Classical and quantum scattering for a class of long range random potentials

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1 Introduction

In this paper we prove existence of modified wave operators, with probability one, for the family of random operators on $L^2(\mathbb{R}^d)$, $d \geq 2$,

$$H = \frac{1}{2}\triangle - V \tag{1.1}$$

where

$$V(x) = \sum_{n\in\mathcal{N}} \omega_n (1+|n|)^{-\frac{3}{4}} \phi\Big(\frac{x-n}{|n|^\beta}\Big) \tag{1.2}$$

with uniformly bounded independent ω_n with mean 0, and $\beta > \frac{1}{2}$. The most important example are Bernoulli variables $\omega_n = \pm 1$, although we would like to point out that no assumption is made about identical distribution of the ω_n. Here ϕ is a standard C^∞ bump function with small support and $\mathcal{N} \subset \mathbb{R}^d$ is a set of points with the property that $\{R < |x| < 2R\} \cap \mathcal{N}$ is a maximally R^β separated set of points so that the summands in (1.2) have disjoint supports. We refer to (1.1) with V as in (1.2) as the $\frac{3}{4}$-model. Our methods also apply to more general potentials than (1.2) for which the individual bumps are not rescaled versions of a single bump function ϕ. More precisely, consider

$$V(x) = \sum_{n\in\mathcal{N}} \omega_n (1+|n|)^{-\frac{3}{4}} \phi_n(x-n) \tag{1.3}$$

where ϕ_n is a C^∞ function supported in a ball $B(0, |n|^\beta)$ satisfying the derivative bounds

$$\sup_{x\in\mathbb{R}^d} |D^\gamma \phi_n(x)| \leq C_\gamma\, |n|^{-|\gamma|\beta} \quad \text{for all multi-indices} \quad \gamma$$

and all $n \in \mathcal{N}$. The net $\mathcal{N}$ is just as above, C_γ is uniform in $\mathcal{N}$, and the functions $\{\phi_n(\cdot - n)\}_{n\in\mathcal{N}}$ have disjoint supports. For simplicity, we shall restrict ourselves to the model (1.2), but all arguments apply just as well to the case (1.3).

Recall that wave operators of (1.1) are defined as the strong L^2 limit

$$\Omega^+ := s - \lim_{t\to\infty} e^{-itH} e^{itH_0} \tag{1.4}$$

provided it exists. Here $H_0 = \frac{1}{2}\triangle$. These are the quantum mechanical analogues of classical wave operators, which allow one to parameterize the trajectories with positive energy in the force field $-\nabla V$ by means of the free trajectories. The most basic result is that $|V(x)| \le C|x|^{-1-\varepsilon}$ guarantees the existence of the limit (1.4). Dollard [8] showed that wave operators do not exist for the Coulomb potential $V(x) = |x|^{-1}$ in $\mathbb{R}^3$. Alsholm, Kato [1], Buslaev, Matveev [2], and Hörmander [10] then constructed suitable modified wave operators by replacing e^{itH_0} with another evolution. In particular, Hörmander treats potentials V for which $|\nabla V(x)| \le C|x|^{-\frac{3}{2}-\varepsilon}$. Another case for which modified wave operators can be constructed is $|\nabla V(x)| \le C|x|^{-1-\varepsilon}$ and $|\nabla^2 V(x)| \le C|x|^{-2-\varepsilon}$, see [6] for a comprehensive and recent exposition of these results, as well as Reed, Simon [16] and Yafaev [18] for scattering theory in general. Observe that (1.2) merely satisfies the decay rates

$$|D^\gamma V(x)| \le C_\gamma\, |x|^{-\frac{3}{4}-|\gamma|\beta} \ \text{ for any multiindex } \ \gamma,$$

i.e., for $|\gamma| = 1$ one has $|x|^{-\frac{5}{4}-\varepsilon}$, and for $|\gamma| = 2$, $|x|^{-\frac{7}{4}-2\varepsilon}$.

Despite the fact that the derivatives of V decay too slowly for any of the aforementioned results to apply, we show in this paper that one can still construct modified wave operators by means of the so called "correspondence principle" and Cook's method. This principle, which was used by Hörmander [10], refers to the construction of modified wave operators by means of classical scattering trajectories. These trajectories are the solutions to Hamilton's equations from classical mechanics

$$\dot{x} = \xi \qquad \dot{\xi} = -\nabla V(x), \tag{1.5}$$

and it is well-known that they are the characteristics of the Hamilton-Jacobi PDE

$$S_t(t,\xi) = \frac{1}{2}|\xi|^2 + V(DS(t,\xi)). \tag{1.6}$$

The solution to that PDE is then used as a phase function of a parametrix $U(t)$ to the time evolution e^{itH}. More precisely, Hörmander defines

$$(U(t)\phi)(x) := \int e^{i[x\cdot\xi - S(t,\xi)]}\, \hat{\phi}(\xi)\, d\xi \tag{1.7}$$

for Schwartz functions $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\mathrm{supp}(\hat{\phi}) \subset \mathbb{R}^d \setminus \{0\}$. A novel feature of the scattering theory of (1.2) is the introduction of a time-dependent amplitude function in the parametrix. Since $D^2V(x)$ does not decay like $|x|^{-2-\varepsilon}$ one can easily show that the parametrix (1.7) does not apply here. Rather, we define

$$(U(t)\phi)(x) := \int e^{i[x\cdot\xi - S(t,\xi)]}\, a(t,\xi)\, \hat{\phi}(\xi)\, d\xi \tag{1.8}$$

with a suitable $a(t,\xi)$.
Generally speaking, the gain of $\frac{1}{4}$ in terms of the conditions on the potential is achieved throughout this paper by systematically exploiting averaging in the potential. As an example, consider the Hamilton-Jacobi PDE (1.6). It is well-known that initial conditions can be chosen such that $DS(t,\xi)$ is a trajectory that escapes at a linear rate to infinity. Therefore, any integral like

$$\int_T^{2T} V(DS(t,\xi))\,dt$$

is going to be significantly smaller (roughly by $T^{-\frac{1}{4}}$ if $\beta = \frac{1}{2}$) than the one with absolute values inside due to the mean-zero assumption on the randomness. Similarly, integrating the ODE (1.5) will lead to the expression $\int_t^\infty \nabla V(x(s))\,ds$. In contrast to the deterministic theory that only exploits the size of $\nabla V(x(s))$ we need to invoke cancellations.
It is well-known that the existence of (modified) wave operators has strong spectral implications. More precisely, Ω^+ is an isometry that gives a unitary equivalence between $\triangle$ and the restriction of H to a subspace of $L^2(\mathbb{R}^d)$. The Weyl criteria implies that even the deterministic essential spectrum $\sigma_{ess}(H)$ of the Schrödinger operator with potential (1.2) coincides with the essential spectrum of the unperturbed operator $-\Delta$, $\sigma_{ess}(H) = [0,\infty)$. Therefore, one obtains that with probability one

$$\sigma_{a.c.}(H) = [0,\infty), \tag{1.9}$$

where $\sigma_{a.c.}$ denotes the absolutely continuous spectrum of H. It seems natural to believe that there should be no singular spectrum in $[0,\infty)$. This conjecture is motivated by the results of Jakšić, Last [12]. However, their result is for the discrete model on $\mathbb{Z}^d$ and is based on the theory of rank-one perturbations. It is therefore not clear how to prove similar results in this context. Finally, the problem of showing completeness of the modified wave operators for the $\frac{3}{4}$-model is open.
The investigation of spectral properties of random operators is of course a well-established field in its own right starting with the work of Anderson. Recently, several works have been devoted to random decaying potentials. Of these we would like to mention Krishna [14] and Kirsch, Krishna, Obermeit [13]. However, it seems that these investigations are very different from the present one, both in terms of their objectives as well as their techniques. In fact, the results in [14] and [13] do not cover the potential (1.2), as they typically require that $(\mathbb{E}|V(x)|^2)^{\frac{1}{2}} \le C|x|^{-1-\varepsilon}$.
A very interesting model (referred to as the $\frac{1}{2}$-model) is given by (1.1) with the potential

$$V(x) = \sum_{n\in\mathcal{N}} \omega_n (1+|n|)^{-\alpha}\phi(x-n) \tag{1.10}$$

where ω_n are as above, but $\alpha > \frac{1}{2}$ and $\mathcal{N} \subset \mathbb{R}^d$ is a 1-net. For this model we can only show that the classical scattering trajectories exist in the sense that they approach infinity and that their tangent vectors approach a limit, see Proposition 2.3. However, given the total lack of any improved decay of the derivatives of V in this case we are unable to show existence of classical (let alone, quantum mechanical) wave operators for the $\frac{1}{2}$-model. Nevertheless, we start the technical part of this paper with the proof of existence of classical scattering trajectories for the $\frac{1}{2}$-model, as that proof introduces the main idea how to exploit averaging in the potential.

Our interest in this question arose in connection with a conjecture about square integrable potentials, see Simon's review [17]: If

$$\int_{\mathbb{R}^d} |V(x)|^2 \, |x|^{1-d} \, dx < \infty,$$

then $-\triangle + V$ has a.c. spectrum essentially supported on $[0, \infty)$. That conjecture would imply that the $\frac{1}{2}$-model should have a.c. spectrum essentially supported on $[0, \infty)$ for *any realization* of the ω_n. In one dimension this has been shown by Deift, Killip [7]. In a series of several papers Christ, Kiselev [4], [5] had previously settled the one-dimensional problem for potentials satisfying $\int_{-\infty}^{\infty} |V(x)|^p \, dx < \infty$, where $1 \le p < 2$. Moreover, Christ and Kiselev have recently constructed wave operators in the one-dimensional case under basically optimal conditions [3]. Another interesting approach to a.c. spectrum under optimal conditions is Molchanov, Novitskii, Vainberg [15].
This paper is organized as follows: In Section 2 we construct classical scattering trajectories for the $\frac{1}{2}$-model with probability one, see 2.3 (this is the only section that considers the $\frac{1}{2}$-model, all the subsequent ones deal with the $\frac{3}{4}$-model). In Section 3 we derive a similar result for the $\frac{3}{4}$-model, see Proposition 3.1. Moreover, we establish various finer estimates on the derivatives of the trajectories with respect to the initial conditions. These estimates are crucial for the construction of the modified wave operators. Finally, we construct Herbst's classical "wave-like" operators [9], see Theorem 3.9, which conjugate the perturbed dynamics of (1.5) with the free evolution. In Section 4 we solve, with probability one, the Hamilton-Jacobi equation by means of the method of characteristics, which are precisely the solutions of (1.5). The main result of this section is Corollary 4.8 which also provides various estimates on the solution $S(t, \xi)$ which are later needed in the construction of the wave operators ($\langle x, \xi \rangle - S(t, \xi)$ is precisely the phase of the parametrix, see (1.8)). In Section 5 we prove our main result which is Theorem 5.6. It states that the parametrix (1.8) exists almost surely and has the desired properties, i.e, $s - \lim_{t\to\infty} e^{-itH} U(t) = W^+$ exists and W^+ intertwines e^{itH} and $e^{i\frac{t}{2}\triangle}$. Most of the work in this section is devoted to the construction of the amplitude $a(t, \xi)$ by means of the solution of certain transport equations, see Proposition 5.5. These transport equations are solved recursively leading to an asymptotic expansion of $a(t, \xi)$. This procedure depends crucially on averaging arguments. Throughout this paper the basic tool in connection with averaging is a well-known large deviation estimate for martingales with bounded increments, see Lemma 2.1.

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2 The existence of classical scattering trajectories for the $\frac{1}{2}$-model

Let V be as in (1.10). We shall also assume that the supports of the various bump functions are fitted inside shells of the form $\{\ell < |x| \le \ell + 1\}$, where $\ell \in \mathbb{Z}^+$. This is indicated in the second figure below by means of the small dotted circles (we have only filled one shell with the small circles in this picture, but in reality all of them are filled). This allows us to define $\mathcal{F}_\ell$ as the σ-algebra generated by the random variables inside the disk $|x| \le \ell$ for $\ell \in \mathbb{Z}^+$.

In this section we show that with large probability all solutions of Hamilton's equation

$$\begin{aligned} \dot{x} &= \xi & \dot{\xi} &= -\nabla V(x) \\ x(0) &= y & \xi(0) &= \eta \end{aligned} \tag{2.1}$$

with large initial y and small $\sphericalangle(y,\eta)$ are classical scattering trajectories. This means that

$$\lim_{t\to\infty} |x(t)| = \infty \quad \text{and} \quad \lim_{t\to\infty} \xi(t) \quad \text{exists.}$$

The underlying intuition is very simple: If we approximate $x(t)$ by a straight line, say $x(t) \approx y + t\eta$, then

$$\xi(t) \approx \eta - \int_0^t \nabla V(y + s\eta)\, ds.$$

The integral on the right-hand side should behave as (a tail of) the random sum $\sum_k \pm k^{-\frac{1}{2}-\varepsilon}$. By Kolmogoroff's three series theorem, this series converges almost surely. Although this is the correct approach to estimate

$$\int_0^t \nabla V(x(s))\, ds \tag{2.2}$$

for a *fixed* curve $x(s)$, integration of the random Hamilton's equations (2.1) requires controlling (2.2) – up to a small probability – for an entire class of (random) classical trajectories. One natural way to circumvent this difficulty would be to consider a net of curves and then approximate a given curve by one in the net. This, however, does not work as the cardinality of the net would necessarily need to be exponentially large, and thus by far outweigh any bound on the probabilities of bad events which are provided, say, by large deviation estimates. We therefore use a different approach, namely one based on the following well-known large deviation estimate for martingales with bounded increments. We provide a sketch of the proof for the reader's convenience.

Lemma 2.1. *Let $\{Y_m\}_{m=1}^M$ be a martingale difference sequence adapted to a filtration $\{\mathcal{F}_m\}$ of increasing σ-algebras. Then*

$$\mathbb{P}\Big[\Big|\sum_{m=1}^M Y_m\Big| > \lambda\Big(\sum_{m=1}^M \|Y_m\|_\infty^2\Big)^{1/2}\Big] < Ce^{-c\lambda^2}$$

with some absolute constants c, C.

Proof. With $t > 0$ a parameter and $S_n := \sum_{j=1}^n Y_j$, consider

$$\begin{aligned} \mathbb{E}\Big[\exp(tS_n)\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\exp(tS_n)\,|\mathcal{F}_{n-1}\Big]\Big] = \\ = \mathbb{E}\Big[\exp(tS_{n-1})\mathbb{E}[e^{tY_n}\,|\mathcal{F}_{n-1}]\Big] &\le \mathbb{E}\Big[\exp(tS_{n-1})\Big]\exp\big(ct^2\|Y_n\|_\infty^2\big). \end{aligned} \tag{2.3}$$

To pass to the second line, apply the elementary fact $e^x - x \le e^{cx^2}$ and the assumption $\mathbb{E}[Y_n \,|\mathcal{F}_{n-1}] = 0$ to

$$\mathbb{E}[e^{tY_n} \,|\mathcal{F}_{n-1}] = 1 + \sum_{\ell=2}^{\infty} \frac{t^\ell}{\ell!}\mathbb{E}[Y_n^\ell \,|\mathcal{F}_{n-1}] \le 1 + \sum_{\ell=2}^{\infty} \frac{t^\ell}{\ell!}\|Y_n\|_\infty^\ell \le \exp\Big(ct^2\|Y_n\|_\infty^2\Big).$$

Applying (2.3) inductively starting at $n = M$, and invoking Tschebyscheff's inequality with a suitable choice of t, finishes the proof. □

Lemma 2.2 estimates by how much hitting times differ for two paths that remain close to each other. It will be used repeatedly in what follows. In what follows we shall often consider times $\tau > 0$ which are minimal with the property that $|x(\tau)| = r$ for some r, i.e., τ is the *hitting time* of the shell $|x| = r$. Another property that will be used often (without further mention) is *energy conservation*

$$\frac{1}{2}|\xi(t)|^2 + V(x(t)) = \frac{1}{2}|\eta|^2 + V(y) \tag{2.4}$$

for all times t.

