THE METHOD OF CONCENTRATION COMPACTNESS AND DISPERSIVE HAMILTONIAN EVOLUTION EQUATIONS*

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1. Introduction

In this brief survey we describe recent advances on large data results for nonlinear wave equations such as

$$\Box u = F(u, Du),\ F(0) = DF(0) = 0,\ (u(0), \dot{u}(0)) = (f, g)$$

We distinguish two basic scenarios:

- **Small data theory:** (f, g) are small, and F is treated as a perturbation. The main questions are local and global well-posedness, the existence of conserved quantities (energy), their relation to the basic symmetries of the equation (especially the dilation symmetry). The choice of spaces in which to solve can be very challenging, and algebraic properties of F may be essential in order to obtain well-posedness. Specifically, nonlinearities exhibiting a null-form structure appear in geometric/physical equations. The dynamics of small data are typically simple, with the associated wave becoming asymptotically free (this is referred to as "scattering").
- **Large data theory:** For local-in-time existence, energy subcritical problems are easier since the time of existence depends only on the energy norm of the data, so one can then time-step to obtain global existence. The problem with this approach lies with the absence of information on the long-term dynamics such as scattering. Finite-time breakdown (blowup) of solutions may occur as well, and then the problem of classification of possible blowup dynamics poses itself. In general, large data theory is concerned with the classification of all possible types of dynamics that solutions may exhibit at large energies. The structure here is much richer, with the underlying

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geometry, choices of a suitable topology or possibly gauge, playing major parts in the possible dynamics.

In the late 1990's Bourgain introduced the idea of *induction on energy* to pass from small data to large data results in his study of the quintic radial Schrödinger equation in $\mathbb{R}^3$. Around 2006, Kenig and Merle introduced their version of the induction on energy principle in order to obtain global existence and scattering for both focusing and defocusing equations, the former requiring further conditions on the data (as they may blow up in finite time). Their argument is indirect, and the basic ideas are as follows:

- **Critical element:** Since we have global existence and scattering for small energies, the failure of this property for some solutions implies that there exists a minimal energy $E_* > 0$ where it fails. One then proceeds to construct a solution u_* with this energy E_*. This is a rather nontrivial step. For scalar equations, one starts with a sequence of solutions u_n with energies approaching E_*, and which fail to obey the scattering property uniformly in n (in more technical terms, with Strichartz norms becoming unbounded as $n \to \infty$). One then applies a concentration-compactness decomposition to this sequence. If we cannot pass to a limit of the u_n, then this decomposition yields a representation of u_n as a sum of weakly interacting constituents, each of which has energy strictly less than E_* and which therefore scatter under the nonlinear flow by the minimality of E_*. Furthermore, these constituents interact only very weakly. Even though nonlinear equations do not obey the superposition principle, one can still conclude due to this weak interaction, and by means of a suitable perturbation theory, that the original sequence obeys the scattering property uniformly in n, a contradiction.
- **Compactness:** Due to the minimality of E_* one can show furthermore that u_* enjoys compactness properties modulo symmetries. In fact, the forward trajectory $(u_*(t), \partial_t u_*(t))$, $t \geq 0$ is pre-compact up to symmetries in the energy space. This is again done by means of an indirect argument, hinging on a concentration-compactness decomposition and the minimality of E_*.
- **Rigidity:** In this final step one shows that any such u_* with a precompact trajectory necessarily vanishes. Heuristically speaking, such a compact object would need to be a special solution (soliton, harmonic map etc.) which then is excluded by the equation itself or conditions on the data (for example, defocusing equations do not admit solitons other than zero, or negative curvature targets do not allow for harmonic maps other than constants). This hinges on *algebraic features* of the equation, and involves identities obtained by contracting the energy-momentum tensor with suitable (conformal) Killing fields. Typical identities of this type go by the name of virial or Morawetz.

The concentration compactness procedure in the previous outline has turned out to much more versatile. For instance, it has been a key ingredient in the classification of blow-up behavior.

2. Calculus of variations

2.1. *Extremal Sobolev imbedding*

Recall the Sobolev imbedding in $\mathbb{R}^3$ (we consider three dimensions for simplicity): $\|f\|_{L^p(\mathbb{R}^3)} \le C\|f\|_{H^1(\mathbb{R}^3)}$ which is valid for $2 \le p \le 6$. A basic question is as follows: *What are the extremizers, what is the optimal constant?*

We rephrase this as a variational problem

$$\inf\left\{\|f\|_{H^1(\mathbb{R}^3)} \;\middle|\; \|f\|_{L^p(\mathbb{R}^3)} = 1\right\} = \mu > 0$$

to which we would like to find a minimizer. We select a minimizing sequence:

$$\{f_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^3), \quad \|f_n\|_p = 1, \quad \|f_n\|_{H^1(\mathbb{R}^3)} \to \mu$$

The issue here is to pass to a limit $f_n \to f_\infty$ *strongly in* $L^p(\mathbb{R}^3)$. There is a loss of compactness due to translation invariance.

Theorem 2.1. *Suppose $2 < p < 6$. Then there exists a sequence $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^3$ such that $\{f_n(\cdot - y_n)\}_{n=1}^\infty$ is pre-compact in $L^p(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$.*

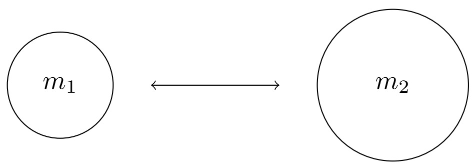

Fig. 1. Separating masses

Let us consider a simplified model, see Fig. 1: Assume that $f_n = g_n + h_n$ where $\|g_n\|_p^p = m_1 > 0$ and $\|h_n\|_p^p = m_2 > 0$, where $m_1 + m_2 = 1$. Further, suppose the supports of g_n, h_n are disjoint. Then

$$\|f_n\|_{H^1}^2 = \|g_n\|_{H^1}^2 + \|h_n\|_{H^1}^2 \ge \mu^2(m_1^{2/p} + m_2^{2/p})$$

Since $2/p < 1$, the right-hand side is larger than μ^2, which is a contradiction.

This example shows that a minimizing sequence cannot separate into separate "bubbles".

2.2. *The profile decomposition*

A much more sophisticated version of this principle is the following concentration compactness decomposition.

