Invariant Manifolds and dispersive Hamiltonian Evolution Equations

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Old-fashioned string theory

How does a guitar string evolve in time?

- Ancient Greece: observed that musical intervals such as an octave, a fifth etc. were based on integer ratios.
- Post Newton: mechanistic model, use calculus and $F = ma$.
- Assume displacement $u = u(t, x)$ is small. Force proportional to curvature: $F = ku_{xx}$.

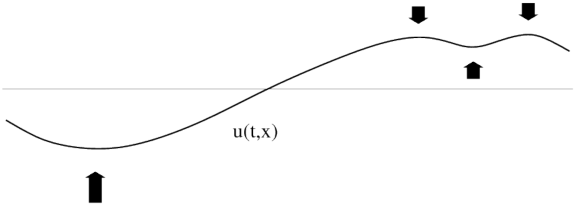


Figure: Forces acting on pieces of string

- Dynamical law $u_{tt} = c^2 u_{xx}$. Write as $\Box u = 0$.
- This is an idealization, or model!

Solving for the string

Cauchy problem:

$$\Box u = 0, u(0) = f, \partial_t u(0) = g$$

d'Alembert solution:

$$(\partial_t^2 - c^2\partial_x^2)u = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

Reduction to first order, transport equations

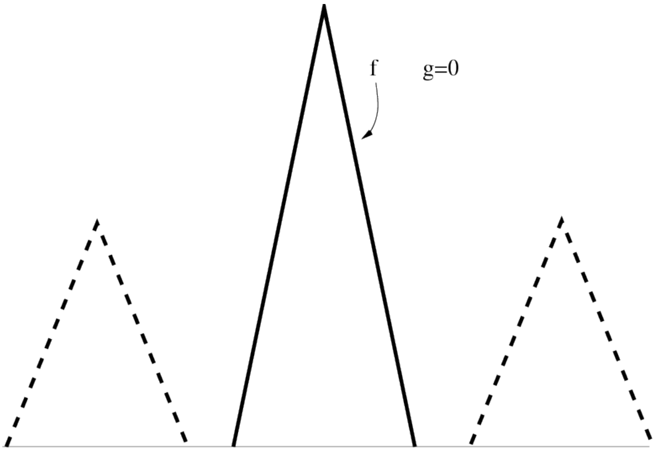
$$u_t + cu_x = 0 \Leftrightarrow u(t,x) = \varphi(x - ct)$$
$$u_t - cu_x = 0 \Leftrightarrow u(t,x) = \psi(x + ct)$$

Adjust for initial conditions, gives d'Alembert formula:

$$u(t,x) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

If $g = 0$, the initial position f splits into left- and right-moving waves.

d'Alembert solution



Standing waves

Clamped string: $u(t,0) = u(t,L) = 0$ for all $t \geq 0$, $\Box u = 0$.

Special solutions (with $c = 1$) with $n \geq 1$ an integer

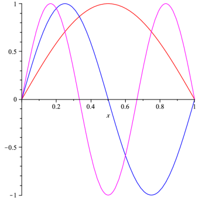
$$u_n(t,x) = \sin(\pi n x/L)\Big[a_n \sin(\pi n t/L) + b_n \cos(\pi n t/L)\Big]$$

Fourier's claim: All solutions are superpositions of these!

$\Omega \subset \mathbb{R}^d$ bounded domain, or compact manifold. Let $-\Delta_\Omega \varphi_n = \lambda_n^2 \varphi_n$, with Dirichlet boundary condition in the former case. Then

$$u(t,x) = \sum_{n \geq 0, \pm} c_{n,\pm} e^{\pm i \lambda_n t} \varphi_n(x)$$

solves $\Box u = 0$ (with boundary condition).



Drum membranes

Two-dimensional waves on a drum: $u_{tt} - \Delta u = 0$ with $u = 0$ on the boundary.

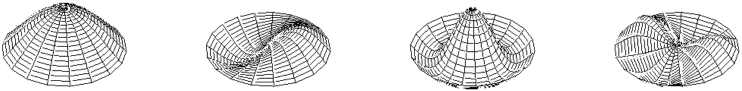


Figure: Four basic harmonics of the drum

First, third pictures $u(t, r) = \cos(t\lambda) J_0(\lambda r)$, where $J_0(\lambda) = 0$.
Second, fourth pictures $u(t, r) = \cos(t\mu) J_m(\mu r) \cos(m\theta)$, where $J_m(\mu) = 0$. The J_m are Bessel functions.

Electrification of waves

Maxwell's equations: $\mathbf{E}(t,x)$ and $\mathbf{B}(t,x)$ vector fields

$$\operatorname{div}\mathbf{E} = \varepsilon_0^{-1}\rho, \quad \operatorname{div}\mathbf{B} = 0$$
$$\operatorname{curl}\mathbf{E} + \partial_t \mathbf{B} = 0, \quad \operatorname{curl}\mathbf{B} - \mu_0\varepsilon_0\partial_t\mathbf{E} = \mu_0\mathbf{J}$$

ε_0 electric constant, μ_0 magnetic constant, ρ charge density, $\mathbf{J}$ current density.
In vaccum $\rho = 0, \mathbf{J} = 0$. Differentiate fourth equation in time:

$$\operatorname{curl}\mathbf{B}_t - \mu_0\varepsilon_0\mathbf{E}_{tt} = 0 \Rightarrow \operatorname{curl}(\operatorname{curl}\mathbf{E}) + \mu_0\varepsilon_0\mathbf{E}_{tt} = 0$$
$$\nabla(\operatorname{div}\mathbf{E}) - \Delta\mathbf{E} + \mu_0\varepsilon_0\mathbf{E}_{tt} = 0 \Rightarrow \mathbf{E}_{tt} - c^2\Delta\mathbf{E} = 0$$

Similarly $\mathbf{B}_{tt} - c^2\Delta\mathbf{B} = 0$.
In 1861 Maxwell noted that c is the speed of light, and concluded that light should be an electromagnetic wave! Wave equation appears as a fundamental equation! Loss of Galilei invariance!