Lemma 2.2. *Let $0 < L < R$ and consider two C^1 curves x_1, x_2 in $\mathbb{R}^d$ so that $x_1(0) = x_2(0)$, $|x_1(0)| = |x_2(0)| = R$, and $\dot{x}_1(0) = \dot{x}_2(0)$. Suppose that $\rho < |\dot{x}_i(t)| < 2\rho$, and $\sphericalangle(x_i(t), \dot{x}_i(t)) < \frac{\pi}{4}$, $i = 1, 2$, for all $0 \le t \le T$ where $T > 10L/\rho$. Finally, assume that*

$$\max_{0\le t\le T} |\dot{x}_1(t) - \dot{x}_2(t)| < \delta. \tag{2.5}$$

Define

$$\tau_i := \min[t \ge 0 \;:\; |x_i(t)| = R + L] \quad \textit{for} \quad i = 1, 2.$$

Then

$$|\tau_1 - \tau_2| \le C\frac{\delta L}{\rho^2}.$$

Proof. Observe that

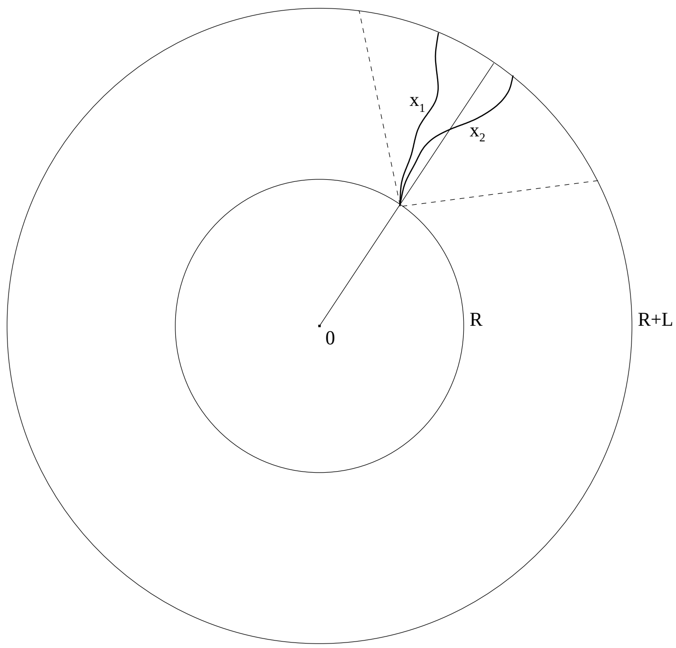
$$\int_0^{\tau_i} 2\dot{x}_i(t) \cdot x_i(t)\, dt = (R + L)^2 - R^2 \quad \text{for} \quad i = 1, 2.$$

Therefore,

$$\left|\int_{\tau_1}^{\tau_2} 2\dot{x}_i(t) \cdot x_i(t)\, dt\right| \le \int_0^{\tau_1} \Big[|x_1(t)||\dot{x}_1(t) - \dot{x}_2(t)| + |\dot{x}_2(t)||x_1(t) - x_2(t)|\Big]\, dt. \tag{2.6}$$

To estimate the right-hand side of (2.6) from above, note firstly that $\tau_1 \le \frac{2L}{\rho}$. Hence, in view of our hypotheses

$$\begin{aligned} &\int_0^{\tau_1} \Big[|x_1(t)||\dot{x}_1(t) - \dot{x}_2(t)| + |\dot{x}_2(t)||x_1(t) - x_2(t)|\Big]\, dt \le \\ &\le \quad (R + L)\delta\tau_1 + 2\rho\frac{\tau_1^2}{2}\delta \le CR\frac{\delta L}{\rho}. \end{aligned} \tag{2.7}$$



To bound the left-hand side of (2.6) from below, one uses that

$$\dot{x}_i(t) \cdot x_i(t) \geq \frac{1}{2}|\dot{x}_i(t)||x_i(t)| \geq \frac{1}{2}R\rho.$$

Combining this with (2.7) yields

$$|\tau_1 - \tau_2| \leq C\frac{\delta L}{\rho^2},$$

as desired. ☐

The following proposition is the main result of this section.

Proposition 2.3. *Let V be as in (1.10) with $\alpha > \frac{1}{2}$. Let $\rho > 0$ be arbitrary but fixed. Then with probability $1 - o(1)$ as $R \to \infty$, any solution $(x(t; y, \eta), \xi(t; y, \eta))$ of (2.1) with $|y| = R$, $|\eta| > \rho$, and $\sphericalangle(y, \eta) < \pi/8$, has the property that $\lim_{t\to\infty} \xi(t; y, \eta) =: \xi(\infty; y, \eta) \neq 0$ exists and, moreover,*

$$|x(t; y, \eta) - y - t\xi(\infty; y, \eta)| \leq C(R + t)^{1-\varepsilon} \tag{2.8}$$

$$|\xi(t; y, \eta) - \xi(\infty; y, \eta)| \leq C\,(R + t)^{-\varepsilon} \tag{2.9}$$

where $\varepsilon > 0$ is so small that $\alpha > \frac{1}{2} + \varepsilon$.

Proof. For notational simplicity we take $\rho = 1$ and $|\eta| \asymp 1$ (the general case being only slightly more complicated). Also, observe that it suffices to prove (2.9), as (2.8) follows by integration. Fix y and η as above and denote the solutions by $x(t), \xi(t)$. Define

$$\tau_k = \inf\{t \geq 0 \;:\; |x(t)| = R + k\}$$

where $\inf \emptyset := \infty$. Our goal is to show that

$$\mathbb{P}\Big[\tau_k \;\text{ exist for }\; 0 \leq k \leq R \text{ and } \sup_{0 \leq k \leq R} |\xi(\tau_k) - \xi(0)| \leq R^{-\varepsilon}\Big] \geq 1 - e^{-R^{\varepsilon}}. \tag{2.10}$$

Observe that $|\xi(\tau_k) - \xi(0)| \leq R^{-\varepsilon}$ for all $0 \leq k \leq R$ implies that $\sphericalangle(x(t), \xi(t)) < \pi/4$ for all times $0 \leq t \leq R$ so that Lemma 2.2 is applicable. Moreover, in this case one also has $\tau_{k+1} - \tau_k \leq 3$. In what follows we shall assume that these two properties hold up to some time and then show that they hold for later times as well. Since they obviously hold for times at least up to $\sqrt{R}$, this is justified. Notice that τ_k is measurable with respect to $\mathcal{F}_{R+k}$. The main idea of the proof is that $\{\xi(\tau_{k+1}) - \xi(\tau_k)\}_{k \geq 0}$ is well-approximated by a martingale difference sequence. More precisely,

$$\begin{aligned} \xi(\tau_{k+1}) - \xi(\tau_k) &= -\int_{\tau_k}^{\tau_{k+1}} \nabla V(x(s))\, ds \\ &= -\int_{\tau_k}^{\tau_{k+1}} \nabla V(x(\tau_k) + (s - \tau_k)\xi(\tau_k))\, ds + O((R+k)^{-2\alpha}). \end{aligned} \tag{2.11}$$

The last line is obtained by Taylor expanding first $x(s)$ and then V (observe that the error term is of the order $(\tau_{k+1} - \tau_k)^3 \, D^2V(\cdot)\nabla V(\cdot)$. Since this is evaluated at points in the shell between $R + k$ and $R + k + 1$, one obtains the stated O-term). In order to make the integral in (2.11) measurable with respect to $\mathcal{F}_{R+k}$, we replace τ_{k+1} with

$$\widetilde{\tau}_{k+1} := \inf\{s + \tau_k \geq \tau_k \;:\; |x(\tau_k) + (s - \tau_k)\xi(\tau_k)| = R + k + 1\}.$$

Since $|\widetilde{\tau}_{k+1} - \tau_{k+1}| \leq C(R + k)^{-\alpha}$ by Lemma 2.2 , one concludes from the preceding that

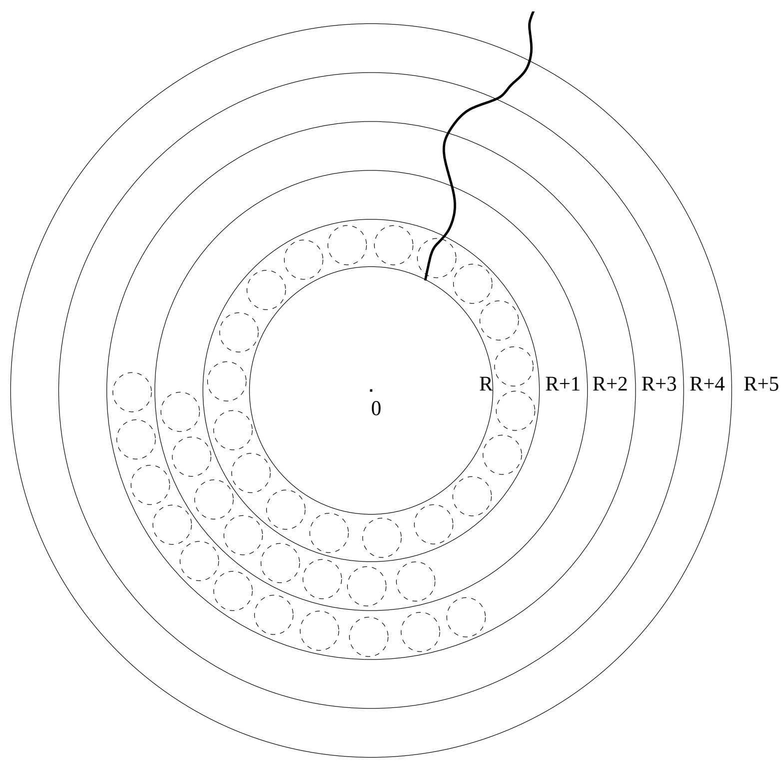
$$\xi(\tau_{k+1}) - \xi(\tau_k) = -\int_{\tau_k}^{\widetilde{\tau}_{k+1}} \nabla V(x(\tau_k) + (s - \tau_k)\xi(\tau_k))\, ds + O((R+k)^{-2\alpha}). \tag{2.12}$$

Denote the integral on the right-hand side by Y_{k+1}. Then Y_{k+1} is measurable with respect to $\mathcal{F}_{R+k}$, and

$$\mathbb{E}[Y_{k+1} \,|\, \mathcal{F}_{R+k}] = 0 \;\text{ for any }\; k \geq 0$$

provided τ_{k+1} exists. In other words, $\{Y_k\}$ is a (vector valued) martingale difference sequence. Also $\|Y_k\|_\infty \leq C(R + k)^{-\alpha}$, since we are assuming that the ω_n are bounded. Hence Lemma 2.1 (applied to the components of Y_k) with $\lambda = R^{\varepsilon}$ and (2.12) imply that

$$\mathbb{P}[\sup_{0 \leq \ell < k \leq R} |\xi(\tau_k) - \xi(\tau_\ell)| > R^{\frac{1}{2} - \alpha + \varepsilon}] \leq CR^2 e^{-cR^{2\varepsilon}} \tag{2.13}$$



provided $\alpha - \frac{1}{2} > 2\varepsilon > 0$. Observe that this implies the existence of the hitting times τ_k all the way up to τ_R. Passing to continuous times allows one to write

$$\mathbb{P}[\sup_{0\le t<s\le\tau_{2R}} |\xi(t)-\xi(s)| > R^{-\varepsilon}] \le C\, R^2 e^{-R^{2\varepsilon}}. \tag{2.14}$$

The bad event on the left-hand side depends on the initial conditions y, η. To remove that dependence one discretizes the initial conditions (y,η) where $|y| = R$ and $|\eta| \asymp 1$ on a scale R^{-10}, say. This then leads to a bound like (2.14) with $R^C \exp^{-R^{2\varepsilon}}$ on the right-hand side. Summing up over dyadic scales $R, 2R, 4R, 8R, \ldots$ finally yields

$$\mathbb{P}[\sup_{y,\eta}\sup_{0\le t<s} (R+t)^{\varepsilon}|\xi(t;y,\eta)-\xi(s;y,\eta)| > 1] \le C\, R^C \exp^{-R^{2\varepsilon}} \tag{2.15}$$

where the sup over y, η is taken only over admissible initial conditions, i.e., $|y| = R$, $|\eta| \asymp 1$, $\sphericalangle(y,\eta) \le \frac{\pi}{4}$. This clearly proves (2.9) for those initial conditions provided R is chosen large enough. It is a simple matter to generalize this to $|\xi| \asymp \rho$ where $\rho = 2^j$, $j = 1, 2, \ldots$. Indeed, we leave it to the reader to check that for $|\eta| \asymp \rho$ the previous argument yields

$$\mathbb{P}[\sup_{y,|\eta|\asymp\rho}\sup_{0\le t<s}(R+t)^{\varepsilon}|\xi(t;y,\eta)-\xi(s;y,\eta)|>1] \le C\,R^{C}\exp^{-\rho^2\,R^{2\varepsilon}}.$$

This allows one to sum over dyadic $\rho \ge 1$. Finally, the case of small (but nonzero!) η simply requires to take R sufficiently large. $\square$

As already remarked in the introduction, the lack of decay of the derivatives of V makes it difficult to deduce any further properties of these scattering trajectories. A detailed analysis of classical scattering trajectories for slowly decaying potentials (in the nonrandom case) was carried out by Herbst [9]. Since his analysis requires at least $|x|^{-2-\varepsilon}$ decay for the second derivatives $D^2V(x)$ (whereas (1.10) only satisfies $|x|^{-\frac{1}{2}-\varepsilon}$ decay for D^2V), his results do not apply. For example, consider the important question of differentiability of $\xi(t;y,\eta)$ with respect to η. Thus let $X(t) = D_\eta x(t;y,\eta)$ and $Y(t) = D_\eta\xi(t;y,\eta)$. In view of (2.1)

$$\dot{X} = Y \qquad \dot{Y} = -D^2V(x(t;y,\eta))\,X \tag{2.16}$$

with initial conditions $X(0) = 0, Y(0) = I$. For a variety of reasons it is very natural to demand that $Y(t)$ remain in a small neighborhood of I. This is desirable, for example, in order to solve the Hamilton-Jacobi equation by the methods of characteristics (see [10]). However, it is clearly not going to be the case for the $\frac{1}{2}$-model, since $X(t)$ will typically grow like t so that $\dot{Y}(T)$ is of size $t^{\frac{3}{2}}$. In fact, it appears that a satisfactory understanding of the classical scattering trajectories has only been obtained under the condition $|D^2V(x)| \le C|x|^{-2-\varepsilon}$, see [9], [16], [6]. Observe that under this condition $|Y(t) - I|$ remains small.

3 Classical scattering for the $\frac{3}{4}$-model

For the remainder of this paper we restrict ourselves to potentials as in (1.2). Define $m(j)$ by means of

$$|D^\gamma V(x)| \le C(1+|x|)^{-m(j)} \quad \text{for multiindices of length} \quad |\gamma| = j.$$

Clearly, we can take $m(j) = \frac{3}{4} + j\beta$. Hence $m(1) > \frac{5}{4}, m(2) > \frac{7}{4}, m(3) > \frac{9}{4}$, and $m(1) + m(3) > \frac{7}{2}$. In [10] Hörmander requires that $m(1) + m(3) > 4$. We again assume that the individual bumps in (1.2) are arranged into disjoint shells. More precisely, for each $R = 2^j$ split $\{R < |x| \le 2R\}$ into shells

$$\mathcal{S}_{\ell,R} := \{x \in \mathbb{R}^d \; : \; R + \ell R^\beta < |x| \le R + (\ell+1)R^\beta\}. \tag{3.1}$$

We assume that $\mathcal{N}$ in (1.2) is chosen so that the terms in (1.2) are supported entirely inside these shells. They define a filtration $\{\mathcal{F}_k\}$ of increasing σ-algebras in the obvious way. The following proposition is the analogue of Proposition 2.3 for the $\frac{3}{4}$-model.

Proposition 3.1. *Let V be as in (1.2) with $\beta > \frac{1}{2}$. Let $\rho > 0$ and $\varepsilon > 0$ be arbitrary. Then up to probability $e^{-R^{\varepsilon}}$, any solution $(x(t;y,\eta), \xi(t;y,\eta))$ of (2.1) with $|y| = R$, $\rho < |\eta| < 2\rho$, and $\sphericalangle(y,\eta) < \pi/8$, has the property that $\lim_{t\to\infty} \xi(t;y,\eta) =: \xi(\infty;y,\eta) \neq 0$ exists and*

$$
\begin{aligned}
|x(t;y,\eta) - y - t\xi(\infty;y,\eta)| &\leq C(R+t)^{\frac{3}{4}-\frac{\beta}{2}+\varepsilon} && (3.2)\\
|\xi(t;y,\eta) - \xi(\infty;y,\eta)| &\leq C\,(R+t)^{-\frac{1}{4}-\frac{\beta}{2}+\varepsilon}, && (3.3)
\end{aligned}
$$

provided $R \geq R_0(\rho,\varepsilon)$.