Proposition 2.1. *Let $\{f_n\}_{n=1}^\infty \subset H^1(\mathbb{R}^3)$ be an arbitrary bounded sequence. Then $\forall j \geq 1$ there $\exists$ (up to subsequence) $\{x_n^j\}_{n=1}^\infty \subset \mathbb{R}^3$ and $V^j \in H^1$ such that*

- *for all $J \geq 1$ one has $f_n = \sum_{j=1}^J V^j(\cdot - x_n^j) + w_n^J$*
- *$\forall j \neq k$ one has $|x_n^j - x_n^k| \to \infty$ as $n \to \infty$*
- *$\limsup_{n\to\infty} \|w_n^J\|_{L^p(\mathbb{R}^3)} \to 0$ as $J \to \infty$ for all $2 < p < 6$.*

Moreover, as $n \to \infty$,

- $\|f_n\|_2^2 = \sum_{j=1}^J \|V^j\|_2^2 + \|w_n^J\|_2^2 + o(1)$
- $\|\nabla f_n\|_2^2 = \sum_{j=1}^J \|\nabla V^j\|_2^2 + \|\nabla w_n^J\|_2^2 + o(1)$

This result is due to P. Gérard [17], see also [20], and is a more explicit form of P. L. Lions' concentration compactness trichotomy for measures. The proof proceeds by considering all possible weak limits of the form $\{f_n(\cdot - z_n)\}_{n=1}^\infty$ in $H^1(\mathbb{R}^3)$, where $\{z_n\}_{n=1}^\infty$ is any sequence of translations. The profiles are obtained by choosing these sequences such that the limits are as large as possible in H^1. Seminal work in this direction was also done by Lieb [29], as well as Brezis-Coron [4] and Struwe [43].

It makes the failure of compactness due to the translation symmetry explicit. Note that it immediately implies the compactness claim of Theorem 2.1 for minimizing sequences: Indeed, there can be only one nonzero profile V^j, by exactly the same argument as in the simplistic model from above. Finally, it is important to realize that only noncompact symmetry groups matter, in this case the group of translations $\mathbb{R}^3$. The rotation symmetries $SO(3)$ can be ignored, since they constitute a compact group. In fact, from any sequence $R_{j,n} \in SO(3)$ we can pass to a limit (up to subsequences) $R_{j,n} \to R_{j,\infty}$ as $n \to \infty$. But then $R_{j,\infty}$ can be included in the profile V^j.

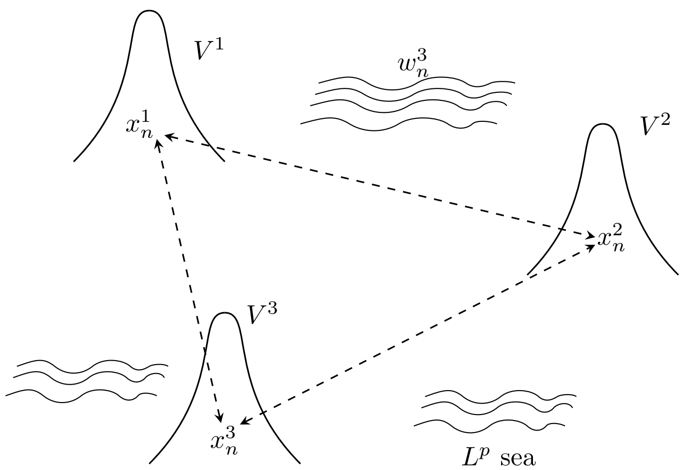

Fig. 2. We fish for more profiles from the sea.

2.3. *The extremizers*

By Theorem 2.1 we may pass to the limit $f_n(\cdot - y_n) \to f_\infty$ in $H^1(\mathbb{R}^3)$, $\|f_\infty\|_p = 1$, $\|f_\infty\|_{H^1} = \mu$. We can further assume that $f_\infty \geq 0$. Then there $\exists \lambda > 0$, a Lagrange multiplier, so that the Euler-Largrange equation

$$-\Delta f_\infty + f_\infty = \lambda |f_\infty|^{p-2} f_\infty$$

holds. One sees immediately that $\lambda > 0$ by multiplication with f_∞ and integration. Next, we remove $\lambda > 0$ since $p > 2$. Then $f_\infty = Q > 0$ solves

$$-\Delta Q + Q = |Q|^{p-2} Q \tag{1}$$

One can further show that $Q \in H^1$, $Q > 0$ is *unique* up to translation amongst all solutions of (1), see [11, 27, 28]. Moreover, it is radial about some point; this is a deep result of [18]. Q is exponentially decaying, radial, and smooth. For $dim = 1$ there is an explicit formula, and the only solutions to (1) in $H^1(\mathbb{R})$ are $0, \pm Q$. This is in contrast to higher dimensions $d > 1$, where one has infinitely many radial solutions to (1) that change sign (these are called nodal solutions), see [2].

2.4. *The critical case*

The decomposition from above fails at $p = 6$ due to the dilation symmetry. The correct setting here is $\dot{H}^1(\mathbb{R}^3)$ since

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C \|f\|_{\dot{H}^1(\mathbb{R}^3)} = C \|\nabla f\|_2 \tag{2}$$

This inequality is translation and scaling invariant, which both constitute noncompact group actions. The analogue of Proposition 2.1 reads as follows, see [17].

Proposition 2.2. *Let* $\{f_n\}_{n=1}^\infty \subset \dot{H}^1(\mathbb{R}^3)$ *a bounded sequence. Then* $\forall j \geq 1$ *there* $\exists$ *(up to subsequence)* $\{x_n^j\}_{n=1}^\infty \subset \mathbb{R}^3$, $\{\lambda_n^j\}_{n=1}^\infty \in \mathbb{R}^+$ *and* $V^j \in \dot{H}^1$ *such that*

- *for all* $J \geq 1$ *one has* $f_n = \sum_{j=1}^J \sqrt{\lambda_n^j} V^j(\lambda_n^j(\cdot - x_n^j)) + w_n^J$
- $\forall j \neq k$ *one has* $\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \lambda_n^j |x_n^j - x_n^k| \to \infty$ *as* $n \to \infty$
- $\limsup_{n\to\infty} \|w_n^J\|_{L^6(\mathbb{R}^3)} \to 0$ *as* $J \to \infty$.

Moreover, as $n \to \infty$,

$$\|\nabla f_n\|_2^2 = \sum_{j=1}^J \|\nabla V^j\|_2^2 + \|\nabla w_n^J\|_2^2 + o(1)$$

The natural variational problem associated with (2) is the following:

$$\inf \left\{ \|f\|_{\dot{H}^1(\mathbb{R}^3)} \;\middle|\; \|f\|_{L^6(\mathbb{R}^3)} = 1 \right\} = \mu > 0$$

Once again, we select a minimizing sequence

$$\{f_n\}_{n=1}^\infty \subset \dot{H}^1(\mathbb{R}^3), \quad \|f_n\|_{L^6(\mathbb{R}^3)} = 1, \quad \|f_n\|_{\dot{H}^1(\mathbb{R}^3)} \to \mu$$

We may assume that $f_n \geq 0$. From the concentration compactness decomposition of Proposition 2.2 and the minimizing property of the sequence, we conclude that there is exactly one profile. Therefore, we have the following analogue of Theorem 2.1.