Visualization of EM fields

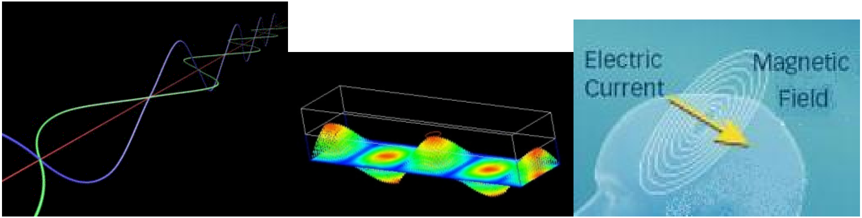


Figure: *E*&*B* fields

Least action

Principle of Least Action: Paths $(x(t), \dot{x}(t))$, for $t_0 \le t \le t_1$ with endpoints $x(t_0) = x_0$, and $x(t_1) = x_1$ fixed. The physical path determined by kinetic energy $K(x, \dot{x})$ and potential energy $P(x, \dot{x})$ minimizes the action:

$$S := \int_{t_0}^{t_1} (K - P)(x(t), \dot{x}(t))\, dt = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))\, dt$$

with L the Lagrangian.

In fact: equations of motion equal Euler-Lagrange equation

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0$$

and the physical trajectories are the critical points of S.

For $L = \frac{1}{2} m\dot{x}^2 - U(x)$, we obtain $m\ddot{x}(t) = -U'(x(t))$, which is Newton's $F = ma$.

Waves from a Lagrangian

Let

$$\mathcal{L}(u,\partial_t u) := \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\Big(-u_t^2 + |\nabla u|^2\Big)(t,x)\, dtdx \tag{1}$$

Substitute $u = u_0 + \varepsilon v$. Then

$$\mathcal{L}(u,\partial_t u) = \mathcal{L}_0 + \varepsilon \int_{\mathbb{R}^{1+d}_{t,x}} (\Box u_0)(t,x) v(t,x)\, dtdx + O(\varepsilon^2)$$

where $\Box = \partial_{tt} - \Delta$. In other words, u_0 is a critical point of $\mathcal{L}$ if and only if $\Box u_0 = 0$.

Significance:

- Underlying symmetries $\Rightarrow$ invariances $\Rightarrow$ Conservation laws
  Conservation of energy, momentum, angular momentum
- Lagrangian formulation has a universal character, and is flexible, versatile.

Wave maps

Let (M, g) be a Riemannian manifold, and $u : \mathbb{R}^{1+d}_{t,x} \to M$ smooth.
What is a wave into M? Lagrangian

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\Big(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2\Big)\, dtdx$$

Critical points $\mathcal{L}'(u, \partial_t u) = 0$ satisfy "manifold-valued wave equation". $M \subset \mathbb{R}^N$ imbedded, this equation is $\Box u \perp T_u M$ or $\Box u = A(u)(\partial u, \partial u)$, A being the second fundamental form.
For example, $M = S^{n-1}$, then

$$\Box u = u(-|\partial_t u|^2 + |\nabla u|^2)$$

Note: Nonlinear wave equation, null-form! Harmonic maps are solutions. Intrinsic formulation: $D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0$, in coordinates

$$-u^i_{tt} + \Delta u^i + \Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0$$

$\eta = (-1, 1, 1, \ldots, 1)$ Minkowski metric

Maxwell from Lagrangian

To formulate electro-magnetism in a Lagrangian frame work, introduce vector potential: $A = (A_0, \mathbf{A})$ with

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \mathbf{E} = \nabla A_0 - \partial_t \mathbf{A}$$

Define curvature tensor $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$
Maxwell's equations: $\partial^\alpha F_{\alpha\beta} = 0$. Lagrangian:

$$\mathcal{L} = \int_{\mathbb{R}^{1+3}_{t,x}} \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \, dtdx$$

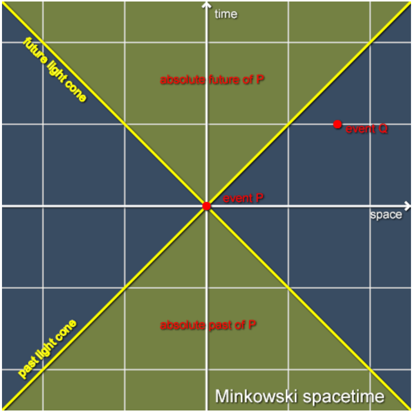
Lorentz invariance: Minkowski metric

$$[x, y] := \eta^{\alpha\beta} x_\alpha y_\beta = -x_0 y_0 + \Sigma_{j=1}^3 x_j y_j$$

Linear maps $S : \mathbb{R}^4 \to \mathbb{R}^4$ with $[Sx, y] = [x, y]$ for all $x, y \in \mathbb{R}^{1+3}_{t,x}$ are called Lorentz transforms. Note: $\Box u = 0 \Leftrightarrow \Box(u \circ S) = 0$.
For $\mathcal{L}$: $\xi = (t, x) \mapsto \eta = (s, y)$, $F_{\alpha\beta} \mapsto \tilde{F}_{\alpha'\beta'} = F_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial \eta^{\alpha'}} \frac{\partial \xi^\beta}{\partial \eta^{\beta'}}$.

