Long-term dynamics of nonlinear wave equations

W. Schlag (University of Chicago)

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Wave maps

Let $(M, g)$ be a Riemannian manifold, and $u : \mathbb{R}^{1+d}_{t,x} \to M$ smooth. Wave maps defined by Lagrangian

$$L(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2} (-|\partial_t u|^2_g + \sum_{j=1}^{d} |\partial_j u|^2_g) \, dt \, dx$$

Critical points $L'(u, \partial_t u) = 0$ satisfy “manifold-valued wave equation”. $M \subset \mathbb{R}^N$ embedded, this equation is

$$\Box u \perp T_u M \quad \text{or} \quad \Box u = A(u)(\partial u, \partial u),$$

$A$ being the second fundamental form.

For example, $M = S^{n-1}$, then

$$\Box u = u(|\partial_t u|^2 - |
\nabla u|^2)$$

Note: Nonlinear wave equation, null-form! Harmonic maps are solutions.
Wave maps

Intrinsic formulation: $D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0$, in coordinates

$$-\partial_{tt} u^i + \Delta u^i + \Gamma^i_{jk}(u) \partial_\alpha u^j \partial_\alpha u^k = 0$$

$\eta = (-1, 1, 1, \ldots, 1)$ Minkowski metric

- Similarity with geodesic equation: $u = \gamma \circ \varphi$ is a wave map provided $\Box \varphi = 0$, $\gamma$ a geodesic.

- Energy conservation: $E(u, \partial_t u) = \int_{\mathbb{R}^d} \left( |\partial_t u|^2_g + \sum_{j=1}^d |\partial_j u|^2_g \right) dx$ is conserved in time.

- Cauchy problem:

$$\Box u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

smooth data. Does there exist a smooth local or global-in-time solution?

Local: Yes. Global: depends on the dimension of Minkowski space and the geometry of the target.
Criticality and dimension

If \( u(t, x) \) is a wave map, then so is \( u(\lambda t, \lambda x), \forall \lambda > 0 \).

Data in the Sobolev space \( \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d) \). For which \( s \) is this space invariant under the natural scaling? Answer: \( s = \frac{d}{2} \).

Scaling of the energy: \( u(t, x) \mapsto \lambda \frac{d-2}{2} u(\lambda t, \lambda x) \) same as \( \dot{H}^1 \times L^2 \).

- **Subcritical case:** \( d = 1 \) the natural scaling is associated with less regularity than that of the conserved energy. Expect global existence. Logic: local time of existence only depends on energy of data, which is preserved.

- **Critical case:** \( d = 2 \). Energy keeps the balance with the natural scaling of the equation. For \( \mathbb{S}^2 \) can have finite-time blowup, whereas for \( \mathbb{H}^2 \) have global existence. Krieger-S.-Tataru 06, Krieger-S. 09, Rodnianski-Raphael 09, Sterbenz-Tataru 09.

- **Supercritical case:** \( d \geq 3 \). Poorly understood. Self-similar blowup \( Q(r/t) \) for sphere as target, Shatah 80s. Also negatively curved manifolds possible in high dimensions: Cazenave, Shatah, Tahvildar-Zadeh 98.
Wellposedness of Wave Maps

- Energy methods: $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ with $s > \frac{n}{2} + 1$.

- Klainerman-Machedon, Klainerman-Selberg 1990s: $X^{s,b}$ spaces which use symbols of both $\Delta, \Box$, bilinear estimates involving null forms of Christodoulou, Klainerman gives $s > \frac{n}{2}$.

- Push to $s = \frac{n}{2}$ since then local=global. Tataru 1998 introduced null frame spaces and achieved small data global regularity, Tao 2000 employed gauge invariance and obtained the desired small energy result, with $d = 2$ the hardest case. Shatah-Struwe 2003: simpler proof, Coulomb gauge in $d \geq 4$.