Theorem 2.2. *There* $\exists \{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^3, \exists \{\lambda_n\}_{n=1}^{\infty} \in \mathbb{R}^+$ *such that* $\{\sqrt{\lambda_n} f_n(\lambda_n(\cdot - y_n))\}_{n=1}^{\infty}$ *is pre-compact in* $L^6(\mathbb{R}^3)$ *and* $\dot{H}^1(\mathbb{R}^3)$.

Note that this theorem identifies the noncompact symmetries as the *only* source of loss of compactness in a minimizing sequence. Passing to the limit $\sqrt{\lambda_n} f_n(\lambda_n(\cdot - y_n)) \to f_\infty \geq 0$, we obtain the Euler-Lagrange equation for $\varphi = c f_\infty$ with $c > 0$

$$\Delta\varphi + \varphi^5 = 0$$

The only ***radial*** $\dot{H}^1$ solutions to this equation are $\pm W$, 0 up to dilation symmetry, where

$$W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$$

The uniqueness follows from the Pohozaev identity.

3. Wave equations

3.1. *Lagrangians*

Consider the Lagrangian

$$\mathcal{L}(u, \partial_t u) := \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\big(- u_t^2 + |\nabla u|^2\big)(t,x)\, dtdx \tag{3}$$

Substitute $u = u_0 + \varepsilon v$. Then

$$\mathcal{L}(u, \partial_t u) = \mathcal{L}(u_0, \partial_t u_0) + \varepsilon \int_{\mathbb{R}^{1+d}_{t,x}} (\Box u_0)(t,x) v(t,x)\, dtdx + O(\varepsilon^2)$$

where $\Box = \partial_{tt} - \Delta$. Thus u_0 is a critical point of $\mathcal{L}$ if and only if $\Box u_0 = 0$. The wave equation is also a *Hamiltonian equation* with conserved energy

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^d} \big(|u_t|^2 + |\nabla u|^2\big)\, dx$$

Amongst other things, the Lagrangian formulation has the following significance:

- Nöther's theorem: Underlying symmetries $\to$ invariances $\to$ Conservation laws
  Conservation of energy, momentum, angular momentum are a result of time-translation, space-translation, and rotation invariance of the Lagrangian.
- Lagrangian formulation has a universal character, and is flexible, versatile.

To illustrate the latter point, let (M, g) be a Riemannian manifold, and $u : \mathbb{R}^{1+d}_{t,x} \to M$ a smooth map. What does it mean for u to satisfy a wave equation?

While it is very non-obvious how to define such an object on the level of the equation, it is easy by modifying (3):

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\big(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2\big)\, dtdx$$

The critical points $\mathcal{L}'(u, \partial_t u) = 0$ satisfy "manifold-valued wave equation". If $M \subset \mathbb{R}^N$ is imbedded, this equation is

$$\Box u \perp T_u M \quad \text{or} \quad \Box u = A(u)(\partial u, \partial u),$$

A being the second fundamental form. This is the *extrinsic formulation.* For example, if $M = \mathbb{S}^{n-1}$, then

$$\Box u = u(|\partial_t u|^2 - |\nabla u|^2).$$

This gives rise to a nonlinear wave equation in a canonical way, the nonlinearity exhibits a so-called null-form structure. Harmonic maps are time-independent solutions.

There is also an *intrinsic formulation:* $D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0$, in coordinates

$$-u^i_{tt} + \Delta u^i + \Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0$$

with $\eta = (-1, 1, 1, \ldots, 1)$ being the Minkowski metric. Note the following points:

- Similarity with geodesic equation: $u = \gamma \circ \varphi$ is a wave map provided $\Box\varphi = 0$, γ a geodesic.
- Energy conservation: $E(u, \partial_t u) = \displaystyle\int_{\mathbb{R}^d} \big(|\partial_t u|_g^2 + \sum_{j=1}^d |\partial_j u|_g^2\big)\, dx$ is conserved in time.
- Cauchy problem:

$$\Box u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

 are smooth data, with u_0 a fixed map into the manifold, and u_1 a vectorfield. Basic problem: *Does there exist a smooth local or global-in-time solution? Local in time:* Yes. *Global in time:* depends on the dimension of Minkowski space and the geometry of the target.

For more background, see the book by Shatah and Struwe, [36].

3.2. *Symmetries*

The wave equation is invariant under the Poincaré group. However, conformal invariance is also essential for the understanding of these equations. Of particular importance to the well-posedness problem is the dilation symmetry. If $u(t, x)$ is a wave map, then so is $u(\lambda t, \lambda x)$ $\forall \lambda > 0$. Suppose the data belong to the Sobolev space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$. The unique s for which this space remains invariant under the natural scaling is $s = \frac{d}{2}$. On the other hand, the energy remains invariant under the following scaling: $u(t, x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$ same as $\dot{H}^1 \times L^2(\mathbb{R}^d)$. The interplay between the natural scaling of the wave-map equation and the scaling of the energy is essential for the solution theory.

- *Subcritical case* $d = 1$. The natural scaling is associated with less regularity than that of the conserved energy. Expect global existence. Logic: local time of existence only depends on energy of data, which is preserved.
- *Critical case* $d = 2$. Energy keeps the balance with the natural scaling of the equation. For $\mathbb{S}^2$ can have finite-time blowup, whereas for $\mathbb{H}^2$ have global existence, see [25, 26, 33, 39, 40].
- *Supercritical case* $d \geq 3$. Poorly understood. Self-similar blowup $Q(r/t)$ for sphere as target [35]. Also negatively curved manifolds possible in high dimensions [5].

From a mathematical perspective, the study of nonlinear Hamiltonian evolution equations focuses on the following problems, broadly stated:

- *Well-posedness:* Existence, uniqueness, continuous dependence on the data, persistence of regularity. At first, one needs to understand this locally in time.
- *Global behavior:* Finite time break down (some norm, such as L^∞, becomes unbounded in finite time)? Or global existence: smooth solutions for all times for smooth data?
- *Blow up dynamics:* If the solution breaks down in finite time, can one describe the mechanism by which it does so? For example, via energy concentration at the tip of a light cone? Often, symmetries (in a wider sense) play a crucial role here.
- *Scattering to a free wave:* If the solutions exists for all $t \geq 0$, does it approach a free wave? $\Box u = N(u)$, then $\exists v$ with $\Box v = 0$ and $(\vec{u} - \vec{v})(t) \to 0$ as $t \to \infty$ in a suitable norm? Here $\vec{u} = (u, \partial_t u)$. If scattering occurs, then we have local energy decay.

Of great importance are equations that admit special "soliton" solutions. For wave maps, these would be given by harmonic maps.