Lorentz transformations 1

$$\begin{bmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



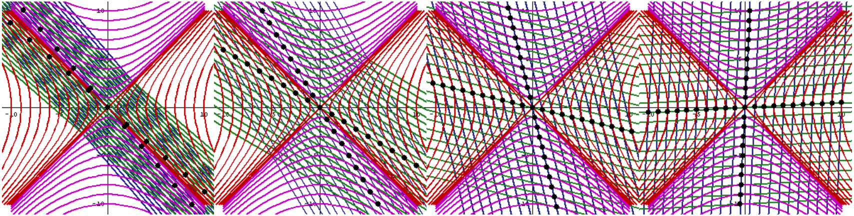


Figure: Snapshots of Lorentz transforms

Lorentz transforms (hyperbolic rotations) are for the d'Alembertian what Euclidean rotations are for the Laplacian.

Gauge invariance

We obtain the same **E**, **B** fields after $A \mapsto A + (\phi_t, \nabla\phi)$.

$$\mathbf{B} = \operatorname{curl}(\mathbf{A} + \nabla\phi) = \operatorname{curl}(\mathbf{A})$$
$$\mathbf{E} = \nabla(A_0 + \phi_t) - \partial_t(\mathbf{A} + \nabla\phi) = \nabla A_0 - \partial_t \mathbf{A}$$

Curvature $F_{\alpha\beta}$ invariant under such gauge transforms.
Impose a gauge: $\partial^\alpha A_\alpha = 0$ (Lorentz), $\operatorname{div}\mathbf{A} = 0$ (Coulomb). These pick out a unique representative in the equivalence class of vector potentials.
Make Klein-Gordon equation $\Box u - m^2 u = \partial_\alpha \partial^\alpha u - m^2 u = 0$ gauge invariant: $u \mapsto e^{i\varphi} u$ with $\varphi = \varphi(t,x)$ does not leave solutions invariant. How to modify? KG-Lagrangian is

$$\mathcal{L}_0 := \int_{\mathbb{R}^{1+3}_{t,x}} \frac{1}{2}\Big(\partial_\alpha u \overline{\partial^\alpha u} + m^2 |u|^2\Big)\, dtdx$$

Need to replace ∂_α with $D_\alpha = \partial_\alpha - iA_\alpha$. Bad choice:

$$\mathcal{L}_1 := \int_{\mathbb{R}^{1+3}_{t,x}} \frac{1}{2}\Big(D_\alpha u \overline{D^\alpha u} + m^2 |u|^2\Big)\, dtdx$$

Maxwell-Klein-Gordon system

How is A_α determined? Need to add a piece to the Lagrangian to rectify that: a "simple" and natural choice is the Maxwell Lagrangian. So obtain

$$\mathcal{L}_{MKG} := \int_{\mathbb{R}^{1+3}_{t,x}} \Big(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} D_\alpha u \overline{D^\alpha u} + \frac{m^2}{2} |u|^2\Big)\, dtdx$$

Dynamical equations, as Euler-Lagrange equation of $\mathcal{L}_{MKG}$:

$$\partial^\alpha F_{\alpha\beta} = \mathrm{Im}\,(\phi\, \overline{D_\beta \phi})$$
$$D^\alpha D_\alpha \phi - m^2 \phi = 0$$

Coupled system, Maxwell with current $J_\beta = \mathrm{Im}\,(\phi\, \overline{D_\beta \phi})$ which is determined by scalar field ϕ. Lorentz and $U(1)$ gauge invariant. Maxwell-Klein-Gordon system.

Nonabelian gauge theory: $\mathcal{G}$ Lie (matrix) group, Lie algebra $\mathfrak{g}$.
Connection 1-form: $A = A_\alpha\, dx^\alpha$ with $A_\alpha : \mathbb{R}^{1+d} \to \mathfrak{g}$.
Covariant differentiation: $D_\alpha = \partial_\alpha + A_\alpha$.
Gauge transform: $\tilde{A}_\alpha = GA_\alpha G^{-1} - (\partial_\alpha G)G^{-1}$.
Curvature is gauge invariant:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = D_\alpha D_\beta - D_\beta D_\alpha - D_{[\partial_\alpha, \partial_\beta]}$$

Yang-Mills equation (nonlinear!):

$$\mathcal{L} := \int_{\mathbb{R}^{1+d}} \frac{1}{4}\mathrm{trace}(F_{\alpha\beta}F^{\alpha\beta})\, dtdx, \qquad D^\alpha F_{\alpha\beta} = 0$$

In Lorentz gauge $-\partial_t A_0 + \sum_{j=1}^3 \partial_j A_j = 0$ one has schematically

$$\Box A = [A, \nabla A] + [A, [A, A]]$$

Eardley-Moncrief: global existence, Klainerman-Machedon: same in energy topology, null-forms!