Proof. It is well-known and very easy to check that forward trajectories have a terminal velocity and satisfy (3.3) with $(R+t)^{1-m(1)}$. This only uses that $m(1) > 1$. This property implies that the following stopping times are always well-defined. We first assume that y, η are fixed. Define stopping times

$$
\begin{aligned}
\tau_\ell &:= \inf\{t \geq 0 \,:\, |x(t)| = R + \ell R^{\beta}\}\\
\widetilde{\tau}_\ell &:= \inf\{s \geq \tau_{\ell-1} \,:\, |x(\tau_{\ell-1}) + (s-\tau_{\ell-1})\xi(\tau_{\ell-1})| = R + \ell R^{\beta}\} && (3.4)
\end{aligned}
$$

for those ℓ with $0 \leq \ell \leq R^{1-\beta}$. Thus, by Taylor expansion,

$$
\begin{aligned}
\xi(\tau_{\ell+1}) - \xi(\tau_\ell) = &- \int_{\tau_\ell}^{\widetilde{\tau}_{\ell+1}} \nabla V(x(\tau_\ell) + (s-\tau_\ell)\xi(\tau_\ell))\, ds\\
&+ O\Big(|\tau_{\ell+1} - \tau_\ell|^3\, \|D^2 V\|_{L^\infty(\mathcal{S}_{\ell,R})}\, \|DV\|_{L^\infty(\mathcal{S}_{\ell,R})}\Big) + O\Big(|\widetilde{\tau}_{\ell+1} - \tau_{\ell+1}|\, \|DV\|_{L^\infty(\mathcal{S}_{\ell,R})}\Big). && (3.5)
\end{aligned}
$$

Denote the integral in (3.5) by $Y_{\ell+1}$. Since $|\tau_{\ell+1} - \tau_\ell| \leq C\frac{R^\beta}{\rho}$ and by Lemma 2.2, with $L = R^\beta$ and $\delta = CR^{-m(1)}\frac{L}{\rho}$,

$$
|\widetilde{\tau}_{\ell+1} - \tau_{\ell+1}| \leq C\frac{R^{2\beta - m(1)}}{\rho^3},
$$

one obtains

$$
\xi(\tau_{\ell+1}) - \xi(\tau_\ell) = Y_{\ell+1} + O\Big(\frac{R^{3\beta}}{\rho^3} R^{-m(1)-m(2)}\Big) + O\Big(\frac{R^{2\beta-2m(1)}}{\rho^3}\Big) = Y_{\ell+1} + O(\rho^{-3}\, R^{-\frac{3}{2}}). \tag{3.6}
$$

Clearly,

$$
\|Y_\ell\|_\infty \leq C|\widetilde{\tau}_{\ell+1} - \tau_\ell|\, \|DV\|_{L^\infty(\mathcal{S}_{\ell,R})} \leq C\,\frac{R^{\beta - m(1)}}{\rho}
$$

and

$$
\sum_{\ell=0}^{\lfloor R^{1-\beta}\rfloor} \|Y_\ell\|_\infty^2 \leq C\,\frac{R^{1+\beta-2m(1)}}{\rho^2} = \frac{R^{-\frac{1}{2}-\beta}}{\rho^2}. \tag{3.7}
$$

Summing up (3.6) yields

$$\Big|\xi(\tau_{\ell_1}) - \xi(\tau_{\ell_2}) - \sum_{\ell=\ell_2+1}^{\ell_1} Y_\ell\Big| \le C\rho^{-3} R^{-\frac12-\beta}.$$

Therefore, Lemma 2.1 and (3.7) imply that

$$\mathbb{P}[|\xi(\tau_{\ell_1}) - \xi(\tau_{\ell_2})| > \lambda\rho^{-1}R^{-\frac14-\frac{\beta}{2}} + C\rho^{-3} R^{-\frac12-\beta}] \le Ce^{-c\lambda^2}$$

for any two ℓ_1, ℓ_2 as above. Setting $\lambda = R^\varepsilon$ one concludes that

$$\mathbb{P}[\sup_{0\le t,s\le 10R/\rho} |\xi(t;y,\eta) - \xi(s;y,\eta)| > R^{-\frac14-\frac{\beta}{2}+\varepsilon}] \le C_\rho\, R^2\, e^{-cR^{2\varepsilon}}$$

provided $R \ge R_0(\rho,\varepsilon)$. The dependence on y,η on the left-hand side can be removed by means of discretization. It is easy to see that this leads to

$$\mathbb{P}[\sup_{|y|=R,|\eta|\asymp\rho}\ \sup_{0\le t,s\le R} |\xi(t;y,\eta) - \xi(s;y,\eta)| > R^{-\frac14-\frac{\beta}{2}+\varepsilon}] \le C\, R^C\, e^{-cR^{2\varepsilon}} < e^{-R^\varepsilon}$$

provided $R \ge R_0(\rho,\varepsilon)$. Finally, summing over $R, 2R, \dots$ yields (3.3), whereas (3.2) follows by integrating in time. We skip the details. □

Remark 3.2. Inspection of the proof reveals that the statement of the proposition remains correct with the same $R_0(\rho,\varepsilon)$ for the entire range $\rho < |\xi|$ and not just in $\rho < |\xi| < 2\rho$. But we shall make no use of this fact.

In [9], Herbst discovered in the long range case that scattering trajectories x_1, x_2 with the same terminal velocity $\lim_{t\to\infty}\xi_1(t) = \lim_{t\to\infty}\xi_2(t)$ posses a limiting difference $\lim_{t\to\infty}(x_1(t) - x_2(t))$. His proof crucially used that $|D^2V(x)| \le C|x|^{-2-\varepsilon}$. As explained in [9], this property allows one to label scattering trajectories by means of $\lim_{t\to\infty}\xi_1(t)$ and $\lim_{t\to\infty}(x_1(t) - x_2(t))$. We discuss this issue below in more detail. It turns out that one can recover this property for the $\frac34$-model. Averaging allows one to make up for the slower decay of D^2V. We start with a simple technical lemma.

Lemma 3.3. *Let $\rho, \varepsilon, R_0(\rho,\varepsilon)$ be as in Proposition 3.1. For any $R_1 \ge R_0(\rho,\varepsilon)$ the following holds up to probability at most $e^{-R_1^\varepsilon}$: Let $x(t)$ be any solution with $|x(0)| = R_1$, and $\rho < |\dot x(0)| < 2\rho$ as constructed in Proposition 3.1. Then, for any α with $|\alpha| \ge 1$,*

$$\Big|\int_t^\infty D^\alpha V(x(s))\,ds\Big| < C_1\,(R_1+t)^{-\frac14-(|\alpha|-\frac12)\beta+\varepsilon} \tag{3.8}$$

for all $t \ge 0$ with a constant C_1 that depends on $\rho, \varepsilon, \alpha$.

Proof. We first consider the portion of the path $x(t)$ that lies between the spheres of radii R and $2R$. Let $\tau_\ell, \tilde{\tau}_\ell$ be defined as in (3.4) and denote the hitting time of the sphere $2R$ by τ^* and that of R by τ_*. Then

$$\begin{aligned} &\textstyle\int_{\tau_*}^{\tau^*} D^\alpha V(x(t))\,dt = \sum_\ell \int_{\tau_\ell}^{\tilde{\tau}_{\ell+1}} D^\alpha V(x(\tau_\ell)+(s-\tau_\ell)\xi(\tau_\ell))\,ds \\ &+\textstyle\sum_\ell O\Big(|\tilde{\tau}_{\ell+1}-\tau_{\ell+1}|\,\|D^\alpha V\|_{L^\infty(\mathcal{S}_{\ell,R})} + |\tau_{\ell+1}-\tau_\ell|^3\|\nabla D^\alpha V\|_{L^\infty(\mathcal{S}_{\ell,R})}\|\nabla V\|_{L^\infty(\mathcal{S}_{\ell,R})}\Big) \end{aligned} \tag{3.9}$$

where the sum extends over all shells $\mathcal{S}_{\ell,R}$ that lie between R and $2R$. Lemma 2.2 with $\delta = R^{-m(1)}\frac{R^\beta}{\rho}$ and $L = R^\beta$ shows that

$$|\tilde{\tau}_{\ell+1}-\tau_{\ell+1}| \le CR^{-\frac{3}{4}+\beta}\rho^{-3}.$$

Combining this with the obvious bounds on the derivatives of V implies that the sum of the error terms in (3.9) is

$$O\Big(R^{-\frac{1}{2}-|\alpha|\beta}\,\rho^{-3}\Big). \tag{3.10}$$

Denote the integral in (3.9) by $Z_{\ell,\alpha}$. Then $\{Z_{\ell,\alpha}\}_\ell$ form a martingale difference sequence with respect to the obvious filtration. Moreover,

$$\|Z_{\ell,\alpha}\|_\infty \le C\rho^{-1}\,R^{-\frac{3}{4}-(|\alpha|-1)\beta}.$$

The corresponding square function (with $R^{1-\beta}$ summands) is therefore bounded by

$$\Big(\sum_\ell \|Z_{\ell,\alpha}\|_\infty^2\Big)^{\frac{1}{2}} \le C\,\rho^{-1}\,R^{-\frac{1}{4}-(|\alpha|-\frac{1}{2})\beta}. \tag{3.11}$$

Since (3.11) is much larger than (3.10), Lemma 2.1 implies that up to probability e^{-R^ε}

$$\Big|\int_{\tau_*}^{\tau^*} D^\alpha V(x(s))\,ds\Big| < C_1\,R^{-\frac{1}{4}-(|\alpha|-\frac{1}{2})\beta+\varepsilon},$$

provided $R_0(\rho,\varepsilon)$ is sufficiently large. The lemma follows by discretizing in the initial conditions and summing over 2^jR. This has been carried out in detail above and we do not repeat it. $\square$

Remark 3.4. The reader will have no difficulty verifying by the same methods that

$$\Big|\int_0^t V(x(s))\,ds\Big| < C_1\,(R_1+t)^{-\frac{1}{4}+\frac{1}{2}\beta+\varepsilon} \tag{3.12}$$

up to probability at most $e^{-R_1^\varepsilon}$.

Lemma 3.5. *Let $\rho, \varepsilon, R_0(\rho,\varepsilon)$ be as in Proposition 3.1. For any $R_1 \geq R_0(\rho,\varepsilon)$ the following holds up to probability at most $e^{-R_1^\varepsilon}$: Let x_0, x_1 be any two solutions with $|x_0(0)| = |x_1(0)| = R_1$, and $\rho < |\dot{x}_0(0)|, |\dot{x}_1(0)| < 2\rho$ as constructed in Proposition 3.1 such that $\lim_{t\to\infty} \dot{x}_0(t) = \lim_{t\to\infty} \dot{x}_1(t)$. Then*

$$\lim_{t\to\infty} (x_0(t) - x_1(t)) =: \Delta(\infty)$$

exists. Moreover, if $\Delta(\infty) = 0$, then $x_0 \equiv x_1$.

Proof. By (3.3) (with $\xi = \dot{x}$) and the assumption of equal terminal velocity

$$|\xi_0(t) - \xi_1(t)| \leq C(R_1 + t)^{-\frac{1}{4} - \frac{\beta}{2} + \varepsilon} \tag{3.13}$$

and thus, since $\beta > \frac{1}{2}$,

$$|x_0(t) - x_1(t)| < C(R_1 + t)^{\frac{1}{2} - \varepsilon} + |x_0(0) - x_1(0)|. \tag{3.14}$$

Let $x_\theta := x_0 + \theta(x_1 - x_0)$, where $0 \leq \theta \leq 1$. Let $\Delta(t) = x_1(t) - x_0(t)$. In view of (3.14),

$$|\Delta(t)| \leq C\,(R_1 + t)^{\frac{1}{2} - \varepsilon}. \tag{3.15}$$

For the remainder of this proof t will be large. Moreover, constants will be allowed to depend on R_1. From (2.1) and Taylor's formula one obtains

$$\begin{aligned}
\dot{\Delta}(t) &= \xi_1(t) - \xi_0(t) = -\int_t^\infty \Big[\nabla V(x_1(s)) - \nabla V(x_0(s))\Big]\,ds \\
&= -\sum_{1\leq|\alpha|\leq N} \frac{1}{\alpha!} \int_t^\infty \nabla D^\alpha V(x_0(s))\Delta(s)^\alpha\,ds && (3.16) \\
&\quad - \sum_{|\alpha|=N+1} \frac{1}{\alpha!} \int_0^1 \int_t^\infty \nabla D^\alpha V(x_\theta(s))\,\Delta(s)^\alpha\,ds\,d\theta. && (3.17)
\end{aligned}$$

Here N is chosen so large that the error term (3.17) is on the order of

$$\begin{aligned}
O\Big(\int_t^\infty s^{-\frac{3}{4}-(N+2)\beta}\, s^{N(\frac{1}{2}-\varepsilon)} |\Delta(s)|\,ds\Big) &= O\Big(\int_t^\infty s^{-3}\,|\Delta(s)|\,ds\Big). \\
&= O\Big(\Delta(t)t^{-2} + \int_t^\infty s^{-2}\,|\dot{\Delta}(s)|\,ds\Big) && (3.18)
\end{aligned}$$

Observe that in order to obtain this estimate we have used the obvious bound on the derivatives of V, as well as (3.15). Now fix any α with $1 \leq |\alpha| \leq N$ and denote

$$U_\alpha(t) := -\int_t^\infty \nabla D^\alpha V(x_0(s))\,ds.$$

By Lemma 3.3

$$|U_\alpha(t)| \le C\, t^{-\frac{1}{4}-(|\alpha|+\frac{1}{2})\beta+\varepsilon}$$

up to probability at most $e^{-R_1^\varepsilon}$. Integrating by parts in the α-term in (3.16) therefore yields

$$\begin{aligned}
&\Big|\int_t^\infty \nabla D^\alpha V(x_0(s))\Delta(s)^\alpha\, ds\Big| \le \Big|\int_t^\infty U_\alpha(s)\frac{d}{ds}\Big(\Delta(s)^\alpha\Big)\, ds\Big| + \Big|U_\alpha(t)\Delta(t)^\alpha\Big| \\
&\le\ C\int_t^\infty s^{-\frac{1}{4}-(|\alpha|+\frac{1}{2})\beta+\varepsilon}\, s^{(\frac{1}{2}-\varepsilon)(|\alpha|-1)}\, |\dot\Delta(s)|\, ds + C\, t^{-\frac{1}{4}-(|\alpha|+\frac{1}{2})\beta+\varepsilon}\, t^{(\frac{1}{2}-\varepsilon)(|\alpha|-1)}\, |\Delta(t)| \\
&\le\ C\int_t^\infty s^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}\, |\dot\Delta(s)|\, ds + C\, t^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}\, |\Delta(t)|. \qquad (3.19)
\end{aligned}$$

Here we have used that $|\alpha| = 1$ is the worst case. Now fix $\beta > \frac{1}{2}$ and choose $\varepsilon > 0$ so small that $-\frac{1}{4} - \frac{3\beta}{2} + \varepsilon < -1 - \delta$ for some $\delta > 0$. Combining (3.16)–(3.19) yields

$$|\dot\Delta(t)| \le Ct^{-1-\delta}|\Delta(t)| + C\int_t^\infty s^{-1-\delta}|\dot\Delta(s)|\, ds. \qquad (3.20)$$

Starting from the trivial bound $|\dot\Delta(t)| \le C$ and assuming (as we may) that $\frac{1}{\delta}$ is not an integer iteration of (3.20) proves that

$$|\dot\Delta(t)| \le Ct^{-(n+1)\delta} \ \text{ and } \ |\Delta(t)| \le Ct^{1-(n+1)\delta}$$

where n is maximal so that $n\delta < 1$. This shows that $\lim_{t\to\infty}\Delta(t)$ exists. For the final statement of the lemma, observe that (3.20) implies that

$$\sup_{t\ge T}|\dot\Delta(t)| \le C\, T^{-1-\delta} \sup_{t\ge T}|\Delta(t)|.$$

If $\Delta(\infty) = 0$, then this implies that

$$\sup_{t\ge T}|\Delta(t)| \le CT^{-\delta} \sup_{t\ge T}|\Delta(t)|.$$

So $\Delta(t) = 0$ for all large t and $x_0 \equiv x_1$ by the uniqueness theorem for ODEs. $\square$

Following common practice (see for example [16]), we set

$$\Sigma^+ := \Big\{(y,\eta)\in\mathbb{R}^{2d}\ :\ V(y) + \frac{1}{2}|\eta|^2 > 0, \quad (x(t),\xi(t)) \ \text{ satisfies } \ \limsup_{t\to\infty}|x(t)| = \infty\Big\},$$

where of course $(x(t),\xi(t)) = (x(t;y,\eta),\xi(t;y,\eta))$ is a solution of (2.1). The following (deterministic) lemma is a standard characterization of Σ^+.