- Large data have dichotomy between blowup/global regularity. For the latter use induction on energy (Bourgain 1990s), for example via Kenig-Merle 2007 concentration compactness (Krieger-S. 2009). Implementation is very complicated: gauge, Tataru/Tao spaces, no linear profile extraction possible (manifold valued functions, no superposition principle).

- Equivariant case (discussed later) is more accessible, Christodoulou, Shatah, Tahvildar-Zadeh, Struwe 1990s made fundamental contributions. Many open problems remain for the non-equivariant case (more about this later).

- Next: Concentration-compactness illustrated by a more elementary semi-linear model.
A nonlinear defocusing Klein-Gordon equation

Consider in $\mathbb{R}^{1+3}_{t,x}$

$$\Box u + u + u^3 = 0, \quad (u(0), \dot{u}(0)) = (f, g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3)$$

Conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\dot{u}|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 + \frac{1}{4} |u|^4 \right) dx$$

With $S(t)$ the linear propagator of $\Box + 1$ we have

$$\ddot{u}(t) = (u, \dot{u})(t) = S(t)(f, g) - \int_0^t S(t-s)(0, u^3(s)) ds$$

whence by a simple energy estimate, $I = (0, T)$

$$\|\ddot{u}\|_{L^\infty(I; \mathcal{H})} \lesssim \|(f, g)\|_{\mathcal{H}} + \|u^3\|_{L^1(I; L^2)} \lesssim \|(f, g)\|_{\mathcal{H}} + \|u\|_{L^3(I; L^6)}^3$$

$$\lesssim \|(f, g)\|_{\mathcal{H}} + T\|\ddot{u}\|_{L^\infty(I; \mathcal{H})}$$

Contraction for small $T$ implies local wellposedness for $\mathcal{H}$ data.
Defocusing NLKG3

\( T \) depends only on \( \mathcal{H} \)-size of data. From energy conservation we obtain global existence by time-stepping.

Scattering (as in linear theory): \( \| \tilde{u}(t) - \tilde{v}(t) \|_{\mathcal{H}} \to 0 \) as \( t \to \infty \) where \( \Box v + v = 0 \) energy solution.

\[
\tilde{v}(0) := \tilde{u}(0) - \int_0^\infty S(-s)(0, u^3)(s) \, ds \quad \text{provided} \quad \|u^3\|_{L^1_t L^2_x} < \infty
\]

Strichartz estimate uniformly in intervals \( I \)

\[
\|\tilde{u}\|_{L^\infty(I; \mathcal{H})} + \|u\|_{L^3(I; L^6)} \leq \|(f, g)\|_{\mathcal{H}} + \|u\|_{L^3(I; L^6)}^3
\]

Small data scattering: \( \|\tilde{u}\|_{L^3(I; L^6)} \leq \|(f, g)\|_{\mathcal{H}} \ll 1 \) for all \( I \). So \( I = \mathbb{R} \) as desired.

Large data scattering valid; induction on energy, concentration compactness (Bourgain, Bahouri-Gerard, Kenig-Merle).
Let $\tilde{u}$ be nonlinear solution with data $(u_0, u_1) \in \mathcal{H}$. Forward scattering set

$$S_+ = \{(u_0, u_1) \in \mathcal{H} | \tilde{u}(t) \text{ exists globally, scatters as } t \to +\infty\}$$

We claim that $S_+ = \mathcal{H}$. This is proved via the following outline:

- **(Small data result):** $\|(u_0, u_1)\|_\mathcal{H} < \varepsilon$ implies $(u_0, u_1) \in S_+$

- **(Concentration Compactness):** If scattering fails, i.e., if $S_+ \neq \mathcal{H}$, then construct $\tilde{u}_*$ of minimal energy $E_* > 0$ for which $\|u_*\|_{L^3_t L^6_x} = \infty$. There exists $x(t)$ so that the trajectory

$$K_+ = \{\tilde{u}_*(\cdot - x(t), t) | t \geq 0\}$$

is pre-compact in $\mathcal{H}$.