- *Special solutions:* If a global solution does not approach a free wave, does it scatter to something else? A stationary nonzero solution, for example? Focusing equations often exhibit nonlinear bound states.
- *Stability theory:* If special solutions exist such as stationary or time-periodic ones, are they orbitally stable? Are they asymptotically stable?
- *Multi-bump solutions:* Is it possible to construct solutions which asymptotically split into moving "solitons" plus radiation? Lorentz invariance dictates the dynamics of the single solitons.
- *Resolution into multi-bumps:* Do all solutions decompose in this fashion (as in linear *asymptotic completeness*)? Suppose solutions $\exists$ for all $t \geq 0$: either scatter to a free wave, or the energy collects in "pockets" formed by such

"solitons"? Quantization of energy.

3.3. *Dispersion*

In $\mathbb{R}^3$, the Cauchy problem $\Box u = 0$, $u(0) = 0, \partial_t u(0) = g$ has solution

$$u(t,x) = t \fint_{tS^2} g(x+y)\sigma(dy)$$

If g is supported on $B(0,1)$, then $u(t,x)$ is supported on $\big||t| - |x|\big| \le 1$. We have Huygens' principle, see Figure 3. Decay of the wave:

$$\|u(t,\cdot)\|_\infty \le Ct^{-1}\|Dg\|_1 \tag{4}$$

In general dimensions the decay is $t^{-\frac{d-1}{2}}$. Generally speaking, (4) is not suitable for nonlinear problems, since $L^1(\mathbb{R}^d)$ is not invariant under the nonlinear flow. Rather, one uses the following energy based variant

$$\| u\|_{L^p_t L^q_x(\mathbb{R}^3)} \lesssim \|(u(0),\dot u(0))\|_{\dot H^1\times L^2(\mathbb{R}^3)} + \|\Box u\|_{L^1_t L^2_x(\mathbb{R}^3)}$$

where $\frac{1}{p}+\frac{1}{q} \le \frac{1}{2}$, $\frac{1}{p}+\frac{3}{q} = \frac{1}{2}$. These are *Strichartz estimates* which play a fundamental role in the study of nonlinear problems. Examples of these estimates are given by $L^\infty_t L^6_x(\mathbb{R}^{1+3})$, $L^8_{t,x}(\mathbb{R}^{1+3})$. In principle, $L^2_t L^\infty_x(\mathbb{R}^{1+3})$ is also in this class although this particular endpoint fails. The original references are [19, 41], and the endpoint is in [21].

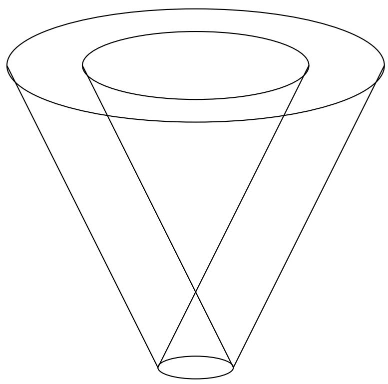

Fig. 3. Huygens principle

4. The cubic Klein-Gordon equation

4.1. *Basic existence theory and small data scattering*

In $\mathbb{R}^{1+3}_{t,x}$ consider the cubic defocusing Klein-Gordon equation

$$\Box u + u + u^3 = 0, \quad (u(0),\dot u(0)) = (f,g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3) \tag{5}$$

with conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{4}|u|^4\Big)\, dx$$

With $S(t)$ denoting the linear propagator of $\Box + 1$ we have

$$\vec{u}(t) = (u, \dot{u})(t) = S(t)(f, g) - \int_0^t S(t-s)(0, u^3(s))\, ds \tag{6}$$

whence by a simple energy estimate, over the time interval $I = (0, T)$

$$\begin{aligned} \|\vec{u}\|_{L^\infty(I;\mathcal{H})} &\lesssim \|(f,g)\|_{\mathcal{H}} + \|u^3\|_{L^1(I;L^2)} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u\|^3_{L^3(I;L^6)} \\ &\lesssim \|(f,g)\|_{\mathcal{H}} + T\|\vec{u}\|^3_{L^\infty(I;\mathcal{H})} \end{aligned}$$

By contraction mapping for small T this implies local well-posedness for $\mathcal{H}$ data. This means that there is a unique solution $(u, \dot{u}) \in C([0,T]; H^1) \times C([0,T]; L^2)$ which satisfies (5) in the Duhamel sense. Note that T depends only on the $\mathcal{H}$-size of data. From energy conservation we obtain global existence by time-stepping.

At this point it is natural to ask about the asymptotic state of the solution as $t \to \infty$. Does it behave like a free wave? Specifically, we are asking about scattering (as in linear theory): does there exist $\vec{v}(t) = (v(t), \dot{v}(t)) \in H^1 \times L^2(\mathbb{R}^3)$ such that $\|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \to 0$ as $t \to \infty$ where $\Box v + v = 0$ is energy solution. If such $\vec{v}(t)$ exists, then necessarily

$$\vec{v}(0) := \vec{u}(0) - \int_0^\infty S(-s)(0, u^3)(s)\, ds \quad \text{provided} \quad \|u^3\|_{L^1_t L^2_x} < \infty$$

Where should the finiteness of $\|u\|_{L^3_t L^6_x}$ be coming from? Since L^3_t expresses decay of the solution, such a property requires *dispersion.* The free Klein-Gordon propagator satisfies the Strichartz estimate

$$\|S(t)(f,g)\|_{L^3_t L^6_x} \le C\|(f,g)\|_{H^1 \times L^2(\mathbb{R}^3)}$$

which implies, via the Duhamel formula (6), the nonlinear Strichartz estimate,

$$\|\vec{u}\|_{L^\infty(I;\mathcal{H})} + \|u\|_{L^3(I;L^6)} \lesssim \|(f,g)\|_{\mathcal{H}} + \|u\|^3_{L^3(I;L^6)}$$

uniformly in intervals I. This immediately yields small data scattering:

$$\|\vec{u}\|_{L^3(I;L^6)} \lesssim \|(f,g)\|_{\mathcal{H}} \ll 1 \ \text{ for all } \ I.$$

So $I = \mathbb{R}$ as desired.

4.2. *Large data scattering*

For large data the previous argument completely fails. Nevertheless, large data scattering does hold for (5). The key is again to show finiteness of $\sup_{I \subset \mathbb{R}} \|u\|_{L^3(I;L^6)}$, which does not simply follow perturbatively, i.e., from the Duhamel formula. One classical method is to use Morawetz estimates, see [19]. We shall now sketch a different route, namely that of induction on energy, which was first considered by

Bourgain [3]. Kenig and Merle [22, 23] introduced a general and robust version of this method, based on the concentration compactness decomposition of Bahouri, Gérard [1].