Invariances and conservation laws: Noether's theorem

Scalar field φ, Lagrangian

$$\mathcal{L} := \int_{\mathbb{R}^{1+d}_{t,x}} L(\varphi, d\varphi)\, dtdx$$

1-parameter groups of symmetries $\Rightarrow$ conservation laws

$\Psi_\varepsilon(t,x) = (t',x')$, $\varphi'(t',x') = \varphi(t,x)$, and for all regions V

$$\int_{V'} L(\varphi', d'\varphi')\, dt'dx' = \int_V L(\varphi, d\varphi)\, dtdx \quad \forall\ |\varepsilon| \ll 1$$

Then stress-energy tensor $\Theta_\alpha^\beta = \frac{\partial L}{\partial(\partial_\beta \varphi)} \partial_\alpha \varphi - \delta_\alpha^\beta L$ satisfies

$$\partial_\beta J^\beta = 0, \quad J^\beta = \Theta_\alpha^\beta \xi^\alpha, \quad \xi^\alpha = \partial_\varepsilon \Psi_\varepsilon^\alpha \Big|_{\varepsilon=0}$$

provided $\mathcal{L}'(\varphi) = 0$

For example: $(t,x) \mapsto (t+\varepsilon, x)$, gives $\partial_t \Theta_0^0 = \partial_j \Theta_0^j$

Energy conservation! $\partial_t \int_{\mathbb{R}^d} \Theta_0^0\, dx = 0$.

spatial translations: $(t, x) \mapsto (t, x + \varepsilon e_j)$, one has $\partial_t \Theta_j^0 = \partial_k \Theta_j^k$

Momentum conservation! $\partial_t \int_{\mathbb{R}^d} \Theta_j^0 \, dx = 0$ for all $1 \le j \le d$.

Energy conservation for specific Lagrangians:

- $\Box u - m^2 u = 0$, $\Theta_0^0 = \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2)$
- Wave maps, $\Theta_0^0 = \frac{1}{2}(|\partial_t u|_g^2 + |\nabla u|_g^2)$
- Maxwell equations, $\Theta_0^0 = \frac{1}{2}(|E|^2 + |B|^2)$
- MKG, $\Theta_0^0 = \frac{1}{2}(|E|^2 + |B|^2 + \sum_{\alpha=0}^{d} |D_\alpha^{(A)} u|^2 + m^2 |u|^2)$

Hamiltonian equations refers to the existence of a conserved energy (in contrast to dissipative systems).

Momentum conservation for KG: $\partial_t \int_{\mathbb{R}^d} u_t \nabla u \, dx = 0$

Basic mathematical questions (for nonlinear problems)

- Wellposedness: Existence, uniqueness, continuous dependence on the data, persistence of regularity. At first, one needs to understand this locally in time.
- Global behavior: Finite time break down (some norm, such as L^∞, becomes unbounded in finite time)? Or global existence: smooth solutions for all times if the data are smooth?
- Blow up dynamics: If the solution breaks down in finite time, can one describe the mechanism by which it does so? For example, via energy concentration at the tip of a light cone? Often, symmetries (in a wider sense) play a crucial role here.
- Scattering to a free wave: If the solutions exists for all $t \geq 0$, does it approach a free wave? $\Box u = N(u)$, then $\exists v$ with $\Box v = 0$ and $(\vec{u} - \vec{v})(t) \to 0$ as $t \to \infty$ in a suitable norm? Here $\vec{u} = (u, \partial_t u)$. If scattering occurs, then we have local energy decay.

Basic questions 2

- Special solutions: If the solution does not approach a free wave, does it scatter to something else? A stationary nonzero solution, for example? Some physical equations exhibit nonlinear bound states, which represent elementary particles.
- Stability theory: If special solutions exist such as stationary or time-periodic ones, are they orbitally stable? Are they asymptotically stable?
- Multi-bump solutions: Is it possible to construct solutions which asymptotically split into moving “solitons” plus radiation? Lorentz invariance dictates the dynamics of the single solitons.
- Resolution into multi-bumps: Do all solutions decompose in this fashion? Suppose solutions $\exists$ for all $t \geq 0$: either scatter to a free wave, or the energy collects in “pockets” formed by such “solitons”? Quantization of energy.

The wave map system 1

$u : \mathbb{R}^{1+d} \to S^{N-1} \subset \mathbb{R}^N$, smooth, solves

$$\Box u = u(-|\partial_t u|^2 + |\nabla u|^2), \quad u(0) = u_0, \ \partial_t u(0) = u_1$$

How to solve Cauchy problem? Data in $X := H^\sigma \times H^{\sigma-1}$,

$$u(t) = S_0(t)(u_0, u_1) + \int_0^t S_0(t-s)A(u)(\partial u, \partial u)(s)\, ds$$

$$S_0(t)(u_0, u_1) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1$$

Energy estimate: provided $\sigma > \frac{d}{2} + 1$, via Sobolev embedding,

$$\begin{aligned}\|\vec{u}(t)\|_X &\le C\Big(\|\vec{u}(0)\|_X + \int_0^t \|A(u)(\partial u, \partial u)(s)\|_{H^{\sigma-1}}\, ds\Big)\\ &\le C\Big(\|\vec{u}(0)\|_X + \int_0^t \|\vec{u}(s)\|_X^3\, ds\Big)\end{aligned}$$

Small time well-posedness

The wave map system 2

What can we say about all times $t \geq 0$? The energy method is very weak, does not allow for global solutions of small energy.