- **(Rigidity Argument):** If a forward global evolution $\tilde{u}$ has the property that $K_+$ pre-compact in $\mathcal{H}$, then $u \equiv 0$.

Let \( \{u_n\}_{n=1}^{\infty} \) free Klein-Gordon solutions in \( \mathbb{R}^3 \) s.t.
\[
\sup_n \|\vec{u}_n\|_{L_\infty^t H} < \infty
\]

\exists \text{ free solutions } v^j \text{ bounded in } \mathcal{H}, \text{ and } (t^i_n, x^i_n) \in \mathbb{R} \times \mathbb{R}^3 \text{ s.t.}
\[
u_n(t, x) = \sum_{1 \leq j < J} v^j(t + t^i_n, x + x^i_n) + w^j_n(t, x)
\]
satisfies \( \forall \ j < J, \tilde{w}^j_n(-t^i_n, -x^i_n) \to 0 \) in \( \mathcal{H} \) as \( n \to \infty \), and

- \( \lim_{n \to \infty} (|t^j_n - t^k_n| + |x^j_n - x^k_n|) = \infty \ \forall \ \ j \neq k \)

- dispersive errors \( w^k_n \) vanish asymptotically:
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|w^j_n\|_{(L_\infty^t L^p_x \cap L_6^t L_6^x)(\mathbb{R} \times \mathbb{R}^3)} = 0 \ \forall \ 2 < p < 6
\]

- orthogonality of the energy:
\[
\|\vec{u}_n\|^2_{\mathcal{H}} = \sum_{1 \leq j < J} \|\vec{v}^j\|^2_{\mathcal{H}} + \|\tilde{w}^j_n\|^2_{\mathcal{H}} + o(1)
\]
Profiles and Strichartz sea

We can extract further profiles from the Strichartz sea if $w_n^4$ does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n \to \infty} ||w_n^4||_{L^\infty_t L_x^p(\mathbb{R}^3)} > 0$. 
Lorentz transformations

\[
\begin{bmatrix}
    t' \\
    x'_1 \\
    x'_2 \\
    x'_3
\end{bmatrix}
= \begin{bmatrix}
    \cosh \alpha & \sinh \alpha & 0 & 0 \\
    \sinh \alpha & \cosh \alpha & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    t \\
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
\]

Figure: Causal structure of space-time
Further remarks on Bahouri-Gérard

- **Noncompact symmetry groups:** space-time translations and Lorentz transforms.

  Compact symmetry groups: Rotations
  Lorentz transforms do not appear in the profiles: Energy bound compactifies them.

- **Dispersive error** $w_n^j$ is not an energy error!

- In the **radial case** only need time translations
The focusing NLKG equation

The focusing NLKG

\[ \Box u + u = \partial_{tt} u - \Delta u + u = u^3 \]

has indefinite conserved energy

\[ E(u, \dot{u}) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\dot{u}|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 - \frac{1}{4} |u|^4 \right) dx \]

- Local wellposedness for \( H^1 \times L^2(\mathbb{R}^3) \) data
- Small data: global existence and scattering
- Finite time blowup \( u(t) = \sqrt{2} (T - t)^{-1} (1 + o(1)) \) as \( t \rightarrow T^- \)
  Cutoff to a cone using finite propagation speed to obtain finite energy solution.
- Stationary solutions \(-\Delta \varphi + \varphi = \varphi^3\), ground state \( Q(r) > 0 \)
Payne-Sattinger theory; saddle structure of energy near \( Q \)

Criterion: finite-time blowup/global existence?
Yes, provided the energy is less than the ground state energy Payne-Sattinger 1975.

