This is the identity $\mathbb{S}^2 \to \mathbb{S}^2$ with stereographic projection onto $\mathbb{R}^2$ as domain (conformal map!).

Large data results for equivariant wave maps

Theorem (Côte, Kenig, Lawrie, S. 2012)
Let (ψ_0, ψ_1) be smooth data.

1. *Let $E(\psi_0, \psi_1) < 2E(Q, 0)$, degree 0. Then the solution exists globally, and scatters (energy on compact sets vanishes as $t \to \infty$). For any $\delta > 0$ there exist data of energy $< 2E(Q, 0) + \delta$ which blow up in finite time.*

2. *Let $E(\psi_0, \psi_1) < 3E(Q, 0)$, degree 1. If the solution $\psi(t)$ blows up at time $t = 1$, then there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}$ with $E(\vec{\varphi}) = E(\vec{\psi}) - E(Q, 0)$, and a decomposition*

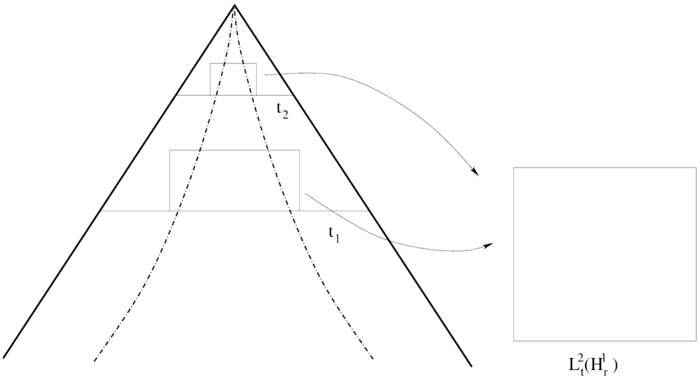
$$\vec{\psi}(t) = \vec{\varphi} + (Q(\cdot/\lambda(t)), 0) + \vec{\epsilon}(t) \quad (\star)$$

s.t. $\vec{\epsilon}(t) \in \mathcal{H}$, $\vec{\epsilon}(t) \to 0$ *in* $\mathcal{H}$ *as* $t \to 1$.

Large data results for equivariant wave maps

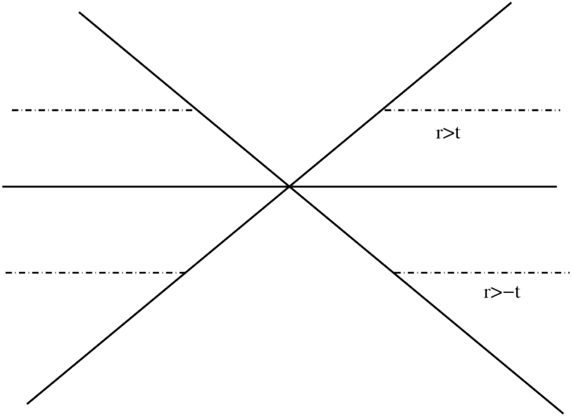
- For degree 1 have an analogous classification to $(\star)$ for global solutions.
- Côte 2013: bubble-tree classification for all energies along a sequence of times.
  Open problems: (A) all times, rather than a sequence (B) construction of bubble trees.
- Duyckaerts, Kenig, Merle 12 established classification results for $\Box u = u^5$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$ instead of Q.
- Construction of $(\star)$ by Krieger-S.-Tataru 06 in finite time, Donninger-Krieger 13 in *infinite time* (for critical NLW)
- Crucial role is played by Michael Struwe's bubbling off theorem (equivariant): if blowup happens, then there exists a sequence of times approaching blowup time, such that a rescaled version of the wave map approaches locally in energy space a harmonic map of positive energy.

Struwe's cuspidal energy concentration



Rescalings converge in $L^2_{t,r}$-sense to a stationary wave map of positive energy, i.e., a harmonic map.

Asymptotic exterior energy



$\Box u = 0,\ u(0) = f \in \dot{H}^1(\mathbb{R}^d),\ u_t(0) = g \in L^2(\mathbb{R}^d)$ radial

Duyckaerts-Kenig-Merle 2011: for all $t \geq 0$ or $t \leq 0$ have $E_{ext}(\vec{u}(t)) \geq cE(f,g)$ provided dimension odd. $c > 0$, $c = \frac{1}{2}$

Heuristics: incoming vs. outgoing data.

Exterior energy: even dimensions

Côte-Kenig-S. 2012: This **fails in even dimensions**.

$d = 2, 6, 10, \ldots$ holds for data $(0, g)$ but fails in general for $(f, 0)$.
$d = 4, 8, 12, \ldots$ holds for data $(f, 0)$ but fails in general for $(0, g)$.

Fourier representation, Bessel transform, dimension d reflected in the phase of the Bessel asymptotics, computation of the asymptotic exterior energy as $t \to \pm\infty$.

For our $3E(Q, 0)$ theorem we need $d = 4$ result; rather than $d = 2$ due to repulsive $\frac{\psi}{r^2}$-potential coming from $\frac{\sin(2\psi)}{2r^2}$.

$(f, 0)$ result suffices by Christodoulou, Tahvildar-Zadeh, Shatah results from mid 1990s. Showed that at blowup $t = T = 1$ have vanishing kinetic energy

$$\lim_{t \to 1} \frac{1}{1-t} \int_t^1 \int_0^{1-t} |\dot{\psi}(t, r)|^2 \, r dr \, dt = 0$$

No result for Yang-Mills since it corresponds to $d = 6$

Exterior energy: odd dimensions

Duyckaerts-Kenig-Merle: in radial $\mathbb{R}^3$ one has for all $R \geq 0$

$$\max_{\pm} \lim_{t\to\pm\infty} \int_{|x|>t+R} |\nabla_{t,x} u|^2 \, dr \geq c \int_{|x|>R} [(ru)_r^2 + (ru)_t^2] \, dr$$

Note: RHS is not standard energy! Orthogonal projection perpendicular to Newton potential $(r^{-1}, 0)$ in $H^1 \times L^2(\mathbb{R}^3 : r > R)$.

Kenig-Lawrie-S. 13 noted this projection and extended the exterior energy estimate to $d = 5$: project perpendicular to plane $(\xi r^{-3}, \eta r^{-3})$ in $H^1 \times L^2(\mathbb{R}^5 : r > R)$

Kenig-Lawrie-Liu-S. 14 all odd dimensions, projections off of similar but larger and more complicated linear subspaces.

Relevance: Exterior wave maps in $\mathbb{R}^3$ with arbitrary degree of equivariance lead to all odd dimensions.

Exterior wave maps

Consider equivariant wave maps from $\mathbb{R}^3 \setminus B(0,1) \to \mathbb{S}^3$ with Dirichlet condition at $R = 1$. Supercritical becomes subcritical, easy to obtain global smooth solutions.

Conjecture by Bizon-Chmaj-Maliborski 2011: All smooth solutions scatter to the unique harmonic map in their degree class.

Results:

- Lawrie-S. 2012: Proved for degree 0 and asymptotic stability for degree 1. Follows Kenig-Merle concentration compactness approach with rigidity argument carried out by a virial identity (complicated).
- Kenig-Lawrie-S. 2013: Proved for all degrees in equivariance class 1. Uses exterior energy estimates instead of virial.
- Kenig-Lawrie-Liu-S. 2014: Proved for all degrees and all equivariance classes. Requires exterior energy estimates in all odd dimensions.

Soliton resolution conjecture holds in this case.

THANK YOU FOR YOUR ATTENTION!