Lemma 3.6. *Let V be as in* (1.2) *with an* arbitrary *choice of bounded* $\{\omega_n\}_{n\in\mathcal{N}}$. *Then*

$$(y,\eta)\in\Sigma^+ \qquad\Longleftrightarrow\qquad \lim_{t\to\infty}\xi(t;y,\eta)=:\xi(\infty)\neq 0 \quad \textit{exists.}$$

Moreover, in that case,

$$|\xi(t;y,\eta)-\xi(\infty;y,\eta)|\le Ct^{1-m(1)}, \qquad |x(t;y,\eta)-(y+t\xi(\infty))|\le Ct^{2-m(1)}. \tag{3.21}$$

Finally, the pair $(x(t;y,\eta),\xi(t;y,\eta))$ *belongs to the outgoing region of phase space, i.e.,*

$$\sphericalangle(x(t;y,\eta),\xi(t;y,\eta))<\frac{\pi}{4} \tag{3.22}$$

for all sufficiently large times t.

Proof. This is basically the same as the proof of Lemma II.1 in [9], and will be skipped. □

Combining Lemma 3.6 with Lemma 3.5 yields the following useful fact.

Corollary 3.7. *Almost surely the following holds: Let* (x_0,ξ_0), (x_1,ξ_1) *be arbitrary solutions of* (2.1) *so that* $\lim_{t\to\infty}\xi_0(t)=\lim_{t\to\infty}\xi_1(t)\neq 0$. *Then*

$$\lim_{t\to\infty} x_0(t)-x_1(t)=:\Delta(\infty) \tag{3.23}$$

exists. If $\Delta(\infty)=0$, then $x_0\equiv x_1$.

Proof. Fix some small $\varepsilon>0$ and an arbitrary $\rho>0$. For all positive integers j let $\mathcal{B}_j(\rho)$ be the bad event with respect to the properties stated in Lemma 3.5 with parameters ρ and $R_1=2^j\ge R_0=2^{j_0}$. According to that lemma, $\mathbb{P}[\mathcal{B}_j(\rho)]<e^{-2^{\varepsilon j}}$ for large j so that

$$\sum_{j=j_0}^{\infty}\mathbb{P}[\mathcal{B}_j(\rho)]<\infty.$$

Now let $\mathcal{B}(\rho):=\limsup_j\mathcal{B}_j(\rho)$. This event has probability zero, and similarly the union $\bigcup_\rho\mathcal{B}(\rho)$ where the union is taken over $\rho=2^k$, $k\in\mathbb{Z}$. The corollary will be proved off the event $\bigcup_\rho\mathcal{B}(\rho)$. In other words, fix a potential V from the complement of $\bigcup_\rho\mathcal{B}(\rho)$.
If $x_0=x(\cdot;y_0,\eta_0)$ and $x_1=x(\cdot;y_1,\eta_1)$ are as above, then $(y_0,\eta_0),(y_1,\eta_1)\in\Sigma^+$ by the previous lemma. Let $\xi(\infty)$ be the common terminal velocity and define $\rho>0$ by means of $\rho<|\xi(\infty)|<2\rho$. Select $R_1=2^j$ so large that both the event $\mathcal{B}_j(\rho)$ does not take place and so that (3.22) holds for the curves x_0 and x_1 after they have passed the sphere $\{|x|=R_1\}$. We may assume that $|x_0(t_0)|=|x_1(t_0)|=R_1$. If the hitting times are different, then one can translate one path in time by the time difference. It is clear that this does not change the conclusion (3.23). Applying Lemma 3.5 to the paths starting at time t_0 yields the desired property. □

The remainder of this section is devoted to constructing Herbst's "wave-like" operators, which we denote by $\mathcal{H}^+$, see Theorem 3.9 below. As the details are similar either to those in [9] or, as far as the probabilistic arguments go, closely related to ones we have already presented, we will be somewhat brief. As always, we consider V as in (1.2) with $\beta > \frac{1}{2}$. For the following lemma define a family of curves $\{z_n(\cdot, p)\}_{n\geq 0}$ via $z_0(t,p) = tp$,

$$z_n(0,p) = 0 \qquad\qquad \dot{z}_n(t,p) = p + \int_t^\infty \nabla V(z_{n-1}(s,p))\, ds \quad \text{for} \quad n \geq 1$$

for any given $p \neq 0$.

Lemma 3.8. *Almost surely the following holds: Let $p \neq 0$ be arbitrary. There are (nonrandom) times $T_n = T_n(|p|)$ such that $|z_n(t,p)| \geq \frac{1}{2}|p|t$ for $t \geq T_n$. Moreover, for all n*

$$|\dot{z}_n(t,p) - p| \leq C_n\, t^{-\frac{1}{2}-\delta} \quad \textit{and} \quad |z_n(t,p) - tp| \leq C_n\, t^{\frac{1}{2}-\delta} \tag{3.24}$$

where $\delta = \delta(\beta) > 0$. Finally, there exists $n_0 = n_0(\beta)$ with the property that for any $\rho > 0$

$$\sup_{\rho < |p| \leq 2\rho} |\dot{z}_{n_0+1}(t,p) - \dot{z}_{n_0}(t,p)| \leq C_1\, t^{-1-\delta} \quad \textit{for all} \quad t \tag{3.25}$$

where C_1 is a random constant that also depends on ρ.

Proof. The statement involving linear growth past time T_n only requires that $m(1) > 1$. Indeed, if it is true for z_{n-1}, $n \geq 1$ (and it clearly is if $n = 1$), then

$$|\dot{z}_n(t,p) - p| \leq \int_t^\infty C\big(\frac{1}{2}|p|s\big)^{-\frac{3}{4}-\beta}\, ds \leq C_\rho\, t^{\frac{1}{4}-\beta} \tag{3.26}$$

provided $t \geq T_{n-1}(\rho)$. Hence

$$\begin{aligned} |z_n(t,p) - tp| &\leq \int_0^{T_{n-1}} |z_n(s,p)|\, ds + C_\rho\, t^{\frac{5}{4}-\beta} \\ &\leq C_n(\rho) + C_\rho\, t^{\frac{5}{4}-\beta} \leq \frac{1}{2} t|p| \end{aligned}$$

provided $t \geq T_n(\rho)$. It follows from these properties that $(z_n(t,p), \dot{z}_n(t,p))$ belongs to the outgoing region of phase space for $t \geq T_n(\rho)$ (possibly after increasing T_n). One can easily improve (3.26) by exploiting averaging in the usual way. More precisely, we show inductively that up to probability at most $e^{-t_0^\varepsilon}$ for all $t \geq t_0$ the estimate (3.24) holds. Only the first inequality in (3.24) needs to be proved, and it is obvious for $n = 0$. Suppose that (3.24) holds up to $n-1$, and we want to prove it for n. On possibility is to Taylor expand, which gives

$$\begin{aligned} \dot{z}_n(t,p) &= p + \sum_{|\alpha|\leq N} \frac{1}{\alpha!} \int_t^\infty \nabla D^\alpha V(sp)(z_{n-1}(s,p) - sp)^\alpha\, ds + \\ &\quad + \sum_{|\alpha| = N+1} \frac{1}{\alpha!} \int_0^1 \int_t^\infty \nabla D^\alpha V(\theta sp + (1-\theta) z_{n-1}(s,p))(z_{n-1}(s,p) - sp)^\alpha\, ds \end{aligned} \tag{3.27}$$

where N is taken sufficiently large so that the error term in (3.27) is no bigger than t^{-1}, say. As usual, we need to bound quantities of the form

$$U_\alpha(t) := \int_t^\infty \nabla D^\alpha V(sp)\, ds.$$

A simple application of Lemma 2.1 shows that for any multi-index α

$$|U_\alpha(t)| \le C_\alpha\, t^{-\frac{1}{4}-(|\alpha|+\frac{1}{2})\beta+\varepsilon} \quad \text{for} \quad t \ge t_0 \tag{3.28}$$

up to probability at most $e^{-t_0^\varepsilon}$ (this situation is easier than the previous ones, as the curve inside V is a straight line). Integrating by parts inside the integrals in the first line of (3.27) (similarly to the way it was done in (3.19)) and invoking the previous bound on $U_\alpha(t)$, one checks that

$$|\dot{z}_n(t,p) - p| \le C\, t^{-\frac{1}{2}-\delta}$$

up to probability at most $e^{-t_0^\varepsilon}$, as claimed. The almost sure nature of the statement is now derived by means of the same type of Borel-Cantelli construction as in the proof of Corollary 3.7. More precisely, given a fixed $\rho > 0$, we require that the large deviation bounds (3.28) hold for some $t_0 = t_0(\rho)$. This can clearly be achieved by means of Borel-Cantelli at the cost of random t_0, which in turn leads to random constants in (3.24).

To obtain (3.25), let $\Delta_n(t,p) = z_n(t,p) - z_{n-1}(t,p)$ for $n \ge 1$. Very much in the same spirit as the proofs of Lemmas 3.3 and 3.5 one verifies that

$$|\dot{\Delta}_{n+1}(t,p)| \le C_n \int_t^\infty \frac{|\dot{\Delta}_n(s,p)|}{s^{1+\delta}}\, ds + C_n\, t^{-1-\delta} |\Delta_n(t,p)| \tag{3.29}$$

with random constants C_n and some $\delta > 0$ only depending on β. In fact,

$$\dot{z}_{n+1}(t,p) - \dot{z}_n(t,p) = \int_t^\infty \Big[\nabla V(z_n(s,p)) - \nabla V(z_{n-1}(s,p))\Big]\, ds$$

$$= \sum_{1\le|\alpha|\le N} \frac{1}{\alpha!} \int_t^\infty \nabla D^\alpha V(z_{n-1}(s,p)) \Delta_n(s)^\alpha\, ds \tag{3.30}$$

$$- \sum_{|\alpha|=N+1} \frac{1}{\alpha!} \int_0^1 \int_t^\infty \nabla D^\alpha V(z_{n-1}(s,p) + \theta(z_n(s,p) - z_{n-1}(s,p))\, \Delta_n(s)^\alpha\, ds\, d\theta. \tag{3.31}$$

The error term of the Taylor expansion (3.31) can be estimated without using any randomness, since (3.24) implies that

$$|\Delta_n(t,p)| \le C t^{\frac{1}{2}-\delta},$$

and also because $\beta > \frac{1}{2}$ (but here $\beta \ge \frac{1}{2}$ would equally well work). Controlling the main terms (3.30) requires estimates of the form

$$\Big|\int_t^\infty \nabla D^\alpha V(z_{n-1}(s,p))\, ds\Big| \le C_\alpha\, t^{-\frac{1}{4}-(|\alpha|+\frac{1}{2})\beta+\varepsilon} \quad \text{for} \quad t \ge t_0 \tag{3.32}$$

up to probability at most $e^{-t_0^\varepsilon}$. These are proved by the method from (3.27), i.e., by means of reduction to the linear case via Taylor expansion. Since the details are basically identical, we skip them. With (3.32) at our disposal, (3.30) can now be estimated by the same integration by parts argument as in (3.19). Starting from an estimate like

$$|\Delta_1(t)| \le C_1 t^{\frac{1}{2}} \quad \text{and} \quad |\dot{\Delta}_1(t)| \le C_1 t^{-\frac{1}{2}},$$

say, one iteratively derives that

$$|\dot{\Delta}_{n+1}(t)| \le C_n t^{-\frac{1}{2}-n\delta}$$

as long as $n\delta < \frac{1}{2}$. Assuming as we may that $\frac{1}{\delta}$ is not an integer, we let n_0 be maximal with the property that $(n_0-1)\delta < \frac{1}{2}$. The lemma follows. □

In what follows let $\Sigma := \{(a,p) \in \mathbb{R}^d \times \mathbb{R}^d \ : \ p \neq 0\}$. Also, we write $z(t,p)$ instead of $z_{n_0+1}(t,p)$ and $\tilde{z}(t,p)$ instead of $z_{n_0}(t,p)$ for the approximate solutions constructed above.

Theorem 3.9. *Almost surely the following holds: Given $(a,p) \in \Sigma$, there exists a unique solution $(x_{(a,p)}, \xi_{(a,p)})$ of (2.1) such that*

$$|x_{(a,p)}(t) - a - z(t,p)| \to 0 \quad \text{and} \quad |\xi_{(a,p)}(t) - p| \to 0 \quad \text{as} \quad t \to \infty. \tag{3.33}$$

The map

$$\mathcal{H}^+ : \begin{cases} \Sigma & \to \quad \Sigma^+ \\ (a,p) & \mapsto \quad (x_{(a,p)}(0), \xi_{(a,p)}(0)) \end{cases}$$

is one-to-one and onto.

Proof. The almost sure nature of this statement is explained just as in the previous proof. To construct $x_{(a,p)}, \xi_{(a,p)}$ satisfying (3.33) define

$$\dot{x}_{(a,p)}(t) = p + \int_t^\infty \nabla V(x(s))\, ds.$$

Since also

$$\dot{z}(t,p) = p + \int_t^\infty \nabla V(\tilde{z}(s,p))\, ds,$$

one then concludes that

$$\dot{x}_{(a,p)}(t) - \dot{z}(t,p) = \int_t^\infty \Big[\nabla V(a + z(s,p) - y(s)) - \nabla V(\tilde{z}(s,p))\Big]\, ds.$$

Here we have set $x_{(a,p)}(t) = a + z(t,p) - y(t)$ with some y that is continuous and goes to zero at infinity. If the first "boundary condition at infinity" holds, see (3.33), then necessarily

$$a - (x_{(a,p)}(t) - z(t,p)) = y(t) = \int_t^\infty \int_s^\infty \Big[\nabla V(a + z(\tau,p) - y(\tau)) - \nabla V(\tilde{z}(\tau,p))\Big]\, d\tau\, ds. \tag{3.34}$$

We interpret this as an integral equation for $y(t)$ with $y(t) \to 0$ as $t \to \infty$. By definition, $x_{(a,p)}(t) = a + z(p,t) - y(t)$ is the desired classical trajectory. Since the integrand only decays like $\tau^{-\frac{7}{4}-\varepsilon}$, some care is needed in interpreting the right–hand side of (3.34). For this purpose, define

$$U(t) := \int_t^\infty D^2V(\tilde{z}(s,p))\,ds.$$

We claim that, up to a bad event of probability at most $e^{-t_0^\varepsilon}$,

$$\sup_{\rho<|p|<2\rho} |U(t)| \le t^{-1-\delta} \ \text{ for all } \ t \ge t_0.$$

Here $t_0 = t_0(\rho, \varepsilon)$ and $\varepsilon > 0$ is very small so that $\delta = \delta(\beta) > 0$. This can again be verified by means of Taylor expansion. More precisely,

$$U(t) = \sum_{|\gamma|\le N} \frac{1}{\gamma!} \int_t^\infty D^{2+\gamma}V(sp)[\tilde{z}(s,p) - sp]^\gamma\,ds \tag{3.35}$$

$$+ \int_0^1 \sum_{|\gamma|=N+1} \frac{1}{\gamma!} \int_t^\infty D^{2+\gamma}V(\theta sp + (1-\theta)\tilde{z}(s,p))[\tilde{z}(s,p) - sp]^\gamma\,ds\,d\theta. \tag{3.36}$$