\[
J := J(u) > J(Q) =: J_0
\]

\[
K := K(\varphi) = \int_{\mathbb{R}^3} \left( |\nabla \varphi|^2 + |\varphi|^2 - |\varphi|^4 \right) dx
\]

\[
J(\varphi) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} |\varphi|^2 - \frac{1}{4} |\varphi|^4 \right) dx
\]

Uniqueness of \( Q \) is the foundation!
Payne-Sattinger theory

\[ j_\varphi(\lambda) := J(e^\lambda \varphi), \varphi \neq 0 \text{ fixed.} \]

Normalize so that \( \lambda_\ast = 0 \). Then \( \partial_\lambda j_\varphi(\lambda) \bigg|_{\lambda=\lambda_\ast} = K(\varphi) = 0 \).

“Trap” the solution in the well on the left-hand side: need \( E < \inf \{ j_\varphi(0) \mid K(\varphi) = 0, \varphi \neq 0 \} = J(Q) \) (lowest mountain pass). Expect global existence in that case.
Above the ground state energy

Theorem (Nakanishi-S. 2010)
Let \( E(u_0, u_1) < E(Q, 0) + \varepsilon^2 \), \((u_0, u_1) \in \mathcal{H}_{rad}\). In \( t \geq 0 \) for NLKG:

1. finite time blowup
2. global existence and scattering to 0
3. global existence and scattering to \( Q \): \( u(t) = Q + v(t) + o_{\mathcal{H}^1}(1) \) as \( t \to \infty \),
   and \( \dot{u}(t) = \dot{v}(t) + o_{L^2}(1) \) as \( t \to \infty \), \( \Box v + v = 0 \), \((v, \dot{v}) \in \mathcal{H}\).

All 9 combinations of this trichotomy allowed as \( t \to \pm \infty \).

- Applies to \( \dim = 3, |u|^{p-1}u, 7/3 < p < 5 \), or \( \dim = 1, p > 5 \).

- Third alternative forms the center stable manifold associated with \((\pm Q, 0)\). Linearized operator \( L_+ = -\Delta + 1 - 3Q^2 \) has spectrum \( \{-k^2\} \cup [1, \infty) \) on \( L^2_{rad}(\mathbb{R}^3) \). Gap \([0, 1)\) difficult to verify, Costin-Huang-S., 2011.

- \( \exists 1\)-dim. stable, unstable manifolds at \((\pm Q, 0)\). Stable manifolds:
  Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko 2009
The invariant manifolds

**Figure:** Stable, unstable, center-stable manifolds
Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$

- $\langle L_+ Q|Q \rangle = -2\|Q\|^4_4 < 0$
- $L_+\rho = -k^2\rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: $L_+$ has no eigenvalues in $(0, 1]$, no threshold resonance (delicate!) Use Kenji Yajima’s $L^p$-boundedness for wave operators.

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}$$

Then $\text{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ simple evals. Formally:

$X_s = P_1L^2$, $X_u = P_{-1}L^2$, $X_c$ is the rest.
Spectrum of matrix Hamiltonian

Figure: Spectrum of nonselfadjoint linear operator in phase space
Numerical 2-dim section through $\partial S_+$ (with R. Donninger)

Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, $(A, B)$ vary in $[-9, 2] \times [-9, 9]$
- **RED**: global existence, **WHITE**: finite time blowup, **GREEN**: $\mathcal{PS}_+$, **BLUE**: $\mathcal{PS}_-$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)
Variational structure above $E(Q, 0)$

$E := E(u, u) > J(Q) + \varepsilon^2 =: J$

- Solution can pass through the balls. Energy is no obstruction anymore as in the Payne-Sattinger case.

- Key to description of the dynamics: One-pass (no return) theorem. The trajectory can make only one pass through the balls.