Problems: Sobolev spaces H^σ defined via the Laplacian Δ, so they are elliptic objects. We need to invoke dispersion. This refers to property of waves in higher dimensions to spread.

In $\mathbb{R}^3$: $\Box u = 0$, $u(0) = 0, \partial_t u(0) = g$,

$$u(t,x) = t \fint_{tS^2} g(x+y)\,\sigma(dy)$$

If g supported on $B(0,1)$, then $u(t,x)$ supported on $\big||t| - |x|\big| \leq 1$.
Huygens' principle.

Decay of the wave: $\|u(t,\cdot)\|_\infty \leq Ct^{-1}\|Dg\|_1$ In general dimensions the decay is $t^{-\frac{d-1}{2}}$.

To invoke this dispersion, we introduce hyperbolic Sobolev spaces, called $X^{\sigma,b}$ spaces (Beals, Bourgain, Klainerman-Machedon).

Domain of influence

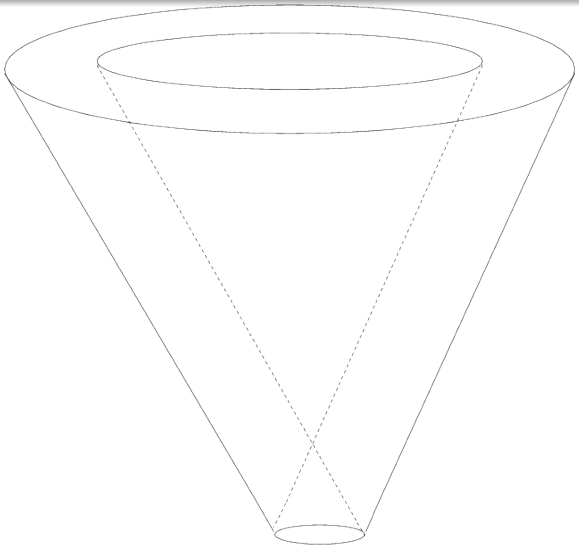


Figure: Huygens principle

The wave map system 3

Idea: To solve $\Delta u = f$ write $u = \Delta^{-1} f$. To solve $\Box u = F$ write $u = \Box^{-1} F$.

Characteristic variety of Δ is $\xi = 0$, but of $\Box$ is $|\tau| - |\xi| = 0$.

This leads to the norm

$$\|F\|_{X^{\sigma,b}} = \left\| \langle \xi \rangle^{\sigma} \langle |\tau| - |\xi| \rangle^{b} \hat{F} \right\|_{L^2_{\tau,\xi}}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

Using $X^{\sigma,b}$ and null forms, Klainerman-Machedon around 1993 were able to show local wellposedness of WM in $H^{\sigma} \times H^{\sigma-1}$ with $\sigma > \frac{d}{2}$. Nonlinearity is special: anihilates self-interactions of waves.

Scaling critical exponent $\sigma_c = \frac{d}{2}$. Rescaling $u(t,x) \rightsquigarrow u(\lambda t, \lambda x)$ preserves solutions. The Sobolev space which is invariant under this scaling is $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$.

So what? Any local existence result in this space is automatically global! Just rescale. Hence the hunt for a low-regularity solution theory.

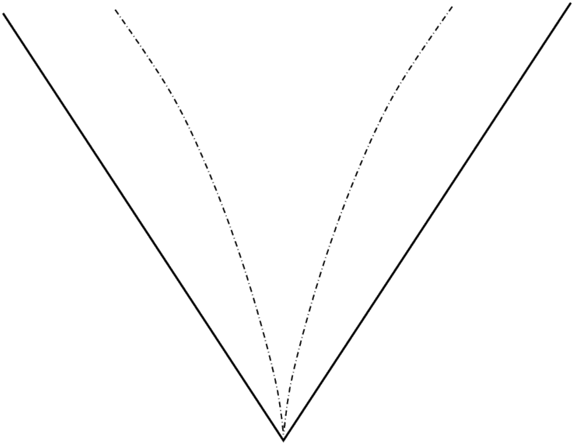
The wave map system 4

Tataru, Tao around 2000: Showed that smooth data of small $(\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1})(\mathbb{R}^d)$ norm lead to global smooth solutions. Low dimensions, especially $d = 2$ are particularly difficult due to the slow dispersion. Shatah-Struwe 2004 found much simplified argument for $d \geq 4$ using moving frames in Coulomb gauge.
Large data: Conserved energy gives $\dot{H}^1 \times L^2$ a priori control:

$$\mathcal{E} = \int_{\mathbb{R}^d} \frac{1}{2}\Big(|\partial_t u|_g^2 + |\nabla u|_g^2\Big)\, dx$$

So $d = 2$ is energy critical, $d \geq 3$ energy super critical.
Shatah 1987: $\exists$ self-similar blowup solution $u(t, r, \phi) = (\phi, q(r/t))$ for $d \geq 3$ (sphere as target).
Struwe 2000: for $d = 2$ equivariant wave maps, if blowup happens, then it must be via a rescaling of a harmonic map. Self-similar excluded, follows from a fundamental estimate of Christodoulou, Tahvildar-Zadeh. Cuspidal rescaling.