The term with $\gamma = 0$ in (3.35) is $O(t^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon})$ by Lemma 2.1. Integrating by parts and using (3.24) with $n = n_0$ allows one to show that all terms in (3.35) with $|\gamma| \ge 1$ are no larger than the term with $\gamma = 0$. Finally, N is chosen so large that (3.36) is less than $O(t^{-2})$, say, for any realization of V, which establishes our claim. By means of the usual Borel-Cantelli construction, see the proof of Corollary 3.7 above, one obtains that almost surely

$$\sup_{\rho<|p|<2\rho} |U(t)| \le C_V\, t^{-1-\delta} \ \text{ for all } \ t \tag{3.37}$$

where C_V is random and also depends on ρ. For a given $a \in \mathbb{R}^d$ and function $y(t)$ set $w(t) := a + z(t,p) - \tilde{z}(t,p) - y(t)$. Taylor expanding and integrating by parts shows that the double integral in (3.34) can be interpreted as

$$\begin{aligned}(Ay)(t) &:= \int_t^\infty \int_s^\infty U(\tau)\dot{w}(\tau)\,d\tau\,ds + \int_t^\infty U(s)w(s)\,ds \\ &+ \int_t^\infty \int_s^\infty \int_0^1 \frac{1}{2}\langle \nabla D^2V(\theta\tilde{z}(s,p) + (1-\theta)(a + z(\tau,p) - y(\tau)))\,w(\tau), w(\tau)\rangle\,d\theta\,d\tau\,ds.\end{aligned} \tag{3.38}$$

Observe that the triple integral in (3.38) is no larger than

$$\int_t^\infty \int_s^\infty C\tau^{-\frac{3}{4}-3\beta}|w(\tau)|^2\,d\tau ds \le C \sup_{t\ge T} |w(t)|^2\, t^{\frac{5}{4}-3\beta}. \tag{3.39}$$

The single integral converges almost surely in view of (3.37) provided w is bounded. In fact,

$$\Big|\int_t^\infty U(s)w(s)\,ds\Big| \le \frac{1}{\delta} C_V\, t^{-\delta} \sup_{s\ge t}(|a| + |z(s,p) - \tilde z(s,p)| + |y(s)|). \tag{3.40}$$

To insure convergence of the double integral we impose the assumption $|\dot y(t)| \le t^{-1}$, say. In view of (3.37) and (3.25) one then obtains that

$$\Big|\int_t^\infty \int_s^\infty U(\tau)\dot w(\tau)\,d\tau\,ds\Big| \le \frac{1}{\delta} C_V\, t^{-\delta} \sup_{s\ge t}(s|\dot z(s,p) - \dot{\tilde z}(s,p)| + s|\dot y(s)|) \le \frac{2}{\delta} C_V\, t^{-\delta}. \tag{3.41}$$

This shows that A is a well-defined operator on the complete metric space

$$Y_T := \Big\{ y \in C^1([T,\infty)) \;:\; y(t) \to 0 \;\text{ as }\; t\to\infty, \quad |y(t)| \le 1, \quad |\dot y(t)| \le t^{-1} \Big\},$$

provided T is chosen large enough (it is random and depends on ρ). The metric here is

$$d_T(y_1,y_2) = \sup_{t\ge T}\Big[|y_1(t) - y_2(t)| + t|\dot y_1(t) - \dot y_2(t)| \Big].$$

We will now check that $A : Y_T \to Y_T$ for T large, and that A is a contraction on that space. It then follows that there is a (unique) fixed point in Y_T. It is evident from (3.39)–(3.41) that $|(Ay)(t)| \le 1$ for large t. Furthermore, it is clear that $\frac{d}{dt}(Ay)$ will satisfy bounds like (3.39)–(3.41) that are better by one power of t. We skip the details. Hence $A : Y_T \to Y_T$. As far as the contraction is concerned, it is easy to see that one has

$$d_T(Ay_1, Ay_2) \le \delta^{-1} C_V\, T^{-\delta}\, d_T(y_1,y_2) + C\, T^{\frac{5}{4} - 3\beta}\, d_T(y_1,y_2) \sup_{t\ge T}(|a| + |z(t,p) - \tilde z(t,p)| + 1),$$

which gives the desired property for large T. Now let $y \in Y_T$ satisfy $Ay = y$ and define $x = a + z(t,p) - y(t)$. It is evident that this function x satisfies (3.33). It now remains to show that x is a solution of (2.1). Differentiating (3.38) twice with respect to t, one obtains that

$$\begin{aligned}\ddot y(t) &= D^2V(\tilde z(t,p))w(t) + \int_0^1 \frac{1}{2}\langle \nabla D^2 V(\theta \tilde z(s,p) + (1-\theta)(a + z(\tau,p) - y(\tau)))\, w(\tau), w(\tau)\rangle\, d\theta \\ &= \nabla V(a + z(t,p) - y(t)) - \nabla V(\tilde z(t,p)).\end{aligned}$$

This implies that $\ddot x = \ddot z - \ddot y = -\nabla V(\tilde z) - \nabla V(x) + \nabla V(\tilde z) = -\nabla V(x)$. Thus x is indeed a solution. That $\mathcal{H}^+$ is one-to-one and onto is a simple consequence of Corollary 3.7, see [9] Theorem II.1 for details. $\square$

Remark 3.10. Following [9], let

$$E_t^{(0)}(a,p) = (a + z(t,p), p) \;\text{ and }\; S_t^{(0)}(a,p) = (a + pt, p).$$

The flow of (2.1) is denoted by U_t. By the theorem,

$$\lim_{t\to\infty} U_t \circ \mathcal{H}^+(a,p) - E_t^{(0)}(a,p) = 0$$

for all $(a,p) \in \Sigma$. Moreover, $\mathcal{H}^+$ intertwines U_t and $S_t^{(0)}$, i.e.,

$$U_t \circ \mathcal{H}^+ = \mathcal{H}^+ \circ S_t^{(0)}.$$

In this respect, $\mathcal{H}^+$ enjoys many properties of a classical wave operator. However, it is in general not a canonical transformation. It is important to realize that it is necessary to consider iterations z_n for many n, see Lemma 3.8. This is in contrast to [1] and [2], where only one iteration is used.

4 Solving the Hamilton-Jacobi equation

In this section we construct solutions, with probability one, of the equation

$$\partial_t S(t,\xi) - \frac{1}{2}|\xi|^2 - V(D_\xi S(t,\xi)) = 0 \tag{4.1}$$

on a subset of $(0,\infty) \times \{\xi \in \mathbb{R}^d \ : \ \xi \neq 0\}$. As in [10], the method of characteristics is used. This requires bounding derivatives of the characteristics, which are precisely the trajectories of (2.1), with respect to the initial conditions. Abusing notation slightly (in the previous section this notation is used with a slightly different meaning), we let $x(t;y,\eta), \xi(t;y,\eta)$ denote a solution of (2.1) with initial data $x(t_0;y,\eta) = t_0 y$ and $\xi(t_0;y,\eta) = \eta$. For the sake of simplicity, we shall restrict ourselves to the case $1 \le |\eta| \le 2$. Then one requires $|y-\eta| < c_0$ for some small constant c_0, say $c_0 = \frac{1}{10}$. We refer to such y,η as *admissible*. The extension to other dyadic shells $\rho < |\xi| < 2\rho$ is straightforward and is analogous to the previous section. The reader will easily verify that small ρ can be compensated for by taking t_0 large.

Lemma 4.1. *Let $\varepsilon > 0$ be sufficiently small depending on $\beta > \frac{1}{2}$. Then there exists a large time $t_0 = t_0(\varepsilon)$ so that up to probability at most $e^{-t_0^\varepsilon}$ the following property holds: Any solution $x(t;y,\eta), \xi(t;y,\eta)$ as specified above satisfies*

$$\left| D^\alpha_{y,\eta}(\xi(t;y,\eta) - \eta)\right| \le t_0^{\frac{3}{4}-\frac{3\beta}{2}+\varepsilon} \quad \textit{for any} \quad |\alpha| = 1. \tag{4.2}$$

Proof. Fix admissible y,η. For simplicity, let $x(t) = x(t;y,\eta)$ and $\xi(t) = \xi(t;y,\eta)$. Denote $X = D_\eta x$ and $Y = D_\eta \xi$. Differentiating (2.1) with respect to η shows that

$$\begin{aligned} \dot X = Y \quad &, \quad \dot Y(t) = -D^2 V(x(t))X \\ X(t_0) = 0 \quad &, \quad Y(t_0) = I \end{aligned} \tag{4.3}$$

where I is the $d \times d$ identity. Our goal is to show that Y remains in a small neighborhood of I. Let

$$U(t) = -\int_t^\infty D^2V(x(s))\,ds.$$

Then $|U(t)| \leq C\,t^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}$ for $t \geq t_0$ up to probability at most $e^{-t_0^\varepsilon}$, see Lemma 3.3. Integrating out (4.3) yields

$$\begin{aligned} Y(t) - Y(t_0) &= -\int_{t_0}^t D^2V(x(s))X(s)\,ds = \int_{t_0}^t U(s)\dot{X}(s)\,ds - U(t)X(t) \\ &= \int_{t_0}^t U(s)Y(s)\,ds - U(t)\int_{t_0}^t Y(s)\,ds. \end{aligned}$$

This clearly implies that

$$\|Y(t) - I\| \leq C\int_{t_0}^t s^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}\|Y(s)\|\,ds$$

so that

$$\max_{t_0\leq t\leq T}\|Y(t) - I\| \leq C\,t_0^{\frac{3}{4}-\frac{3\beta}{2}+\varepsilon}\max_{t_0\leq t\leq T}\|Y(t)\|. \tag{4.4}$$

Hence

$$\max_{t_0\leq t\leq T}\|Y(t)\| \leq 2,$$

say, provided t_0 was chosen large enough. Reinserting this bound into (4.4) and letting $T \to \infty$ shows that

$$\sup_{t\geq t_0}\|Y(t) - I\| \leq C\,t_0^{\frac{3}{4}-\frac{3\beta}{2}+\varepsilon},$$

as desired. Now let $X = D_y x$ and $Y = D_y \xi$. These quantities satisfy the system (4.3) with initial conditions $X(t_0) = t_0\,I$ and $Y(t_0) = 0$. Virtually the same argument as above now implies that Y satisfies (4.2), as claimed. $\square$

The following corollary is basically an integrated version of the previous lemma. It will be used in the construction of solutions of (4.1). The exact form of the exponent in (4.5) is not relevant, only that it can be made strictly less than one.

Corollary 4.2. *Under the same hypotheses as in the previous lemma one has the bound*

$$\Big|D^\alpha_{y,\eta}(x(t;y,\eta) - y - (t-t_0)\xi(t;y,\eta))\Big| \leq C\,t^{\frac{7}{4}-\frac{3\beta}{2}+\varepsilon} \tag{4.5}$$

for any $|\alpha| = 1$ and all times $t \geq t_0$.

Proof. Fix some α of length one. As above, let

$$U(t) := -\int_t^\infty D^2V(x(s;y,\eta))\,ds.$$

Using (2.1) and integrating by parts one obtains

$$\begin{aligned}
&D^\alpha_{y,\eta}(x(t;y,\eta)-y-(t-t_0)\xi(t;y,\eta)) = D^\alpha_{y,\eta}\int_{t_0}^t\Big[\xi(s;y,\eta)-\xi(t;y,\eta)\Big]\,ds\\
=\;& D^\alpha_{y,\eta}\int_{t_0}^t (s-t_0)\,\nabla V(x(s;y,\eta))\,ds = \int_{t_0}^t (s-t_0)\,D^2V(x(s;y,\eta))D^\alpha_{y,\eta}x(s;y,\eta)\,ds\\
=\;& -\int_{t_0}^t U(s)\Big[D^\alpha_{y,\eta}x(s;y,\eta)+(s-t_0)D^\alpha_{y,\eta}\xi(t;y,\eta)\Big]\,ds + (t-t_0)U(t)\,D^\alpha_{y,\eta}x(t;y,\eta).
\end{aligned}$$

Since

$$D^\alpha_{y,\eta}x(t;y,\eta) = D^\alpha_{y,\eta}\,y + \int_{t_0}^t D^\alpha_{y,\eta}\xi(s;y,\eta)\,ds,$$

Lemma 4.1 implies the bound for all $t \ge t_0$

$$|D^\alpha_{y,\eta}x(t;y,\eta)| \le Ct$$

up to probability at most $e^{-t_0^\varepsilon}$ and provided t_0 is large. By Lemma 3.3 and Lemma 4.1 therefore, up to probability at most $e^{-t_0^\varepsilon}$,

$$\Big|D^\alpha_{y,\eta}(x(t;y,\eta)-y-(t-t_0)\xi(t;y,\eta))\Big| \le C\int_{t_0}^t s^{-\frac14-\frac{3\beta}{2}+\varepsilon}\,s\,ds + Ct^{\frac74-\frac{3\beta}{2}+\varepsilon} \le Ct^{\frac74-\frac{3\beta}{2}+\varepsilon},$$

as claimed. $\square$

In the following lemma we estimate derivatives of higher order. Let

$$\mu(j) = (1-\beta)j - \frac14 - \frac{\beta}{2} \quad \text{for} \quad j \ge 2. \tag{4.6}$$

Lemma 4.3. *Let $\varepsilon = \varepsilon(\beta) > 0$ be sufficiently small depending on β. Then there exists a large time $t_0 = t_0(\varepsilon)$ so that up to probability at most $e^{-t_0^\varepsilon}$ the following property holds: Any solution $x(t;y,\eta), \xi(t;y,\eta)$ satisfies the estimates*

$$\Big|D^\alpha_{y,\eta}(\xi(t;y,\eta), x(t;y,\eta)/t)\Big| \le C\,t^{\mu(|\alpha|)+\varepsilon} \tag{4.7}$$

for any multi-index $|\alpha| \ge 2$ and $t \ge t_0$. The constant C depends on β, ε.

Proof. First consider the case $|\alpha| = 2$ and derivatives in η. Then one has the system

$$\begin{aligned} \frac{\partial^2 \dot{x}}{\partial\eta_j\partial\eta_k} &= \frac{\partial^2 \xi}{\partial\eta_j\partial\eta_k} \\ \frac{\partial^2 \dot{\xi}}{\partial\eta_j\partial\eta_k} &= -\sum_{p,q=1}^{d} \frac{\nabla\partial^2 V(x)}{\partial x_p\partial x_q}\frac{\partial x_p}{\partial\eta_j}\frac{\partial x_q}{\partial\eta_k} - \sum_{p=1}^{d}\frac{\nabla\partial V(x)}{\partial x_p}\frac{\partial^2 x_p}{\partial\eta_j\partial\eta_k} \end{aligned} \tag{4.8}$$

with initial conditions at time t_0 both equal to zero. Let

$$\begin{aligned} U_{pq}(t) &= -\int_t^\infty \frac{\nabla\partial^2 V(x(s))}{\partial x_p\partial x_q}\, ds \\ W_p(t) &= -\int_t^\infty \frac{\nabla\partial V(x(s))}{\partial x_p}\, ds. \end{aligned}$$

By Lemma 3.3

$$|U_{pq}(t)| \le C\, t^{-\frac{1}{4}-\frac{5\beta}{2}+\varepsilon} \quad\text{and}\quad |W_p(t)| \le C\, t^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon} \tag{4.9}$$

up to probability $e^{-t_0^\varepsilon}$ where $t_0 = t_0(\varepsilon)$ is large. Integrating out (4.8) and then integrating by parts in the resulting expression leads to (with $D_j = \frac{\partial}{\partial\eta_j}$ etc. and using the summation convention)

$$\begin{aligned} D^2_{jk}\xi(t) &= 2\int_{t_0}^t U_{pq}(s)D_j\xi_p(s)D_k x_q(s)\, ds - U_{pq}(t)D_j x_p(t)D_k x_q(t) \\ &\quad + \int_{t_0}^t W_p(s)D^2_{jk}\xi_p(s)\, ds - W_p(t)D^2_{jk}x_p(t). \end{aligned} \tag{4.10}$$

Here we have used that $D_j x(t_0) = 0$ and $D^2_{jk}x(t_0) = 0$. The reader will easily verify that combining (4.10) with (4.9) and (4.2) leads to

$$\begin{aligned} |D^2_\eta\xi(t)| &\le C\int_{t_0}^t s^{-\frac{1}{4}-\frac{5\beta}{2}+\varepsilon}\, s\, ds + C\, t^{-\frac{1}{4}-\frac{5\beta}{2}+\varepsilon}\, t^2 \\ &\quad + C\int_{t_0}^t s^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}\, |D^2_\eta\xi(s)|\, ds + C\, t^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}\int_{t_0}^t |D^2_\eta\xi(s)|\, ds \\ &\le C\, t^{\frac{7}{4}-\frac{5\beta}{2}+\varepsilon} + C\int_{t_0}^t s^{-\frac{1}{4}-\frac{3\beta}{2}+\varepsilon}|D^2_\eta\xi(s)|\, ds. \end{aligned} \tag{4.11}$$

Gronwall's inequality therefore implies

$$|D^2_\eta\xi(t)| \le C\, t^{\frac{7}{4}-\frac{5\beta}{2}+\varepsilon} + C\int_{t_0}^t s^{\frac{3}{2}-4\beta+2\varepsilon}\, ds \le C\, t^{\frac{7}{4}-\frac{5\beta}{2}+\varepsilon},$$

as claimed. The same argument also applies to derivatives in y, since (4.2) continues to hold for y. The only modification that needs to be made arises from $\frac{\partial x}{\partial y} = t_0 I$, cf. (4.10) and (4.11). But this can easily be seen to be insignificant. Finally, the bound on $D^2_{y,\eta}(x/t)$ follows by integrating the bound we just obtained for ξ in time.