- Point: Stabilization of the sign of $K(u(t))$. 
Such trajectories are excluded by means of an indirect argument using a variant of the virial argument that was essential to the rigidity step of concentration compactness.
**One-pass theorem**

**Crucial no-return property:** Trajectory does **not return to balls around** \((\pm Q, 0)\). Suppose it did; Use **virial identity**

\[
\partial_t \langle w \dot{u} \mid Au \rangle = - \int_{\mathbb{R}^3} (|\nabla u|^2 - \frac{3}{4} |u|^4) \, dx + \text{error}, \quad A = \frac{1}{2} (x \nabla + \nabla x)
\]

where \(w = w(t, x)\) is a space-time cutoff that lives on a rhombus, and the “error” is controlled by the **external energy**.

Finite propagation speed \(\Rightarrow\) error controlled by **free energy outside large balls** at times \(T_1, T_2\).
Integrating between \(T_1, T_2\) gives **contradiction**; the **bulk** of the integral of \(K_2(u(t))\) here comes from exponential ejection mechanism near \((\pm Q, 0)\).

**Non-perturbative argument.**
Figure: Space-time cutoff for the virial identity
Complete description of possible long-term dynamics: Given focusing NLKG3 in $\mathbb{R}^3$ with radial energy data, show that the solution either

- blows up in finite time
- exists globally, scatters to one of the stationary solutions $-\Delta \varphi + \varphi = \varphi^3$ (including 0)

Moreover, describe dynamics, center-stable manifolds associated with $\varphi$.

Evidence: With dissipation given by $-\alpha \partial_t u$ term, result holds (Burq-Raugel-S.).

Critical equation: $\Box u = u^5$ in $\mathbb{R}^3$, Duyckaerts-Kenig-Merle proved analogous result with rescaled ground-state profiles $\sqrt{\lambda} W(\lambda x)$, $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$.

Obstruction: Exterior energy estimates in DKM scheme fail in the KG case due to speed of propagation $< 1$. 
Center-stable manifold, $u^5$ critical equation

Nakanishi-S theorem applies to nonradial NLKG, NLS, different subcritical powers and dimensions. Critical equations exhibit similar, yet qualitatively essentially different phenomena due to scaling symmetry.

$$\ddot{u} - \Delta u = |u|^{2^* - 2}u, \quad u(t, x): \mathbb{R}^{1+d} \to \mathbb{R}, \quad 2^* = \frac{2d}{d - 2},$$

Static Aubin, Talenti solutions

$$W_\lambda = T_\lambda W, \quad W(x) = \left[1 + \frac{|x|^2}{d(d - 2)}\right]^{1 - \frac{d}{2}},$$

$T_\lambda$ is $\dot{H}^1$ preserving dilation

$$T_\lambda \varphi = \lambda^{d/2 - 1} \varphi(\lambda x)$$

Positive radial solutions $-\Delta W - |W|^{2^* - 2}W = 0$. Functionals:

$$J(\varphi) := \int_{\mathbb{R}^d} \left[\frac{1}{2} |\nabla \varphi|^2 - \frac{1}{2^*} |\varphi|^{2^*}\right] dx, \quad K(\varphi) := \int_{\mathbb{R}^d} [|
abla \varphi|^2 - |\varphi|^{2^*}] dx$$
Critical wave equation

Radial $\dot{H}^1 \times L^2$, $E(\varphi) < J(W) + \varepsilon^2$, outside soliton tube

$$S = \{ \pm \tilde{W}_\lambda \mid \lambda > 0 \} + O(\varepsilon)$$

There exists four open disjoint sets which lead to all combinations of FTB/GE and scattering to 0 as $t \to \pm l$.

- Krieger-Nakanishi-S. 2013: complete description of all solutions with energy $E(\varphi) < J(W) + \varepsilon^2$. Type-I conjecture!

- Center-stable manifold exists in $\dot{H}^1 \times L^2$, contains all $W_\lambda$, solutions with $\lambda \to 0, \infty$ (but Krieger-S. 05 showed that in a stronger non-invariant topology exists codim-1 manifold with global solutions, $\lambda(t) \to \lambda_* \in (0, \infty)$).