Cuspidal energy concentration



Krieger-S-Tataru 2006: There exist equivariant blowup solutions $Q(rt^{-\nu}) + o(1)$ as $t \to 0+$ for $\nu > \frac{3}{2}$ and $Q(r) = 2 \arctan r$. Rodnianski-Sterbenz, Raphael-Rodnianski also constructed blowup, but of a completely different nature, closer to the t^{-1} rate.

For negatively curved targets one has something completely different: Cauchy problem for wave maps $\mathbb{R}^{1+2}_{t,x} \to \mathbb{H}^2$, has global smooth solutions for smooth data, and energy disperses to infinity ("scattering"). Krieger-S 2009, based on Kenig-Merle approach to global regularity problem for energy critical equations, to appear as a book with EMS. Tao 2009, similar result, arxiv.

Sterbenz-Tataru theorem, Comm. Math. Physics: Cauchy problem for wave maps $\mathbb{R}^{1+2}_{t,x} \to M$ with data of energy $E < E_0$ which is the smallest energy attained by a non-constant harmonic map $\mathbb{R}^2 \to M$. Then global smooth solutions exist.

Semilinear focusing equations

Energy subcritical equations:

$$\Box u + u = |u|^{p-1} u \text{ in } \mathbb{R}^{1+1}_{t,x}(\text{even}), \mathbb{R}^{1+3}_{t,x}$$
$$i\partial_t u + \Delta u = |u|^2 u \text{ in radial } \mathbb{R}^{1+3}_{t,x}$$

Energy critical case:

$$\Box u = |u|^{2^*-2} u \text{ in radial } \mathbb{R}^{1+d}_{t,x} \tag{2}$$

For $d = 3$ one has $2^* = 6$.

Goals: Describe transition between blowup/global existence and scattering, "Soliton resolution conjecture". Results apply only to the case where the energy is at most slightly larger than the energy of the "ground state soliton".

$\forall\, u[0] \in \mathcal{H}$ there $\exists!$ strong solution $u \in C([0,T); H^1)$, $\dot u \in C^1([0,T); L^2)$ for some $T \geq T_0(\|u[0]\|_{\mathcal{H}}) > 0$. Properties: continuous dependence on data; persistence of regularity; energy conservation:

$$E(u,\dot u) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4\Big)\, dx$$

If $\|u[0]\|_{\mathcal{H}} \ll 1$, then global existence; let $T^* > 0$ be maximal forward time of existence: $T^* < \infty \Longrightarrow \|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} = \infty$. If $T^* = \infty$ and $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} < \infty$, then u scatters: $\exists\,(\tilde u_0, \tilde u_1) \in \mathcal{H}$ s.t. for $v(t) = S_0(t)(\tilde u_0, \tilde u_1)$ one has

$$(u(t), \dot u(t)) = (v(t), \dot v(t)) + o_{\mathcal{H}}(1) \quad t \to \infty$$

$S_0(t)$ free KG evol. If u scatters, then $\|u\|_{L^3([0,\infty),L^6(\mathbb{R}^3))} < \infty$.
Finite prop.-speed: if $\vec u = 0$ on $\{|x - x_0| < R\}$, then $u(t,x) = 0$ on $\{|x - x_0| < R - t, 0 < t < \min(T^*, R)\}$.

Finite time blowup, forward scattering set

$T > 0$, exact solution to cubic NLKG

$$\varphi_T(t) \sim c(T-t)^{-\alpha} \qquad \text{as} \;\; t \to T_+$$

$\alpha = 1$, $c = \sqrt{2}$.
Use finite prop-speed to cut off smoothly to neighborhood of cone $|x| < T - t$. Gives smooth solution to NLKG, blows up at $t = T$ or before.
Small data: global existence and scattering. **Large data:** can have finite time blowup.
Is there a criterion to decide finite time blowup/global existence?

Forward scattering set: $S(t) =$ nonlinear evolution

$$\mathcal{S}_+ := \Big\{(u_0, u_1) \in \mathcal{H} := H^1 \times L^2 \mid u(t) := S(t)(u_0, u_1) \; \exists \; \forall \text{ time}$$
$$\text{and scatters to zero, i.e.,} \quad \|u\|_{L^3([0,\infty);L^6)} < \infty\Big\}$$

Forward Scattering set

$\mathcal{S}_+$ satisfies the following properties:

- $\mathcal{S}_+ \supset B_\delta(0)$, a small ball in $\mathcal{H}$,
- $\mathcal{S}_+ \neq \mathcal{H}$,
- $\mathcal{S}_+$ is an open set in $\mathcal{H}$,
- $\mathcal{S}_+$ is path-connected.

Some natural questions:

1. Is $\mathcal{S}_+$ bounded in $\mathcal{H}$?
2. Is $\partial\mathcal{S}_+$ a smooth manifold or rough?
3. If $\partial\mathcal{S}_+$ is a smooth mfld, does it separate regions of FTB/GE?
4. Dynamics starting from $\partial\mathcal{S}_+$? Any special solutions on $\partial\mathcal{S}_+$?