For larger α one uses induction. Therefore, let α be a multi-index of length $|\alpha| \geq 3$. Taking D^α_η derivatives of (2.1) leads to an expression of the form

$$D^\alpha_\eta \dot{\xi} = \sum \mathrm{coeff} \, \frac{\nabla \partial^{m_1+\ldots+m_\ell} V(x)}{\partial^{m_1} x_{p_1} \cdot \ldots \cdot \partial^{m_\ell} x_{p_\ell}} \prod_{i=1}^{\ell} \prod_{k=1}^{m_i} D_{\eta^{(i)}_k} x_{p_i}, \tag{4.12}$$

where coeff stands for suitable coefficients and the sum extends over all terms so that

$$\sum_{i=1}^{\ell} \sum_{k=1}^{m_i} |\eta^{(i)}_k| = |\alpha| \quad \text{with} \quad |\eta^{(i)}_k| \geq 1. \tag{4.13}$$

There is exactly one term with $\ell = 1$, namely

$$\sum_{p=1}^{d} \frac{\nabla \partial V(x)}{\partial x_p} D^\alpha_\eta x_p, \tag{4.14}$$

cf. the second term in (4.8). The other terms only contain derivatives of x of orders $< |\alpha|$ and are therefore estimated by means of the induction hypothesis. More precisely, consider a typical term with $\ell \geq 2$ as on the right-hand side of (4.12) and let

$$U(t) = -\int_t^\infty \frac{\nabla \partial^{m_1+\ldots+m_\ell} V(x(s))}{\partial^{m_1} x_{p_1} \cdot \ldots \cdot \partial^{m_\ell} x_{p_\ell}} \, ds.$$

By Lemma 3.3 one has

$$|U(t)| \leq C t^{-\frac{1}{4} - (M+\frac{1}{2})\beta + \varepsilon}$$

up to probability at most $e^{-t_0^\varepsilon}$. Here we have set $M = \sum_{i=1}^{\ell} m_i$. Integrating by parts as in (4.11) and invoking the induction hypothesis, as well as (4.13) yields an integral of the form (the boundary term is dominated by the integral as before)

$$\begin{aligned} |U(t)| &\leq C \int_{t_0}^{t} s^{-\frac{1}{4}-(M+\frac{1}{2})\beta+\varepsilon} \, s^{M-1} \prod_{i=1}^{\ell} \prod_{k=1}^{m_i} s^{\mu(|\eta^{(i)}_k|)+\varepsilon_1} \, ds && (4.15) \\ &\leq C t^{-\frac{1}{4}-(M+\frac{1}{2})\beta+\varepsilon+M+(1-\beta)|\alpha|-M(\frac{1}{4}+\frac{\beta}{2})+M\varepsilon_1}. && (4.16) \end{aligned}$$

Here $\varepsilon > 0$ is arbitrary small but fixed, but some care needs to be taken with the choice of ε_1. Observe that we can use the induction hypothesis only when $|\eta^{(i)}_k| \geq 2$. Nevertheless, since $\mu(1) = \frac{3}{4} - \frac{3\beta}{2}$ estimate (4.15) remains correct provided one chooses $\varepsilon_1 = \frac{3\beta}{2} - \frac{3}{4} + \varepsilon$ so that $\mu(1) + \varepsilon_1 = \varepsilon$. It is now a simple matter to check that the exponent in (4.16) is maximized if $M = |\alpha|$ (clearly, that case is the analogue of the double sum in (4.8). In particular, all $|\eta_k^{(i)}| = 1$). Indeed,

$$\begin{aligned} & -\frac{1}{4} - (M + \frac{1}{2})\beta + \varepsilon + M + (1-\beta)|\alpha| - M(\frac{1}{4} + \frac{\beta}{2}) + M\varepsilon_1 \leq \\ \leq \quad & -\frac{1}{4} - \beta(|\alpha| + \frac{1}{2}) + (|\alpha|+1)\varepsilon + |\alpha|. \end{aligned}$$

By our choice of ε_1 this is in fact an equality. The conclusion from the preceding is the bound

$$|D_\eta^\alpha \xi(t)| \leq C \int_{t_0}^t s^{-\frac{1}{4} - \beta(|\alpha| + \frac{1}{2}) + \varepsilon}\, s^{|\alpha|-1}\, ds + C \int_{t_0}^t s^{-\frac{1}{4} - \frac{3\beta}{2} + \varepsilon} |D_\eta^\alpha \xi(s)|\, ds.$$

As before, Gronwall finishes the proof. The changes required for D_y^α and x/t are analogous to those for $|\alpha| = 2$ and will be skipped. □

We are now able to solve equation (4.1) by means of the method of characteristics. This is standard, see for example Section A.3 in [6]. We will call a C^∞- function ψ on $\mathbb{R}^d \setminus \{0\}$ *admissible* if

$$|\nabla\psi(\xi) - \xi| \leq \frac{|\xi|}{10}$$

for all $\xi \neq 0$. The example to keep in mind is of course $\psi(\xi) = \frac{1}{2}|\xi|^2$.

Proposition 4.4. *Given $0 < \rho < 1$ arbitrary and $\varepsilon > 0$ sufficiently small, there exists some large time $t_0(\varepsilon, \rho)$ so that the following holds up to probability $e^{-t_0^\varepsilon}$: For any admissible ψ the equation* (4.1) *has a unique C^∞-solution on the set*

$$\Omega_{t_0,\rho} := [t_0, \infty) \times \Big\{ \xi \in \mathbb{R}^d \ : \ \rho \leq |\xi| \leq 2\rho^{-1} \Big\}$$

with initial condition $S(t_0, \xi) = t_0\psi(\xi)$. Moreover, with μ as in (4.6)*, one has*

$$\begin{aligned} |D_\xi S(t,\xi)| + |D_\xi^2 S(t,\xi)| &\leq Ct \\ |D_\xi^\alpha S(t,\xi)| &\leq C_\alpha\, t^{1+\mu(|\alpha|-1)+\varepsilon} \quad \textit{for all } |\alpha| \geq 3 \end{aligned} \tag{4.17}$$

in $\Omega_{t_0,\rho}$. Moreover, in $\Omega_{t_0,\rho}$

$$|D_\xi S(t,\xi) - t_0\nabla\psi(\xi) - (t-t_0)\xi| \leq C\, t^{\frac{3}{4} - \frac{\beta}{2} + \varepsilon}. \tag{4.18}$$

Proof. Uniqueness is completely standard and will not be presented, see [6] for example. Let $x(t; y, \eta)$ and $\xi(t; y, \eta)$ be the solutions of (2.1) from the beginning of this section. We now specialize to $y = \nabla\psi(\eta)$. Recall that this means solving (2.1) with initial conditions $x(t_0) = t_0\nabla\psi(\eta)$ and $\xi(t_0) = \eta$. We denote these solutions by $x(t;\eta), \xi(t;\eta)$. For simplicity we set $\rho = 1$, but take $\eta \in \mathbb{R}^d$ in a slightly larger region, namely $\frac{1}{2} < |\eta| < 4$. By Proposition 3.1 this insures that $\xi(t;\eta)$ remains in the region $\frac{1}{4} \le |\xi| \le 8$ for all times provided t_0 is taken to be large enough. Now define

$$Q(t,\eta) := t_0\psi(\eta) + \int_{t_0}^{t} \Big[\frac{1}{2}|\xi(s;\eta)|^2 + V(x(s;\eta)) + \langle x(s;\eta), \dot{\xi}(s;\eta)\rangle\Big]\, ds. \tag{4.19}$$

Then

$$\dot{Q}(t,\eta) = \frac{1}{2}|\xi(t;\eta)|^2 + V(x(t;\eta)) + \langle x(t;\eta), \dot{\xi}(t;\eta)\rangle$$

and therefore, with $\partial_j = \frac{\partial}{\partial \eta_j}$ and suppressing the arguments for simplicity,

$$\begin{aligned} \partial_j \dot{Q} &= \langle \xi, \partial_j \xi\rangle + \langle \nabla V(x), \partial_j x\rangle + \langle \partial_j x, -\nabla V(x)\rangle + \langle x, \partial_j \dot{\xi}\rangle \\ &= \frac{d}{dt}\langle x, \partial_j \xi\rangle. \end{aligned}$$

Observe that we have used (2.1) in two places. This implies that

$$\begin{aligned} \partial_j Q(t,\eta) - \langle x(t;\eta), \partial_j \xi(t;\eta)\rangle &= \partial_j Q(t_0,\eta) - \langle x(t_0;\eta), \partial_j \xi(t_0;\eta)\rangle \\ &= t_0 \partial_j \psi(\eta) - \langle t_0 \nabla\psi(\eta), \vec{e_j}\rangle = 0 \end{aligned}$$

where $\vec{e_j}$ denotes the standard j^{th} basis vector. We have thus obtained

$$\partial_j Q(t,\eta) = \langle x(t;\eta), \partial_j \xi(t;\eta)\rangle \tag{4.20}$$

for all times $t \ge t_0$. By Proposition 3.1 and Lemma 4.1 the map $\eta \mapsto \xi(t;\eta)$ is invertible with inverse $\xi = \xi(t;\eta)$. Indeed, Proposition 3.1 implies that any failure of being one-to-one has to be local, whereas local invertibility is assured by the inverse function theorem from Lemma 4.1. It is immediate from Lemma 4.1 that $\partial_\eta \xi(t;\eta) = O(1)$ and therefore also $D_\xi \eta(t;\xi) = O(1)$. Moreover, Lemma 4.3 and the chain rule combined with Leibniz's rule imply that $|D_\xi^\alpha \eta(t;\xi)| \le C_\alpha\, t^{\mu(|\alpha|)+\varepsilon}$ for all $|\alpha| \ge 2$. This depends on the convexity of the $\mu(j)$ (here one of course has linearity) and the details of this general fact can be found in [10]. Now define $S(t,\xi) = Q(t,\eta(t;\xi))$ for $1 < |\xi| < 2$ and $t \ge t_0$ where t_0 will be given by the preceding technical probabilistic lemmas. Observe that $\eta(t;\xi)$ will remain in $\frac{1}{2} < |\eta| < 4$ provided t_0 is large. With $D_\ell := \frac{\partial}{\partial \xi_\ell}$ one obtains from (4.20) that

$$\begin{aligned} D_\ell S(t,\xi) &= \sum_{j=1}^{d} \partial_j Q(t,\eta(t;\xi)) D_\ell \eta_j(t;\xi) \\ &= \Big\langle x(t,\eta(t;\xi)), \sum_{j=1}^{d} \partial_j \xi(t;\eta(t;\xi)) D_\ell \eta_j(t;\xi)\Big\rangle = x_\ell(t,\eta(t;\xi)), \end{aligned}$$

which proves the fundamental relation

$$DS(t,\xi) = x(t;\eta(t;\xi)). \tag{4.21}$$

To check that S solves (4.1) one computes, using (4.20),

$$\begin{aligned}
\partial_t S(t,\xi) &= \partial_t Q(t,\eta(t;\xi)) + \sum_{j=1}^d \partial_j Q(t,\eta(t;\xi))\dot{\eta_j}(t;\xi) \\
&= \partial_t Q(t,\eta(t;\xi)) + \Big\langle x(t;\eta(t;\xi)), \sum_{j=1}^d \partial_j \xi(t;\eta(t;\xi))\dot{\eta_j}(t;\xi)\Big\rangle \\
&= \frac{1}{2}|\xi(t;\eta(t;\xi))|^2 + V\big(\nabla S(t,\xi(t;\eta(t;\xi)))\big) + \langle x(t;\eta(t;\xi)), \dot{\xi}(t;\eta(t;\xi))\rangle \\
&\qquad + \Big\langle x(t;\eta(t;\xi)), \frac{d}{dt}\xi(t;\eta(t;\xi)) - \partial_t \xi(t;\eta(t;\xi))\Big\rangle \\
&= \frac{1}{2}|\xi|^2 + V\big(DS(t,\xi)\big),
\end{aligned}$$

as claimed. The derivative bounds for $|\alpha| = 1, 2$ in (4.17) follow from (4.21) and Lemma 4.1. More precisely, if $|\alpha| = 1$, then

$$|DS(t,\xi)| \le \sup_{\frac{1}{2}<|\eta|<4} |x(t;\eta)| \le Ct.$$

whereas if $|\alpha| = 2$, then

$$|D^2 S(t,\xi)| \le \sup_{\frac{1}{2}<|\eta|<4} |\partial_\eta x(t;\eta)| \sup_{1<|\xi|<2} |D_\xi \eta(t;\xi)| \le Ct.$$

Here we have used Lemma 4.1 and Corollary 4.2, which provide bounds uniform in the initial conditions. For $|\alpha| = 3$ consider

$$\frac{\partial^3 S}{\partial \xi_j \partial \xi_k \partial \xi_\ell} = \sum_{p,q=1}^d \partial^2_{pq} x_\ell D_j \eta_p D_k \eta_q + \sum_{p=1}^d \partial_p x_\ell D^2_{jk} \eta_p.$$

Since $D_j \eta = O(1)$, Lemma 4.3 implies that the right–hand side is $O(t^{1+\mu(2)+\varepsilon})$ as claimed. The case of higher order α is treated via induction. The chain rule arises in this process very much in the same way as in the previous proof. We skip the details.

As for (4.18), observe that by (3.2) and (3.3)

$$\sup_{\frac{1}{2}<|\eta|<4} |x(t;\eta) - t_0 \nabla\psi(\eta) - (t-t_0)\xi(t;\eta)| \le C\, t^{\frac{3}{4}-\frac{\beta}{2}+\varepsilon}$$

for all $t \ge t_0$. In particular, one can set $\eta = \eta(t;\xi)$ where $1 < |\xi| < 2$ which leads to

$$|D_\xi S(t,\xi) - t_0 \nabla\psi(\eta(t;\xi)) - (t-t_0)\xi| \le C\, t^{\frac{3}{4}-\frac{\beta}{2}+\varepsilon}, \tag{4.22}$$

see (4.21). To replace $\eta(t;\xi)$ in $\nabla\psi$, one uses that $|\eta - \xi(t;\eta)| \le C t_0^{-\frac{1}{4}-\frac{\beta}{2}+\varepsilon}$. This in turn implies that $|\xi - \eta(t;\xi)| \le C t_0^{-\frac{1}{4}-\frac{\beta}{2}+\varepsilon}$. Inserting this inequality into (4.22) leads to the desired bound (4.18). □

Below it will be necessary to estimate averages of the form

$$\Big|\int_t^\infty D^\alpha V(D_\xi S(t,\xi))\,ds\Big|$$

where $|\alpha|\ge 1$. This is of course very close to Lemma 3.3. In view of the arguments in Lemma 3.8 and Theorem 3.9 it might seem natural to approximate the path $DS(t,\xi)$ by means of the line $t_0\nabla\psi(\xi)+(t-t_0)\xi$ and integrate by parts. This, however, does not work as well as before. For example, let $z(t,\xi)=t_0\nabla\psi(\xi)+(t-t_0)\xi$ and estimate $\int_T^{2T} V(DS(t,\xi))\,dt$ by means of approximating $DS(t,\xi)$ with $z(t,\xi)$. Taylor expansion gives rise to terms of which the first is

$$\int_T^{2T} DV(z(t,\xi))\,(DS(t,\xi)-z(t,\xi))\,dt.$$

Letting $U(t)=\int_T^t DV(z(\tau,\xi))\,d\tau$ and integrating by parts leads to an expression of the form

$$\int_T^{2T} U(t)\,(DS_t(t,\xi)-\dot z(t,\xi))\,dt=\int_T^{2T} U(t)\,D_xV(DS(t,\xi))D^2S(t,\xi)\,dt \tag{4.23}$$

where we have used the equation (4.1) to pass to the second expression. Putting absolute values inside the right-hand side of (4.23) leads to a bound $T^{1-\frac{3\beta}{2}+\varepsilon}$ which is by roughly $\frac14$ worse than the desired one. This indicates, of course, that further cancellation needs to be exploited in (4.23). Rather than trying to pursue this approach, we return to the martingale method. One of the advantages of this method is that one approximates the path $D_\xi S(t,\xi)$ by piecewise linearization (on a time scale T^β) rather than by linearization over the entire interval $[T,2T]$.