Let $\vec{u}$ be the solution to (5) with data $(u_0, u_1) \in \mathcal{H}$. Define the forward scattering set

$$\mathcal{S}_+ = \{(u_0, u_1) \in \mathcal{H} \,|\, \vec{u}(t) \,\exists \text{ globally, scatters as } t \to +\infty\}$$

We claim that $\mathcal{S}_+ = \mathcal{H}$. This is proved via the following outline:

- (Small data result) $\|(u_0, u_1)\|_{\mathcal{H}} < \varepsilon$ implies $(u_0, u_1) \in \mathcal{S}_+$
- (Concentration Compactness) If scattering fails, i.e., if $\mathcal{S}_+ \neq \mathcal{H}$, then construct $\vec{u}_*$ of minimal energy $E_* > 0$ for which $\|u_*\|_{L^3_t L^6_x} = \infty$. There exists a continuous curve $x(t)$ so that the trajectory

$$K_+ = \{\vec{u}_*(\cdot - x(t), t) \,|\, t \geq 0\}$$

  is pre-compact in $\mathcal{H}$.
- (Rigidity Argument) If a forward global evolution $\vec{u}$ has the property that K_+ is pre-compact in $\mathcal{H}$, then $u \equiv 0$.

This blue print was introduced by Kenig-Merle [22, 23], based on the Bahouri-Gérard [1] decomposition; for the latter see also [30].

4.2.1. *Profile decomposition*

We now formulate a version of the concentration compactness decomposition which is relevant to the study of (5). Note the similarity with Proposition 2.1, the subcritical elliptic profile decomposition.

Proposition 4.1. *Let $\{u_n\}_{n=1}^{\infty}$ free Klein-Gordon solutions in $\mathbb{R}^3$ s.t.*

$$\sup_n \|\vec{u}_n\|_{L^\infty_t \mathcal{H}} < \infty$$

$\exists$ free solutions v^j bounded in $\mathcal{H}$, and translations $(t_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}^3$ s.t.

$$u_n(t, x) = \sum_{1 \leq j < J} v^j(t + t_n^j, x + x_n^j) + w_n^J(t, x)$$

satisfies $\forall\ j < J$, $\vec{w}_n^J(-t_n^j, -x_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$, and

- $\lim_{n\to\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty \;\forall\, j \neq k$
- *dispersive errors w_n^J vanish asymptotically:*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^J\|_{(L^\infty_t L^p_x \cap L^3_t L^6_x)(\mathbb{R}\times\mathbb{R}^3)} = 0 \quad \forall\, 2 < p < 6$$

- *orthogonality of the energy:*

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1 \leq j < J} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{w}_n^J\|_{\mathcal{H}}^2 + o(1)$$

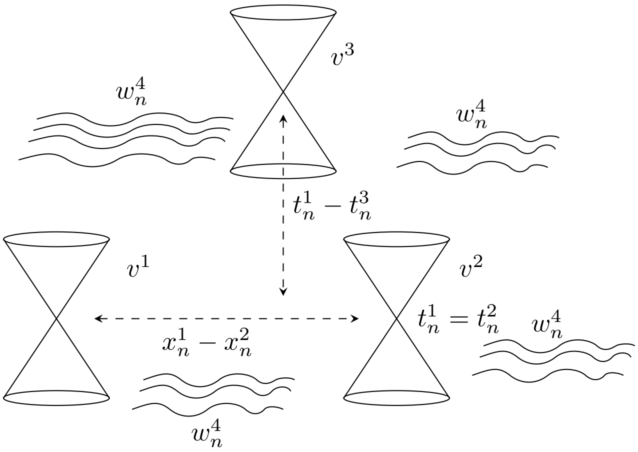

Fig. 4. Profile decomposition

Figure 4 displays three profiles amongst the "Strichartz sea" w_n^4. We can extract further profiles from this Strichartz sea if w_n^4 does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n\to\infty} \|w_n^4\|_{L_t^\infty L_x^p(\mathbb{R}^3)} > 0$.

Several comments are in order:

- Noncompact symmetry groups: space-time translations and Lorentz transforms such as

$$\begin{bmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

  The rotations form a compact symmetry group and can be ignored in Proposition 4.1. Since Lorentz transforms do not constitute a compact group, the question arises as to why they do not appear in the profile decomposition. The reason for this is of course that the assumption of a uniform energy bound compactifies them. In other words, only bounded α come into play.
- Dispersive error w_n^J is not an energy error. In other words, in general one cannot expect that $\limsup_{n\to\infty} \|w_n^J\| \to 0$ as $J \to \infty$.
- In the radial case we only need time translations.

4.2.2. *Critical element*

Key observation in the Kenig-Merle scheme: We can have only one profile due to minimality of the energy E_*. To be more specific, we now quickly review the basic steps in the application of Proposition 4.1 in the large data scattering blue print.

- Select a sequence $\vec{u}_n(0) \in \mathcal{H}$, s.t. $E(\vec{u}_n(0)) \to E_*$ and $\|u_n\|_{L^3_t(\mathbb{R};L^6_x(\mathbb{R}^3))} \to \infty$ as $n \to \infty$. Let $E_* > 0$ be minimal with this property. Here we are using the small data scattering theory.
- Apply the concentration compactness decomposition to $\{\vec{u}_n(0)\}_n$.
- Suppose we have two nontrivial profiles, say $v^1 \neq 0$, $v^2 \neq 0$. Then $E(\vec{v}^j(\cdot + t_n^j)) < E_*$ for all j. Pass to *nonlinear profiles* V^j

$$\|\vec{v}^j(t_n^j) - \vec{V}^j(t_n^j)\|_{\mathcal{H}} \to 0 \quad \text{as} \quad n \to \infty$$

  Then by orthogonality of the energy we have $E(V^j) < E_*$ and therefore V^j scatters.
- Pick J so large that $\|w_n^J\|_{L^3_t L^6_x} < \varepsilon$ for all large n. Perturbation theory implies that we can glue all V^j together with w_n^J whence

$$\|u_n\|_{L^3_t L^6_x} \leq M < \infty \quad \forall\, n$$

  But this is a contradiction, and there can be at most **one profile**. This gives *compactness* as in the elliptic case *up to the symmetries* – in our case space-time translations.
- Gives compactness of forward/backward trajectory. Again proved by contradiction and a profile decomposition.

Radial case, $u_*(t)$ has precompact forward trajectory in $H^1 \times L^2(\mathbb{R}^3)$.