Stationary solutions, ground state

Stationary solution $u(t,x) = \varphi(x)$ of NLKG, weak solution of

$$-\Delta\varphi + \varphi = \varphi^3 \tag{3}$$

Minimization problem

$$\inf\left\{\|\varphi\|_{H^1}^2 \mid \varphi \in H^1,\ \|\varphi\|_4 = 1\right\}$$

has radial solution $\varphi_\infty > 0$, decays exponentially, $\varphi = \lambda\varphi_\infty$ satisfies (3) for some $\lambda > 0$.
Coffman: **unique ground state** Q.
Minimizes the stationary energy (or action)

$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\right) dx$$

amongst all nonzero solutions of (3).
Payne-Sattinger dilation functional

$$K_0(\varphi) = \langle J'(\varphi)|\varphi\rangle = \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4)(x)\, dx$$

Some answers

Theorem (Nakanishi-S)

Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\rm rad}$. In $t \geq 0$ for NLKG:

1. *finite time blowup*
2. *global existence and scattering to* 0
3. *global existence and scattering to Q:* $u(t) = Q + v(t) + O_{H^1}(1)$ *as* $t \to \infty$, *and* $\dot{u}(t) = \dot{v}(t) + O_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 combinations of this trichotomy allowed as $t \to \pm\infty$.

- Applies to dim $= 3$, cubic power, or dim $= 1$, all $p > 5$.
- Under *energy assumption* (EA) $\partial\mathcal{S}_+$ is connected, smooth mfld, which gives (3), separating regions (1) and (2). $\partial\mathcal{S}_+$ contains $(\pm Q, 0)$. $\partial\mathcal{S}_+$ forms the **center stable manifold** associated with $(\pm Q, 0)$.
- $\exists$ 1-dimensional stable, unstable mflds at $(\pm Q, 0)$. Stable mfld: Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko

The invariant manifolds

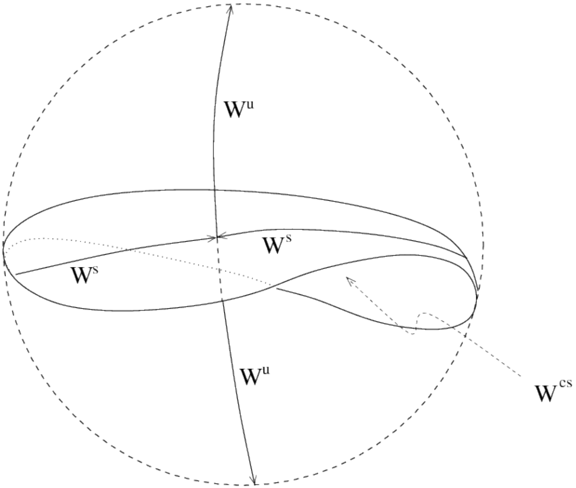


Figure: Stable, unstable, center-stable manifolds

Hyperbolic dynamics

$\dot{x} = Ax + f(x)$, $f(0) = 0$, $Df(0) = 0$, $\mathbb{R}^n = X_s + X_u + X_c$, A-invariant spaces, $A \restriction X_s$ has evals in $\mathrm{Re}\, z < 0$, $A \restriction X_u$ has evals in $\mathrm{Re}\, z > 0$, $A \restriction X_c$ has evals in $i\mathbb{R}$.
If $X_c = \{0\}$, Hartmann-Grobman theorem: conjugation to e^{tA}.

If $X_c \neq \{0\}$, **Center Manifold Theorem:** $\exists$ local invariant mflds M_u, M_s, M_c around $x = 0$, tangent to X_u, X_s, X_c, respectively.

$$M_s = \{|x_0| < \varepsilon \mid x(t) \to 0 \ \text{exponentially as} \quad t \to \infty\}$$
$$M_u = \{|x_0| < \varepsilon \mid x(t) \to 0 \ \text{exponentially as} \quad t \to -\infty\}$$

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x + O(|x|^2)$$

$\mathrm{spec}(A) = \{1, -1, i, -i\}$

Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$.

- $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$
- $L_+\rho = -k^2\rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: L_+ has no eigenvalues in $(0, 1]$, no threshold resonance (delicate! Costin-Huang-S, 2011)

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}$$

Then $\mathrm{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ simple evals. Formally: $X_s = P_1 L^2$, $X_u = P_{-1} L^2$. X_c is the rest.

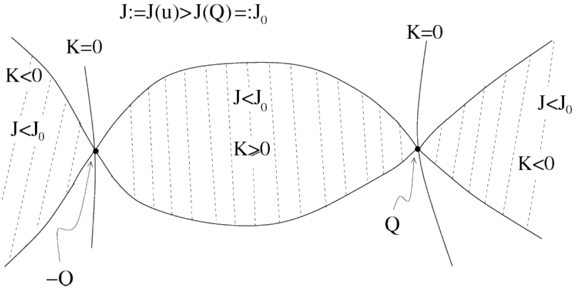


Figure: The splitting of $J(u) < J(Q)$ by the sign of $K = K_0$

- Energy near $\pm Q$ a "saddle surface": $x^2 - y^2 \leq 0$
- Explains mechanism of Payne-Sattinger theorem, 1975
- Similar picture for $E(u, \dot{u}) < J(Q)$. Solution trapped by $K \geq 0$, $K < 0$ in that set.

Variational structure above $J(Q)$ (Noneffective!)