- Inside the soliton tube there exist blowup solutions, as found by Krieger-S.-Tataru 06. Then Duyckaerts-Kenig-Merle 09 showed that all type II blowup are of the KST form, as long as energy below $2J(Q)$. So trapping by the soliton tube cannot mean scattering to $\{W_\lambda\}$ as it did in the subcritical case.
Equivariant wave maps

\( u : \mathbb{R}^{1+2}_{t,x} \to S^2 \) satisfies WM equation

\[ \Box u \perp T_u S^2 \iff \Box u = u(|\partial_t u|^2 - |\nabla u|^2) \]

as well as equivariance assumption \( u \circ R = R \circ u \) for all \( R \in SO(2) \)

Figure: Equivariance and Riemann sphere
Equivariant wave maps

\( u(t, r, \phi) = (\psi(t, r), \phi) \), spherical coordinates, \( \psi \) angle from north pole satisfies

\[
\psi_{tt} - \psi_{rr} - \frac{1}{r} \psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi, \psi_t)(0) = (\psi_0, \psi_1)
\]

- Conserved energy

\[
E(\psi, \psi_t) = \int_0^\infty \left( \psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2} \right) r \, dr
\]

- \( \psi(t, \infty) = n\pi, n \in \mathbb{Z} \), homotopy class = degree = \( n \)

- Stationary solutions = harmonic maps = 0, \( \pm Q(r/\lambda) \), where \( Q(r) = 2 \arctan r \). This is the identity \( \mathbb{S}^2 \to \mathbb{S}^2 \) with stereographic projection onto \( \mathbb{R}^2 \) as domain (conformal map!).
Large data results for equivariant wave maps

Theorem (Côte, Kenig, Lawrie, S. 2012)

Let \((\psi_0, \psi_1)\) be smooth data.

1. Let \(E(\psi_0, \psi_1) < 2E(Q, 0)\), degree 0. Then the solution exists globally, and scatters (energy on compact sets vanishes as \(t \to \infty\)). For any \(\delta > 0\) there exist data of energy \(< 2E(Q, 0) + \delta\) which blow up in finite time.

2. Let \(E(\psi_0, \psi_1) < 3E(Q, 0)\), degree 1. If the solution \(\psi(t)\) blows up at time \(t = 1\), then there exists a continuous function, \(\lambda : [0, 1) \to (0, \infty)\) with \(\lambda(t) = o(1 - t)\), a map \(\vec{\varphi} = (\varphi_0, \varphi_1) \in H\) with \(E(\vec{\varphi}) = E(\vec{\psi}) - E(Q, 0)\), and a decomposition

   \[
   \vec{\psi}(t) = \vec{\varphi} + (Q (\cdot / \lambda(t)), 0) + \vec{\epsilon}(t) \quad (\star)
   \]

   s.t. \(\vec{\epsilon}(t) \in H\), \(\vec{\epsilon}(t) \to 0\) in \(H\) as \(t \to 1\).
Large data results for equivariant wave maps

- For degree 1 have an analogous classification to (⋆) for global solutions.

- Côte 2013: bubble-tree classification for all energies along a sequence of times.
  Open problems: (A) all times, rather than a sequence (B) construction of bubble trees.

- Duyckaerts, Kenig, Merle 12 established classification results for \( \Box u = u^5 \) in \( \dot{H}^1 \times L^2(\mathbb{R}^3) \) with \( W(x) = (1 + |x|^2/3)^{-\frac{1}{2}} \) instead of \( Q \).