4.2.3. *Rigidity*

We begin with the radial case. The essential ingredient in this step is the *virial identity*, $A = \frac{1}{2}(x\nabla + \nabla x)$

$$\partial_t \langle \chi \dot{u}_* \mid A u_* \rangle = -\int_{\mathbb{R}^3} (|\nabla u_*|^2 + \frac{3}{4}|u_*|^4)\, dx + \text{error}$$

$\chi(t,x)$ cutoff to $|x| \leq R$, error is uniformly small due to compactness. Now integrate in time:

$$\langle \chi \dot{u}_* \mid A u_* \rangle \Big|_0^T = -\int_0^T \Big[\int_{\mathbb{R}^3} (|\nabla u_*|^2 + \frac{3}{4}|u_*|^4)\, dx + \text{error}\Big](t)\, dt$$

The left-hand side here is $O(R \times Energy(\vec{u}_*))$, whereas the right-hand side satisfies $\geq T \times Energy(\vec{u}_*)$. This is a contradiction for large T if $u_* \neq 0$.

In the nonradial case, there exists a path $x(t)$ s.t. $\vec{u}_*(t, \cdot - x(t))$ is relatively compact for $t \geq 0$ in $H^1 \times L^2$. We know $|x(t)| \leq Ct$ by finite propagation speed. If optimal, this would clearly destroy virial argument.

The key observation at this point is that u_* has *vanishing momentum:*

$$P(\vec{u}_*) = \langle \dot{u}_* \mid \nabla u_* \rangle = 0$$

Indeed, if this were not the case, then by means of a Lorentz transform we could lower the energy while retaining the property that the solution does not scatter. But this is a contradiction to the minimality of the energy E_*. From the vanishing momentum we conclude that $x(t) = o(t)$. The virial argument now applies to show that $u_* = 0$.

We arrive at the following conclusion.

Theorem 4.1. *For any data* $(f, g) \in H^1 \times L^2(\mathbb{R}^3)$ *there exists a unique global solution* $\vec{u}(t)$ *to the Cauchy problem* $\Box u + u + u^3 = 0$, $\vec{u}(0) = (f, g)$ *which scatters to a free energy solution as* $t \to \pm\infty$.

5. Focusing cubic Klein-Gordon equation

The focusing cubic nonlinear Klein-Gordon equation

$$\Box u + u = \partial_{tt} u - \Delta u + u = u^3 \tag{7}$$

has an *indefinite conserved energy*

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4 \Big)\, dx$$

We briefly review several basic facts about (7):

- Local wellposendness for $H^1 \times L^2(\mathbb{R}^3)$ data.
- Small data global existence and scattering.
- Finite time blowup $u(t) = \sqrt{2}(T - t)^{-1}(1 + o(1))$ as $t \to T-$. Cutoff to a cone using finite propagation speed to obtain finite energy solution, as shown in Figure 5. Dashed line is a smooth cutoff which $= 1$ on $|x| \le T$.
- Eq. (7) admits stationary solutions characterized by $-\Delta\varphi + \varphi = \varphi^3$, amongst these we single out the ground state $Q(r) > 0$.

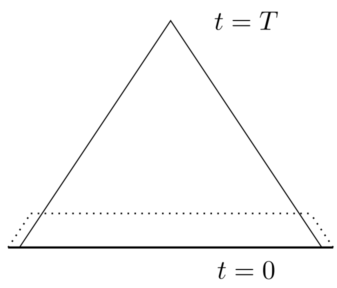

Fig. 5. Cutoff for the blowup solutions

At this point it is natural to ask whether there might be a criterion to decide between finite-time blowup vs. global existence. Although this question turns out to be somewhat too general and vague, there is a clean affirmative answer provided the energy is less than the ground state energy. This criterion was discovered by Payne and Sattinger around 1975, see [32]. Their argument rests on the observation that the energy near $(Q, 0)$ is *a saddle surface*. More specifically, we define the functionals

$$J(\varphi) = \int_{\mathbb{R}^3} \big(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\big)\, dx,$$
$$K(\varphi) = \int_{\mathbb{R}^3} \big(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\big)\, dx.$$

J is the stationary energy, whereas K arises from J via the dilation symmetry. To see this, define $j_\varphi(\lambda) := J(e^\lambda\varphi)$ where $\varphi \neq 0$ is fixed. Figure 6 depicts the

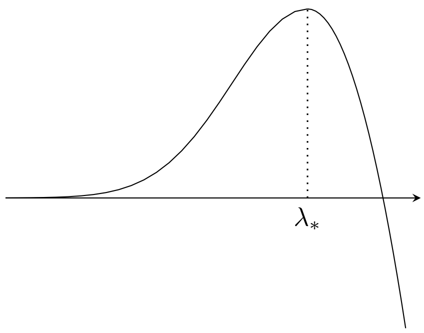

Fig. 6. Payne-Sattinger well

graph of j_φ, with a unique horizontal tangent at $\lambda = \lambda_*$. We normalize so that $\lambda_* = 0$. Then $\partial_\lambda j_\varphi(\lambda)\big|_{\lambda=\lambda_*} = K_0(\varphi) = 0$. We might expect that the well on the left-hand side of λ_* acts to trap the solution, leading to global existence. Due to the arbitrariness of φ, we need to find the smallest summit (or mountain pass) $j_\varphi(\lambda_*)$. This turns out to be

$$\inf\{j_\varphi(0) \mid K_0(\varphi) = 0,\ H^1 \ni \varphi \neq 0\} = J(Q) \tag{8}$$

This infimum is attained uniquely at $\pm Q$ up to translations. What gives rise to (8) is the *uniqueness of* Q as positive solution of the elliptic equation. Figure 7 depicts level sets of J, K and how they relate to $(\pm Q, 0)$. The Payne-Sattinger theorem states that for data $(f, g) \in H^1 \times L^2(\mathbb{R}^3)$ for which $E(f, g) < E(Q, 0)$ one has the following dichotomy:

$$\begin{aligned} K(f) \geq 0 &\Longrightarrow \text{ global existence} \\ K(f) < 0 &\Longrightarrow \text{ finite time blowup} \end{aligned} \tag{9}$$

These two regions are invariant under the nonlinear flow, as shown in Figure 7. The middle region is $K \geq 0$ and it traps the solution. One can immediately check that for $K(u) \geq 0$, the energy $E(u, \dot{u})$ is proportional to $\|(u, \dot{u})\|_{H^1\times L^2}$. But since the latter remains bounded, the solution is automatically global by the standard well-posedness. It is harder, but still elementary, to see that $K < 0$ leads to finite-time blowup.

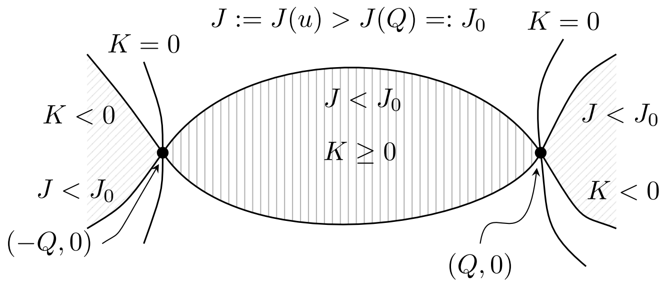

Fig. 7. The saddle structure of the energy near the ground state

In the regime of energies above $E(Q, 0)$ one has the following description of the dynamics, see [31].