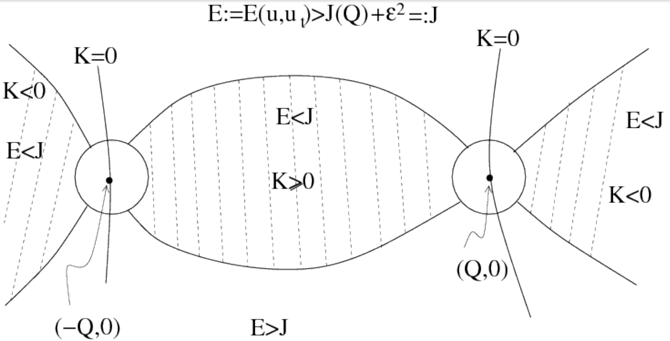


Figure: Signs of $K = K_0$ away from $(\pm Q, 0)$

$\forall\, \delta > 0\ \ \exists\, \varepsilon_0(\delta), \kappa_0, \kappa_1(\delta) > 0$ s.t. $\forall \vec{u} \in \mathcal{H}$ with $E(\vec{u}) < J(Q) + \varepsilon_0(\delta)^2, \quad d_Q(\vec{u}) \geq \delta$, one has following dichotomy:

$$K_0(u) \leq -\kappa_1(\delta) \ \text{ and } \ K_2(u) \leq -\kappa_1(\delta), \qquad \text{or}$$

$$K_0(u) \geq \min(\kappa_1(\delta), \kappa_0 \|u\|_{H^1}^2) \ \text{ and } \ K_2(u) \geq \min(\kappa_1(\delta), \kappa_0 \|\nabla u\|_{L^2}^2).$$

Numerical 2-dim section through $\partial\mathcal{S}_+$ (with R. Donninger)

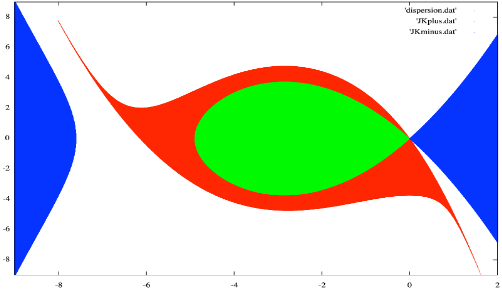


Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, (A, B) vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_-$, BLUE: $\mathcal{PS}_+$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)

One-pass theorem I

Crucial no-return property: Trajectory does **not return to balls around** $(\pm Q, 0)$. Use *virial identity*, $A = \frac{1}{2}(x\nabla + \nabla x)$,

$$\partial_t \langle w\dot{u} | Au \rangle = -K_2(u(t)) + \text{error}, \quad K_2(u) = \int \left(|\nabla u|^2 - \frac{3}{4}|u|^4\right) dx \quad (4)$$

where $w = w(t,x)$ is a space-time cutoff that lives on a rhombus, and the "error" is controlled by the external energy.

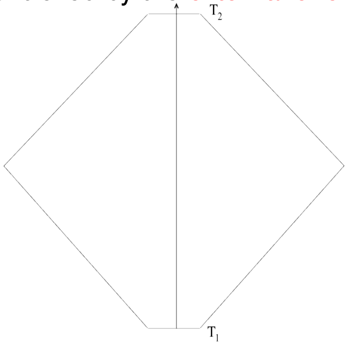


Figure: Space-time cutoff for the virial identity

One-pass theorem II

Finite propagation speed ⇒ error controlled by free energy outside large balls at times T_1, T_2.

Integrating between T_1, T_2 gives contradiction; the **bulk** of the integral of $K_2(u(t))$ here comes from exponential ejection mechanism near $(\pm Q, 0)$.

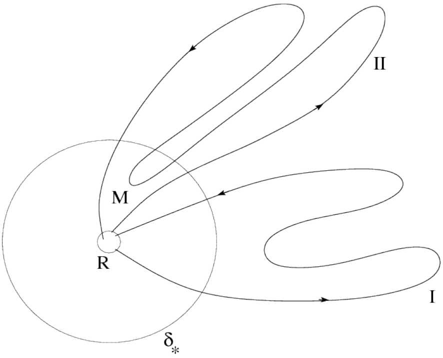


Figure: Possible returning trajectories

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ZURICH LECTURES IN ADVANCED MATHEMATICS

Kenji Nakanishi and Wilhelm Schlag

Invariant Manifolds and Dispersive Hamiltonian Evolution Equations

The notion of an invariant manifold arises naturally in the asymptotic stability analysis of stationary or standing wave solutions of unstable dispersive Hamiltonian evolution equations such as the focusing semilinear Klein–Gordon and Schrödinger equations. This is due to the fact that the linearized operators about such special solutions typically exhibit negative eigenvalues (a single one for the ground state), which lead to exponential instability of the linearized flow and allows for ideas from hyperbolic dynamics to enter.

One of the main results proved here for energy subcritical equations is that the center-stable manifold associated with the ground state appears as a hyper-surface which separates a region of finite-time blowup in forward time from one which exhibits global existence and scattering to zero in forward time. Our entire analysis takes place in the energy topology, and the conserved energy can exceed the ground state energy only by a small amount.

This monograph is based on recent research by the authors and the proofs rely on an interplay between the variational structure of the ground states on the one hand, and the nonlinear hyperbolic dynamics near these states on the other hand. A key element in the proof is a virial-type argument excluding almost homoclinic orbits originating near the ground states, and returning to them, possibly after a long excursion.

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