- Construction of (⋆) by Krieger-S.-Tataru 06 in finite time, Donninger-Krieger 13 in infinite time (for critical NLW)

- Crucial role is played by Michael Struwe’s bubbling off theorem (equivariant): if blowup happens, then there exists a sequence of times approaching blowup time, such that a rescaled version of the wave map approaches locally in energy space a harmonic map of positive energy.
Struwe’s cuspidal energy concentration

Rescalings converge in $L_{t,r}^2$-sense to a stationary wave map of positive energy, i.e., a harmonic map.
\[ \Box u = 0, \ u(0) = f \in \dot{H}^1(\mathbb{R}^d), \ u_t(0) = g \in L^2(\mathbb{R}^d) \] radial

Duyckaerts-Kenig-Merle 2011: for all \( t \geq 0 \) or \( t \leq 0 \) have \( E_{\text{ext}}(\bar{u}(t)) \geq cE(f, g) \) provided dimension odd. \( c > 0, \ c = \frac{1}{2} \)

Heuristics: incoming vs. outgoing data.
Exterior energy: even dimensions

Côte-Kenig-S. 2012: This **fails in even dimensions**.

d = 2, 6, 10, ... holds for data (0, g) but fails in general for (f, 0).
d = 4, 8, 12, ... holds for data (f, 0) but fails in general for (0, g).

Fourier representation, Bessel transform, dimension d reflected in the phase of the Bessel asymptotics, computation of the asymptotic exterior energy as \( t \to \pm \infty \).

For our \( 3E(Q, 0) \) theorem we need \( d = 4 \) result; rather than \( d = 2 \) due to repulsive \( \frac{\psi}{r^2} \)-potential coming from \( \frac{\sin(2\psi)}{2r^2} \).

(\( f, 0 \)) result suffices by Christodoulou, Tahvildar-Zadeh, Shatah results from mid 1990s. Showed that at blowup \( t = T = 1 \) have vanishing kinetic energy

\[
\lim_{t \to 1} \frac{1}{1-t} \int_t^1 \int_0^{1-t} |\dot{\psi}(t, r)|^2 r dr dt = 0
\]

No result for Yang-Mills since it corresponds to \( d = 6 \)
Duyckaerts-Kenig-Merle: in radial $\mathbb{R}^3$ one has for all $R \geq 0$

$$\max_{\pm} \lim_{t \to \pm \infty} \int_{|x| > t + R} |\nabla_{t,x} u|^2 \, dr \geq c \int_{|x| > R} \left[ (ru)_r^2 + (ru)_t^2 \right] \, dr$$

**Note:** RHS is not standard energy! Orthogonal projection perpendicular to Newton potential $(r^{-1}, 0)$ in $H^1 \times L^2(\mathbb{R}^3 : r > R)$.

Kenig-Lawrie-S. 13 noted this projection and extended the exterior energy estimate to $d = 5$: project perpendicular to plane $(\xi r^{-3}, \eta r^{-3})$ in $H^1 \times L^2(\mathbb{R}^5 : r > R)$

Kenig-Lawrie-Liu-S. 14 all odd dimensions, projections off of similar but larger and more complicated linear subspaces.

**Relevance:** Exterior wave maps in $\mathbb{R}^3$ with arbitrary degree of equivariance lead to all odd dimensions.
Exterior wave maps

Consider equivariant wave maps from $\mathbb{R}^3 \setminus B(0, 1) \rightarrow S^3$ with Dirichlet condition at $R = 1$. Supercritical becomes subcritical, easy to obtain global smooth solutions.

Conjecture by Bizon-Chmaj-Maliborski 2011: All smooth solutions scatter to the unique harmonic map in their degree class.

Results:

- **Lawrie-S. 2012**: Proved for degree 0 and asymptotic stability for degree 1. Follows Kenig-Merle concentration compactness approach with rigidity argument carried out by a virial identity (complicated).

- **Kenig-Lawrie-S. 2013**: Proved for all degrees in equivariance class 1. Uses exterior energy estimates instead of virial.

- **Kenig-Lawrie-Liu-S. 2014**: Proved for all degrees and all equivariance classes. Requires exterior energy estimates in all odd dimensions.

Soliton resolution conjecture holds in this case.
THANK YOU FOR YOUR ATTENTION!