Theorem 5.1. *Let* $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\rm rad}$. *In* $t \geq 0$ *for* (7) *there is the following trichotomy:*

(1) finite time blowup
(2) global existence and scattering to 0
(3) global existence and scattering to Q*:* $u(t) = Q + v(t) + o_{H^1}(1)$ *as* $t \to \infty$*, and* $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 *combinations of this trichotomy allowed as* $t \to \pm\infty$.

A few remarks about Theorem 5.1:

- Applies to dim $= 3$, $|u|^{p-1}u$, $7/3 < p < 5$, or dim $= 1$, $p > 5$.
- Third alternative forms the *center stable manifold* associated with $(\pm Q, 0)$, see Figure 8. This is a standard notion from hyperbolic dynamical systems.
- $\exists$ 1-dimensional stable, unstable manifolds at $(\pm Q, 0)$. These arise in the classification of all possible dynamics at energy level $E(u, \dot{u}) = E(Q, 0)$, see [13, 14].

The linearized operator $L_+ = -\Delta + 1 - 3Q^2$ has spectrum $\{-k^2\} \cup [1, \infty)$ on $L^2_{rad}(\mathbb{R}^3)$ where $k > 0$. It is easy to see that there must be negative spectrum since $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$. This implies that there is a simple negative eigenvalue (ground state of L_+). That there is no other negative spectrum and no kernel over radial functions follows from the uniqueness of Q. Much more delicate is the *spectral gap property:* L_+ has no eigenvalues in $(0, 1]$, and no threshold resonance. This is only needed in order to understand the scattering properties of the linearized dynamics. In particular, it allows one to use Kenji Yajima's L^p-boundedness for wave operators, see [47].

To understand the *perturbative, i.e., stable dynamics* of Theorem 5.1, we plug $u = Q + v$ into (7):

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

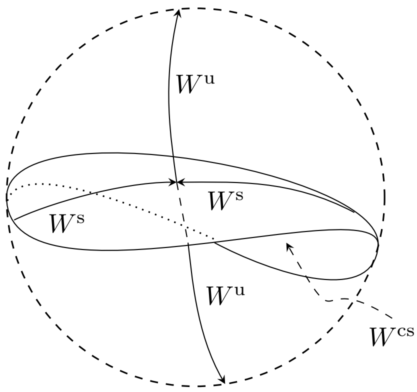

Fig. 8. Stable, unstable, center-stable manifolds

We rewrite this as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix}$$

Then $\mathrm{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ being simple eigenvalues, see Figure 9. The spectrum makes it clear that one should expect 1-dimensional stable/unstable manifolds, as well as a codimension 1 center-stable manifold.

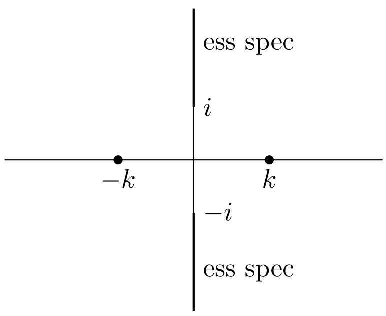

Fig. 9. Spectrum of nonselfadjoint linear operator in phase space

Let us now comment on the non-perturbative aspects of Theorem 5.1, which are most important in describing the dynamics away from the center-stable manifold. Figure 10 shows what happens to Figure 7 at energy levels larger than $E(Q, 0)$: the regions which formerly pinched off at $(\pm Q, 0)$ are now "fattened up" and a solution may pass through small balls surrounding these points. Energy is no obstruction anymore as in the Payne-Sattinger case. The key to the description of the dynamics is the one-pass (or no return) theorem. This establishes that the trajectory can make only one pass through the balls. Returning trajectories are excluded by means of an indirect argument using a variant of the virial argument that was essential to the rigidity step of Kenig-Merle. The point behind the stabilization of the sign of $K(u(t))$ is that we may then essentially fall back on the Payne, Sattinger type argument to decide the long-term fate of the solutions. The scattering, on the other hand, requires the use of concentration compactness ideas, as in Kenig-Merle.

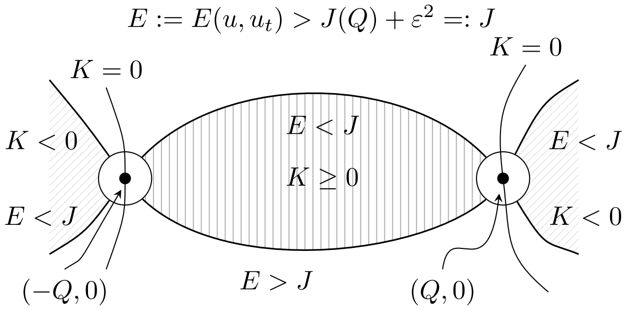

Fig. 10. Signs of $K = K_0$ away from $(\pm Q, 0)$

6. Wave maps

Suppose the smooth map $u : \mathbb{R}^{1+2}_{t,x} \to \mathbb{S}^2$ satisfies the wave map equation

$$\Box u \perp T_u\mathbb{S}^2 \Leftrightarrow \Box u = u(|\partial_t u|^2 - |\nabla u|^2)$$

as well as the equivariance assumption $u \circ R = R \circ u \;\forall\; R \in SO(2)$, see Figure 11.

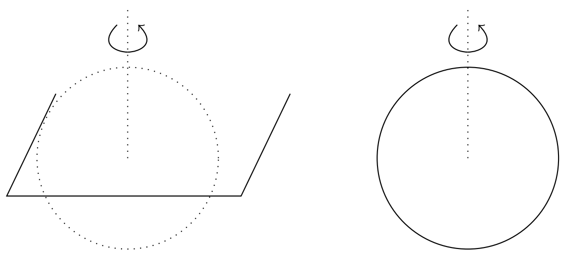

Fig. 11. Equivariance and Riemann sphere

Then u takes the special form $u(t, r, \phi) = (\psi(t, r), \phi)$ in spherical coordinates, where ψ measures the angle from the north pole. This angle then satisfies the equivariant wave map equation

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi, \dot{\psi})(0) = (\psi_0, \psi_1)$$

- Conserved energy

$$E(\psi, \dot{\psi}) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2}\right) r\, dr$$

- $\psi(t, \infty) = n\pi, n \in \mathbb{Z}$, homotopy class = degree = n and we define $\mathcal{H}_n$ to be the set of finite energy data of degree n.
- stationary solutions = harmonic maps = $0, \pm Q(r/\lambda)$, where $Q(r) = 2 \arctan r$. This is the identity $\mathbb{S}^2 \to \mathbb{S}^2$ with stereographic projection onto $\mathbb{R}^2$ as domain.