Lemma 4.5. *Let $S(t,\xi)$ be the solution constructed in the previous proposition. Then, up to probability at most $e^{-t_0^\varepsilon}$, one has the bounds for all $t\ge t_0$*

$$\Big|\int_t^\infty D^\alpha V(D_\xi S(s,\xi))\,ds\Big|\le C_\alpha\, t^{-\frac14-(|\alpha|-\frac12)\beta+\varepsilon}$$

and all multi-indices $|\alpha|\ge 1$, whereas for $\alpha=0$

$$\Big|\int_{t_0}^t V(D_\xi S(s,\xi))\,ds\Big|\le C\,t^{-\frac14+\frac{\beta}{2}+\varepsilon}.$$

Proof. Let $T>0$ be arbitrary but fixed. Let T be such that $|D_\xi S(T,\xi)|=R$ and denote by T^* the time for which $|D_\xi S(t,\xi)|=2R$. As in the proof of Lemma 3.3, define stopping times $\tau_\ell,\widetilde{\tau}_\ell$ by means of

$$\begin{aligned}\tau_\ell\ &:=\ \inf\{t\ge 0\ :\ |D_\xi S(t,\xi)|=R+\ell R^\beta\}\\ \widetilde{\tau}_\ell\ &:=\ \inf\{s\ge \tau_{\ell-1}\ :\ |D_\xi S(\tau_{\ell-1},\xi)+(s-\tau_{\ell-1})D_\xi S_t(\tau_{\ell-1},\xi)|=R+\ell R^\beta\}\end{aligned} \tag{4.24}$$

where $0 \leq \ell \leq R^{1-\beta}$ (recall that the bumps in (1.2) are arranged to lie inside shells $\mathcal{S}_{\ell,R}$, see (3.1)). In view of (4.18) these times are well-defined. We now claim that $S(\tau_\ell;\xi)$ is defined entirely in terms of the random variables inside $|x| \leq R + \ell R^\beta$. This of course requires that the characteristics of the Hamilton-Jacobi PDE belong to the outgoing region of phase space, which is guaranteed by the assumption on the initial data (ψ is admissible). To verify our claim, recall that

$$D_\xi S(\tau_\ell;\xi) = x(\tau_\ell;\eta(\tau_\ell;\xi)),$$

see (4.21). Let $\eta(\tau_\ell;\xi) = \eta_0$ so that $\xi = \xi(\tau_\ell;\eta_0)$. Since by definition $|x(\tau_\ell;\eta_0)| = R + \ell R^\beta$, it follows that $S(\tau_\ell;\xi)$ draws only on those random variables inside $|x| \leq R + \ell R^\beta$, as claimed. This implies that

$$z_\ell(t) := D_\xi S(\tau_\ell,\xi) + (t-\tau_\ell) D_\xi S_t(\tau_\ell,\xi)$$

is measurable with respect to those variables as well. By Taylor expansion

$$\begin{aligned}
&\int_T^{T^*} D^\alpha V(D_\xi S(t,\xi))\,dt = \sum_\ell \int_{\tau_\ell}^{\tilde\tau_{\ell+1}} D^\alpha V(z_\ell(t))\,dt \\
&\quad + \sum_\ell O\Big(|\tilde\tau_{\ell+1}-\tau_{\ell+1}|\,\|D^\alpha V\|_{L^\infty(\mathcal{S}_{\ell,R})} + |\tau_{\ell+1}-\tau_\ell|^3 \|\nabla D^\alpha V\|_{L^\infty(\mathcal{S}_{\ell,R})} \sup_{\tau_\ell \leq t \leq \tilde\tau_{\ell+1}} |D_\xi S_{tt}(t,\xi)|\Big)
\end{aligned} \tag{4.25}$$

where the sum extends over all shells $\mathcal{S}_{\ell,R}$ that lie between R and $2R$. By the equation (using formal notation),

$$D_\xi S_t(t,\xi) = \xi + \nabla V(D_\xi S(t,\xi)) D_\xi^2 S(t,\xi) \tag{4.26}$$

$$D_\xi^2 S_t(t,\xi) = I + D_x^2 V(D_\xi S(t,\xi))(D_\xi^2 S(t,\xi))^2 + \nabla V(D_\xi S) D_\xi^3 S \tag{4.27}$$

$$D_\xi S_{tt}(t,\xi) = D_x^2 V(D_\xi S(t,\xi)) D_\xi S_t(t,\xi) D_\xi^2 S(t,\xi) + \nabla V(D_\xi S(t,\xi)) D_\xi^2 S_t(t,\xi) \tag{4.28}$$

By (4.26) and (4.17) one has $D_\xi S_t = O(1)$, and by (4.27)

$$D_\xi^2 S_t(t,\xi) = O(t^{-\frac34 - 2\beta} t^2 + t^{-\frac34-\beta} t^{1+\mu(2)+\varepsilon}) = O(t^{\frac54 - 2\beta}).$$

Inserting these estimates into (4.28) shows that

$$D_\xi S_{tt}(t,\xi) = O(t^{-\frac34-2\beta} t + t^{-\frac34-\beta} t^{\frac54 - 2\beta}) = O(t^{\frac14 - 2\beta}). \tag{4.29}$$

To estimate the difference $\tilde\tau_{\ell+1} - \tau_{\ell+1}$ via Lemma 2.2, one uses (4.26) to conclude that δ in that lemma satisfies

$$\delta \leq \sup_{T \leq t \leq T^*} |\dot z_\ell(t) - D_\xi S_t(t,\xi)| \leq \sup_{T \leq t \leq T^*} |D_\xi S_t(\tau_\ell,\xi) - D_\xi S_t(t,\xi)| \leq C T^{\frac14 - \beta}.$$

Hence

$$|\tilde\tau_{\ell+1} - \tau_{\ell+1}| \leq C T^{\frac14}. \tag{4.30}$$

The error terms in (4.25) are therefore bounded by

$$\sum_\ell O(T^{\frac14}T^{-\frac34-|\alpha|\beta}+T^{3\beta}T^{-\frac34-(|\alpha|+1)\beta}T^{\frac14-2\beta}) = O(T^{1-\beta}T^{-\frac12-|\alpha|\beta}) = O(T^{\frac12-(|\alpha|+1)\beta}). \tag{4.31}$$

Now let

$$Y_\ell = \int_{\tau_\ell}^{\tilde\tau_{\ell+1}} D^\alpha V(z_\ell(t))\,dt.$$

One checks that $\|Y_\ell\|_\infty \le CT^{-\frac34-(|\alpha|-1)\beta}$ and hence

$$\Big(\sum_\ell \|Y_\ell\|_\infty^2\Big)^{\frac12} \le CT^{-\frac14-(|\alpha|-\frac12)\beta}. \tag{4.32}$$

One easily checks that the exponent in (4.32) is bigger by $\frac{3\beta}{2}-\frac34>0$ than the exponent in (4.31). Hence Lemma 2.1 implies that

$$\Big|\sum_\ell Y_\ell\Big| \le C\,T^\varepsilon T^{-\frac14-(|\alpha|-\frac12)\beta}$$

up to probability at most e^{-T^ε}. Summing up over $t, 2t, 4t, \ldots$ if $|\alpha|\ge 1$, or $t_0, 2t_0, 4t_0, \ldots$ if $\alpha=0$ yields the lemma. $\square$

The following lemma establishes bounds that are similar to (4.18) and will be crucial for applications.

Lemma 4.6. *Up to probability at most $e^{-t_0^\varepsilon}$, the solution $S(t,\xi)$ constructed in Proposition 4.4 satisfies the bounds*

$$\big|D_\xi^\alpha\big[S(t,\xi)-t_0\psi(\xi)-(t-t_0)|\xi|^2/2\big]\big| \le C_\alpha\, t^{1+\nu(|\alpha|)+\varepsilon} \quad \textit{for all } |\alpha|\le 2 \tag{4.33}$$

Here $\nu(0)=\frac{\beta}{2}-\frac54$, $\nu(1)=-\frac14-\frac{\beta}{2}$, and $\nu(2)=\frac34-\frac{3\beta}{2}$.

Proof. The case $|\alpha|=1$ is simply a restatement of (4.18). For $\alpha=0$ one uses the equation. Indeed, this gives

$$S(t,\xi)-t_0\psi(\xi)-\frac12(t-t_0)|\xi|^2 = \int_{t_0}^t V(D_\xi S(s,\xi))\,ds.$$

By the previous lemma the right hand side is $O(t^{-\frac14+\frac{\beta}{2}+\varepsilon})$ which gives the stated value of $\nu(0)$. For simplicity, we do not treat the case $|\alpha|=2$ in a similar way, as this would require averaging more complicated expressions than those considered in the previous lemma. Rather, we use the methods of proof from Proposition 4.4. Thus, let

$$\Delta(t;\xi)=S(t;\xi)-t_0\psi(\xi)-\frac{t-t_0}{2}|\xi|^2 \ \text{ and }\ \widetilde\Delta(t;\eta)=Q(t,\eta)-t_0\psi(\xi(t;\eta))-\frac{t-t_0}{2}|\xi(t;\eta)|^2$$

so that $\Delta(t;\xi) = \widetilde{\Delta}(t;\eta(t;\xi))$. We want to show that

$$\left|D_\xi^\alpha \Delta(t;\xi)\right| \leq Ct^{\nu(|\alpha|)+1} \quad \text{for} \quad |\alpha| = 2. \tag{4.34}$$

By (4.20),

$$\partial_j \widetilde{\Delta}(t;\eta) = \langle x(t;\eta) - t_0 \nabla\psi(\xi(t;\eta)) - (t-t_0)\xi(t;\eta), \partial_j \xi(t;\eta)\rangle. \tag{4.35}$$

Therefore,

$$\begin{aligned}\partial_{jk}\widetilde{\Delta}(t;\eta) &= \langle x(t;\eta) - t_0\nabla\psi(\xi(t;\eta)) - (t-t_0)\xi(t;\eta), \partial_{jk}\xi(t;\eta)\rangle \\ &\quad + \langle \partial_k(x(t;\eta) - t_0\nabla\psi(\xi(t;\eta)) - (t-t_0)\xi(t;\eta)), \partial_j\xi(t;\eta)\rangle.\end{aligned}$$

Invoking Proposition 3.1 and the estimates from this section (namely (4.7), (4.5), and (4.2)) yields the bound

$$\left|\partial_{jk}\widetilde{\Delta}(t;\eta)\right| \leq C(t^{\frac{3}{4}-\frac{\beta}{2}+\varepsilon}t^{\mu(2)+\varepsilon} + t^{\frac{7}{4}-\frac{3\beta}{2}+\varepsilon}) \leq C\,t^{\frac{7}{4}-\frac{3\beta}{2}+\varepsilon}. \tag{4.36}$$

By the chain rule and the summation convention,

$$D^2_{pq}\Delta(t;\xi) = \partial_{jk}\widetilde{\Delta}(t;\eta(t;\xi))D_p\eta_j(t;\xi)D_q\eta_k(t;\xi) + \partial_j\widetilde{\Delta}(t;\eta(t;\xi))D^2_{pq}\eta_j(t;\xi). \tag{4.37}$$

To estimate the second term on the right-hand side one uses (4.35) as follows (with $\eta = \eta(t;\xi)$):

$$\begin{aligned}&\left|\partial_j\widetilde{\Delta}(t;\eta)D^2_{pq}\eta_j(t;\xi)\right| \\ &= \left|\langle x(t;\eta) - t_0\nabla\psi(\xi(t;\eta)) - (t-t_0)\xi(t;\eta), \partial_j\xi(t;\eta)D^2_{pq}\eta_j(t;\xi)\rangle\right| \\ &= \left|\langle x(t;\eta) - t_0\nabla\psi(\xi(t;\eta)) - (t-t_0)\xi(t;\eta), \partial_{j\ell}\xi(t;\eta)D_p\eta_j(t;\xi)D_q\eta_\ell(t;\xi)\rangle\right| \\ &\leq C\,t^{\frac{3}{4}-\frac{\beta}{2}+\varepsilon}t^{\mu(2)+\varepsilon}. \qquad (4.38)\end{aligned}$$

Controlling the right-hand side of (4.37) by means of (4.38) and (4.36) yields (4.34) with $\nu(2) = \frac{3}{4} - \frac{3\beta}{2} < 0$, as claimed. □

Remark 4.7. It is important to note that if β is close to $\frac{1}{2}$, then for some $\delta > 0$ one has $\nu(0) < -1+\delta$, $\nu(1) < -\frac{1}{2} - \delta$, and $\nu(2) < -\delta$.

In the following corollary we construct almost sure solutions to the Hamilton-Jacobi equations without any assumptions on $\xi \in \mathbb{R}^d \setminus \{0\}$. The method of building the solution up from smaller pieces is the same as in [10].

Corollary 4.8. *Given a sufficiently small $\varepsilon > 0$, there exists a C^∞-function $S(t,\xi)$ on a subset of $(0,\infty) \times \{\xi \in \mathbb{R}^d : \xi \neq 0\}$ with the following property: For any compact $K \subset \mathbb{R}^d \setminus \{0\}$ there exists a time $t_0 = t_0(\varepsilon, K)$ such that $S(t,\xi)$ solves the Hamilton-Jacobi equation* (4.1) *on the set $[t_0,\infty) \times K$. Moreover, one has the bounds* (4.17) *and* (4.33) *with random constants C, C_α.*

Proof. Consider the regions

$$\Omega_j := \{\xi \in \mathbb{R}^d \,:\, 2^{-j-1} \le |\xi| \le 2^{j+1}\}$$

for $j = 0, 1, 2 \dots$. Choose smooth cutoff functions χ_j so that $\chi_j = 1$ on Ω_{j-1} and $\mathrm{supp}(\chi_j) \subset \Omega_j$ (with $\Omega_{-1} = \emptyset$). By the proposition, a.s. there is a solution S_0 on $[t_0, \infty) \times \Omega_0$ with initial $\psi(\xi) = \frac{1}{2}|\xi|^2$ for some random t_0 that also depends on ε. Here we have used Borel-Cantelli to pass to an a.s. statement. Now apply the proposition to the region Ω_1 with initial

$$\psi(\xi) = (1 - \chi_0)(\xi)\frac{1}{2}|\xi|^2 + \chi_0(\xi) S_0(t_1, \xi)/t_1$$

at some other large random time t_1. Observe that this ψ will be admissible by (4.33) with $|\alpha| = 0, 1$ provided t_1 is large. Continuing inductively finishes the proof. □

5 Existence of quantum mechanical modified wave operators for the $\frac{3}{4}$ model

Let $H = \frac{1}{2}\triangle - V$ with V given by (1.2). In this section we shall define a bounded operator $U(t)$ on $L^2(\mathbb{R}^d)$ so that almost surely for any $f_0 \in L^2(\mathbb{R}^d)$ there exists $f \in L^2(\mathbb{R}^d)$ such that

$$\left\| e^{itH} f - U(t) f_0 \right\|_2 \to 0 \quad \text{as} \quad t \to \infty. \tag{5.1}$$

Since e^{itH} is unitary and $U(t)$ will have the property that

$$\sup_{t>0} \|U(t) f\|_2 \le C_\rho \|f\|_2 \;\; \text{for all } f \text{ with } \; \mathrm{supp}(\hat{f}) \subset \{\rho < |\xi| < \rho^{-1}\}$$

for any $\rho > 0$, it is clear that (5.1) is equivalent to the existence of the L^2-limit of

$$e^{-itH} U(t)\phi \quad \text{as} \quad t \to \infty$$

for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\mathrm{supp}(\hat{\phi}) \subset \mathbb{R}^d \setminus \{0\}$. The existence of this limit will be established by means of Cook's method, i.e., we shall show that

$$\int_{t_0}^{\infty} \left\| \frac{d}{dt} e^{-itH} U(t) \phi \right\|_2 dt = \int_{t_0}^{\infty} \left\| [-iHU(t) + \dot{U}(t)]\phi \right\|_2 dt < \infty. \tag{5.2}$$