Theorem 6.1. *[7] Let (ψ_0, ψ_1) be smooth data.*

(1) Let $E(\psi_0, \psi_1) < 2E(Q, 0)$, degree 0. Then the solution exists globally, and scatters (energy on compact sets vanishes as $t \to \infty$). For any $\delta > 0$ there exist data of energy $< 2E(Q, 0) + \delta$ which blow up in finite time.

(2) Let $E(\psi_0, \psi_1) < 3E(Q, 0)$, degree 1. If the solution $\psi(t)$ blows up at time $t = 1$, then there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $E(\vec{\varphi}) = E(\vec{\psi}) - E(Q, 0)$, and a decomposition

$$\vec{\psi}(t) = \vec{\varphi} + (Q\left(\cdot/\lambda(t)\right), 0) + \vec{\epsilon}(t) \tag{10}$$

s.t. $\vec{\epsilon}(t) \in \mathcal{H}_0$, $\vec{\epsilon}(t) \to 0$ in $\mathcal{H}_0$ as $t \to 1$.

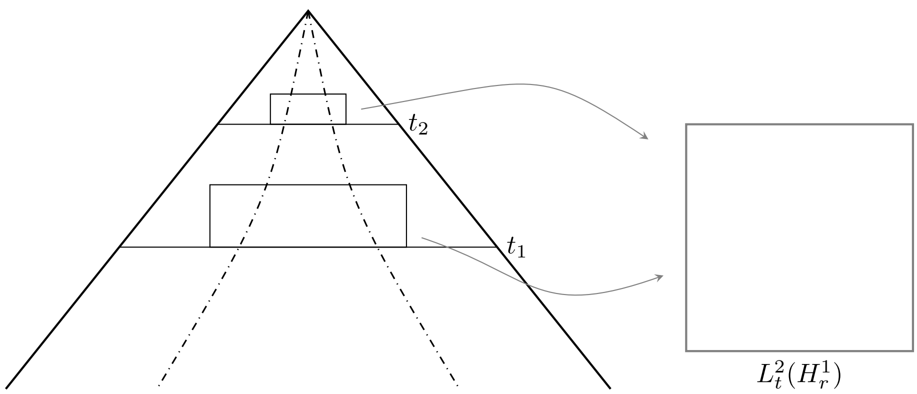

Fig. 12. Struwe's bubbling off theorem

- For degree 1 have an analogous classification to (10) for global solutions, see [8].
- Côte, Kenig, Merle [6] proved the degree 0 result for $E < E(Q, 0) + \delta$. Proof proceeds via the small data scattering/concentration-compactness/rigidity scheme.
- Duyckaerts, Kenig, Merle [16] established analogous classification results for $\Box u = u^5$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$ instead of Q. We use certain parts of their ideology, which is very heavily based on concentration compactness arguments. Note that here we cannot rely in any form on induction on energy.

- Construction of blowup solutions as in (11) by Krieger-S.-Tataru, Donninger-Krieger [12, 26].
- A crucial role in the proof of our degree 1, $3E(Q,0)$ result is played by Struwe's bubbling off theorem in the equivariant setting [42]: if blowup happens, then there exists a sequence of times approaching blowup time, such that a rescaled version of the wave map approaches locally in energy space a harmonic map of positive energy, see Figure 12. The rescalings converge in the local $L^2_{t,r}$-sense to a stationary wave map of positive energy, i.e., a harmonic map.

A fundamental role in the degree 1 argument is played by a property of the linear wave equation. To be specific, consider $\Box u = 0$, $u(0) = f \in \dot{H}^1(\mathbb{R}^d)$, $u_t(0) = g \in L^2(\mathbb{R}^d)$ arbitrary functions. Then Duyckaerts, Kenig, and Merle showed the following: There exists $c > 0$ such that for all $t \geq 0$ or all $t \leq 0$ one has

$$E_{\mathrm{ext}}(\vec{u}(t)) \geq cE(f,g) \tag{11}$$

provided the dimension is odd. Here the exterior energy is computed relative to the region in Figure 13.

In *even dimensions* this property fails, see [9]. To be precise, in dimensions $d = 2, 6, 10, \ldots$ (11) holds for radial data $(0,g)$, but fails in general for data $(f,0)$. On the other hand, for dimensions $d = 4, 8, 12, \ldots$ (11) holds for radial data $(f,0)$ but fails in general for data $(0,g)$.

The proof of both the positive and negative results is based on the Fourier representation, which in our radial context becomes a Bessel transform. The dimension d is then reflected in the phase of the Bessel asymptotics. Due to the monotonicity of the energy over the regions $\{|x| \geq t\}$ the key calculation is that of the asymptotic exterior energy as $t \to \pm\infty$.

For our $3E(Q,0)$ theorem we need the $d = 4$ result rather than $d = 2$ due to the repulsive $\frac{\psi}{r^2}$-potential coming from $\frac{\sin(2\psi)}{2r^2}$. Why does the $(f,0)$ result suffice for our argument? Because of the results by Christodoulou, Tahvildar-Zadeh, and Shatah [10, 37, 38] about equivariant wave maps, see also the book by Shatah, Struwe [36]. Amongst other things, these authors showed that at blowup $t = T = 1$ one has vanishing kinetic energy

$$\lim_{t \to 1} \frac{1}{1-t} \int_t^1 \int_0^t |\dot{\psi}(t,r)|^2 \, r dr \, dt = 0$$

This vanishing (modulo many other arguments) then allows us to work with the more restrictive form of (11) for data $(f,0)$. However, for equivariance class 2 or Yang-Mills our arguments do not apply in their present form, since for these problems one encounters a semi-linear equation in dimension $d = 6$. So we would need to fall back on $(0,g)$ for (11) to hold, which seems impossible to do.

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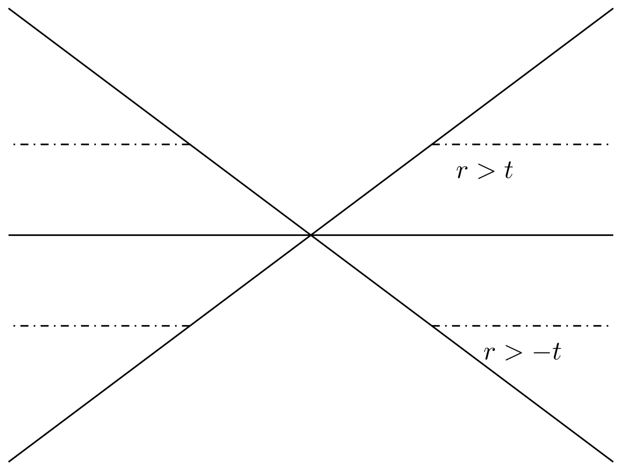

Fig. 13. Exterior energy regions

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