For any ϕ as above the evolution $U(t)$ is defined as

$$U(t)\phi(x) := \int_{\mathbb{R}^d} e^{i[\langle x, \xi\rangle - S(t,\xi)]} \, a(t,\xi)\hat{\phi}(\xi)\, d\xi \tag{5.3}$$

where S is the solution of the Hamilton-Jacobi equation from Section 4 and a is an amplitude that we shall construct in this section. More precisely, this amplitude exists almost surely and it has the form

$$a(t,\xi) = 1 + \sum_{j=1}^{M} a_j(t,\xi) \text{ where } \sup_{\rho<|\xi|<\rho^{-1}} |a_j(t,\xi)| \le C_{j,\rho}\, t^{-j\frac{\delta}{2}} \text{ for all } t \ge t_0(\rho) \tag{5.4}$$

for any $\rho > 0$ with random constants $C_{j,\rho}$. Here $\delta > 0$ satisfies $\beta > \frac{1}{2} + \delta$ and M is a large constant that only depends on β and $M \to \infty$ as $\beta \to \frac{1}{2}$. Moreover, a_j are smooth complex valued functions that satisfy suitable derivative bounds, see below. Since $a(t,\xi) \to 1$ uniformly on compact subsets of $\mathbb{R}^d \setminus \{0\}$ as $t \to \infty$, one concludes that the wave operator

$$W^+ := s - \lim_{t\to\infty} e^{-itH} U(t)$$

is unitary and intertwines H with $\triangle$, i.e.,

$$e^{itH} W^+ = W^+ e^{i\frac{t}{2}\triangle}.$$

This property also relies on the fact that $S(t+\tau,\xi) - S(t,\xi) \to \frac{\tau}{2}|\xi|^2$ as $t \to \infty$, which follows from the equation (4.1). The modified dynamics $U(t)$ given by (5.3) is similar to the one used in [10], but we are forced to include an amplitude. It is not difficult to see that the usual choice $a = 1$ is insufficient for our purposes basically because both $|DV(x)| \lesssim |x|^{-\frac{3}{2}-\varepsilon}$ and $|D^2V(x)| \lesssim |x|^{-2-\varepsilon}$ do not hold for our potential (in this section we use the notation $a \lesssim b$ to denote $a \le Cb$ for some constant and similarly $a \gtrsim b$. Also, $a \asymp b$ means that both $a \gtrsim b$ and $a \lesssim b$).
Inserting (5.3) into (5.2) and exploiting (4.1) shows that one needs to establish

$$\int_{t_0}^{\infty} \Big\| V(x) \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, a(t,\xi)\hat{\phi}(\xi)\, d\xi - \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, [V(\partial S(t,\xi))a(t,\xi) + i a_t(t,\xi)]\hat{\phi}(\xi)\, d\xi \Big\|_{L^2_x} dt < \infty \tag{5.5}$$

where we have set $\Phi(t,x;\xi) = \langle x,\xi\rangle - S(t,\xi)$. It is essential (and standard) that the critical points of the phase are given by $x = DS(t,\xi)$ (in what follows, $D = D_\xi$ whereas D_x will be used as derivatives of V). We are going to pull $V(x)$ inside the first integral. This can be accomplished by means of Taylor expansion and the following simple technical lemma.

Lemma 5.1. *For any multi-index α one has*

$$e^{i\Phi}(D\Phi)^\alpha = (-i)^{|\alpha|} D^\alpha e^{i\Phi} + \sum_{\substack{m\ge1, |\nu_1|,\ldots,|\nu_m|\ge2 \\ \nu_1+\ldots+\nu_m+\gamma=\alpha}} \mathrm{coeff}\, D^{\nu_1}\Phi \cdot \ldots \cdot D^{\nu_m}\Phi\; D^\gamma e^{i\Phi}$$

where coeff *stands for suitable complex constants.*

Proof. This is a standard induction argument. The case $\alpha = 0$ is obvious. If the statement holds up to $|\alpha| = N$, then fox such an α and consider $\alpha + e_\ell$ where $e_\ell = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ is the ℓ^{th} standard basis vector. Thus

$$
\begin{aligned}
e^{i\Phi}(D\Phi)^{\alpha+e_\ell} &= -iD_\ell(e^{i\Phi}(D\Phi)^\alpha) + ie^{i\Phi} D_\ell\Big[(D\Phi)^\alpha\Big] \\
&= -iD_\ell\Big[(-i)^{|\alpha|}D^\alpha e^{i\Phi} + \sum_{\substack{m\geq 1, |\nu_1|,\ldots,|\nu_m|\geq 2 \\ \nu_1+\ldots+\nu_m+\gamma=\alpha}} \mathrm{coeff}\, D^{\nu_1}\Phi \cdot \ldots \cdot D^{\nu_m}\Phi\; D^\gamma e^{i\Phi}\Big] \qquad (5.6) \\
&\quad + ie^{i\Phi} \sum_{j=1}^d \alpha_j (D\Phi)^{\alpha - e_j} D_{j\ell}\Phi. \qquad (5.7)
\end{aligned}
$$

Applying the product rule in (5.6) and the induction hypothesis in (5.7) yields an expression of the desired form. ☐

From Taylor's formula with some large value of N,

$$
\begin{aligned}
&V(x) \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, a(t,\xi)\hat{\phi}(\xi)\, d\xi = \\
&= \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)} \Big\{ V(DS(t,\xi)) + \sum_{1\leq|\alpha|\leq N} \frac{1}{\alpha!} D_x^\alpha V(DS(t,\xi))(x - DS(t,\xi))^\alpha \Big\}\, a(t,\xi)\hat{\phi}(\xi)\, d\xi \qquad (5.8) \\
&+ \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)} \sum_{|\alpha|=N+1} \frac{1}{\alpha!} \int_0^1 D_x^\alpha V(\theta DS(t,\xi) + (1-\theta)x)\, d\theta\, (x - DS(t,\xi))^\alpha\, a(t,\xi)\hat{\phi}(\xi)\, d\xi. \qquad (5.9)
\end{aligned}
$$

The point is of course to construct a in such a way that ia_t in (5.5) cancels most of the terms in (5.8) whereas those terms that remain as well as (5.9) give an integrable contribution to (5.5) in t provided N is large. The latter properties will follow from stationary phase. By Lemma 5.1,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)} D_x^\alpha V(DS(t,\xi))(x - DS(t,\xi))^\alpha\, a(t,\xi)\hat{\phi}(\xi)\, d\xi = \\
&= (-i)^{|\alpha|} \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)} D^\alpha \big[D_x^\alpha V(DS(t,\xi)) a(t,\xi)\hat{\phi}(\xi)\big]\, d\xi \qquad (5.10) \\
&\quad + \sum_{\substack{m\geq 1, |\nu_1|,\ldots,|\nu_m|\geq 2 \\ \nu_1+\ldots+\nu_m+\gamma=\alpha}} \mathrm{coeff} \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)} D^\gamma \big[D^{\nu_1}S \cdot \ldots \cdot D^{\nu_m}S\; D_x^\alpha V(DS(t,\xi))\, a(t,\xi)\hat{\phi}(\xi)\big]\, d\xi. \qquad (5.11)
\end{aligned}
$$

Observe that here we have replaced $D^{\nu_j}\Phi$ with $D^{\nu_j}S$ since $|\nu_j| \geq 2$. In order to obtain a ϕ-independent definition of the amplitude a, one needs to check that those terms in (5.10) and (5.11) in which at least one derivative falls on $\hat{\phi}(\xi)$ are negligible, in the sense that they give a t-integrable contribution to (5.5). This will require using the following bounds on S from the previous section, see (4.17) and (4.33): There is $c < \frac{1}{2}$ (in fact, $c = 1 - \beta$) and $\varepsilon > 0$ so that for all large t

$$|S(t,\xi)| + |DS(t,\xi)| \lesssim t, \ |D^\alpha S(t,\xi)| \lesssim t^{1+(|\alpha|-2)c} \ \text{ for } \ |\alpha| \geq 2 \tag{5.12}$$

$$\left|D^2\left[S(t,\xi)/t - \frac{1}{2}|\xi|^2\right]\right| \leq t^{-\varepsilon}. \tag{5.13}$$

The second inequality in (5.12) can be derived from (4.17) as follows: One has $|D^\alpha S(t,\xi)| \lesssim t^{1+\mu(|\alpha|-1)+\varepsilon}$ with $\mu(j) = (1-\beta)j - \frac{1}{4} - \frac{\beta}{2} = (1-\beta)(j-1) + \frac{3}{4} - \frac{3\beta}{2}$. Since $\frac{3}{4} - \frac{3\beta}{2} + \varepsilon \leq 0$, the estimate above holds. "Large" t means that $t \geq t_0$ with some random t_0 that also depends on the compact subset of $\mathbb{R}^d \setminus \{0\}$ to which ξ belongs. Clearly, (5.13) ensures that the phase Φ has a unique critical point on the support of $\hat{\phi}$ provided t is sufficiently large. We denote this critical point by ξ_0. For the following lemma we need to assume that for all large t and any multi-index γ

$$|D^\gamma a(t,\xi)| + |D^\gamma a_t(t,\xi)| \lesssim t^{c|\gamma|}. \tag{5.14}$$

This bound will be proven below when we construct the amplitude a. The following lemma is an instance of the well-known fact that with high probability a particle lies in the "classical region" $|x| \asymp t$. Because of this property it suffices to restrict L^2_x in (5.5) to $L^2(|x| \asymp t)$.

Lemma 5.2. *Under the assumption* (5.14) *one has*

$$\left\| V(x) \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, a(t,\xi)\hat{\phi}(\xi)\, d\xi - \int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, [V(\partial S(t,\xi))a(t,\xi) + ia_t(t,\xi)]\hat{\phi}(\xi)\, d\xi \right\|_{L^2(|x| \not\asymp t)} \lesssim t^{-2}. \tag{5.15}$$

Here $|x| \not\asymp t$ means that either $|x| \leq A^{-1}t$ or $|x| \geq At$ for some constant $A > 0$ that only depends on $\mathrm{supp}(\hat{\phi})$. *In particular, it suffices to prove* (5.5) *in the region $|x| \asymp t$.*

Proof. Fix some $\phi \in \mathcal{S}$ with $\mathrm{supp}(\hat{\phi}) \subset \{2A^{-1} < |\xi| < A/2\}$, and A some large constant. In view of (4.18) one has

$$|D\Phi(t,x;\xi)| = |x - t\xi| + O(t^{\frac{1}{2}})$$

for large t and $A^{-1} \leq |\xi| \leq A$. Thus,

$$|D\Phi(t,x;\xi)| \asymp |x| + t \ \text{ in the region } \ |x| > At, \ |x| < A^{-1}t.$$

We will integrate each of the ξ-integrals in (5.15) by parts separately. In order to do so, one first checks that

$$|D^\alpha[V(\partial S(t,\xi))a(t,\xi)]| \lesssim t^{c|\alpha| - \frac{3}{4}} \tag{5.16}$$

by (5.14) and (5.12). Indeed,

$$D^\alpha[V(\partial S(t,\xi))] = \sum_{\substack{|\alpha'|=m\geq 1, |\nu_1|,\ldots,|\nu_m|\geq 2\\ |\nu_1|+\ldots+|\nu_m|=|\alpha|+m}} \mathrm{coeff}\, D_x^{\alpha'} V(DS(t,\xi))\, D^{\nu_1}S \cdot \ldots \cdot D^{\nu_m}S(t,\xi)$$

so that (with the sum over the same indices)

$$|D^\alpha[V(\partial S(t,\xi))]| \lesssim \sum t^{-\frac{3}{4}-\beta|\alpha'|}\, t^{m+c(\sum_{j=1}^m |\nu_j|-2m)} \lesssim t^{-\frac{3}{4}+c|\alpha|}.$$

Combining this with (5.14) via the product rule yields (5.16). Moreover, one has $|D^\alpha \Phi(t,x;\xi)| = |D^\alpha S(t;\xi)| \lesssim t^{1+c(|\alpha|-2)}$ for $|\alpha| \geq 2$ by (5.12). Thus each of the ξ-integrals in (5.15) is no larger than $(t+|x|)^{-p}t^{cp}$ where p is the number of partial integrations. Hence, (5.15) itself is controlled by

$$\Big(\int_{\mathbb{R}^d} (t+|x|)^{-2p} t^{2cp}\, dx\Big)^{\frac{1}{2}} \lesssim t^{cp+\frac{d}{2}}\, t^{-p} \lesssim t^{-2}$$

provided p is taken to be large. □

We now return to those terms in (5.10) and (5.11) in which at least one derivative falls on $\hat{\phi}(\xi)$. Since the volume of the region $|x| \asymp t$ is t^d, the following lemma shows that the contribution to $L^2(|x| \asymp t)$ in (5.5) of those terms is integrable in time and therefore negligible. We shall make the restriction $|x| \asymp t$ henceforth without further mention.

Lemma 5.3. *Under the assumption* (5.14) *the integrals in* (5.10) *and* (5.11) *in which at least one derivative falls on* $\hat{\phi}$ *are* $\lesssim t^{-\frac{5}{4}-\frac{d}{2}}$.

Proof. We claim that the integrals in (5.10) and (5.11) in which at least one derivative falls on $\hat{\phi}$ are of the form

$$\int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, b(t,\xi)\, d\xi \quad \text{where} \quad \sup_\xi |D^\rho b(t,\rho)| \lesssim t^{-\frac{5}{4}+c|\rho|} \tag{5.17}$$

for any multi-index ρ. This is easy to see in the case of (5.10), since by Leibniz's rule

$$b(t,\xi) = \sum_{\substack{\gamma_3\neq 0\\ \gamma_1+\gamma_2+\gamma_3=\alpha}} \operatorname{coeff} D^{\gamma_1}[D_x^\alpha V(DS(t,\xi))] D^{\gamma_2} a(t,\xi) D^{\gamma_3}\hat{\phi}(\xi).$$

Expanding the derivatives inside the sum and applying (5.12) and (5.14) yields

$$|b(t,\xi)| \lesssim t^{-\frac{3}{4}-|\alpha|\beta} t^{c(|\alpha|-1)} \lesssim t^{-\frac{5}{4}}$$

and similarly for the derivatives. In the more complicated case (5.11) one has

$$b(t,\xi) = \sum_{\gamma'+\gamma''=\gamma} \operatorname{coeff} D^{\gamma'}[D^{\nu_1} S \cdot \ldots \cdot D^{\nu_m} S\, D_x^\alpha V(DS(t,\xi)) a(t,\xi)] D^{\gamma''}\hat{\phi}(\xi)$$

where $m \geq 1$, $|\nu_j| \geq 2$, and $|\nu_1| + \ldots + |\nu_m| + |\gamma'| < |\alpha|$. These conditions imply that $m \leq |\alpha|/2$. Hence (with max referring to the maximum with respect to all admissible choices of multi-indices)

$$\begin{aligned} |b(t,\xi)| &\lesssim \max t^{m+c\sum_{j=1}^m(|\nu_j|-2)}\, t^{-\frac{3}{4}-|\alpha|\beta}\, t^{c|\gamma'|} \lesssim \max t^{c(|\alpha|-1)-\frac{3}{4}-|\alpha|\beta+(1-2c)|\alpha|/2} \\ &\lesssim \max t^{-\frac{3}{4}-c+(\frac{1}{2}-\beta)|\alpha|} \lesssim t^{-\frac{5}{4}} \end{aligned} \tag{5.18}$$

and similarly for the derivatives (to pass to the final inequality in (5.18) we use that $c = 1 - \beta$ and $|\alpha| \geq 1$). For the remainder of this proof we work with (5.17) without using any other information. Following [10] we change variables to make application of (non)stationary phase easier. Set

$$\Psi(t,x;\eta) = t^{c-1}\langle x,\eta\rangle - t^{2c-1}S(t,\eta/t^c) = t^{2c-1}\Phi(t,x;\eta/t^c) \tag{5.19}$$

so that $|D^\alpha\Psi| \lesssim 1$ for all $|\alpha| \geq 2$. Also pick a smooth bump function χ of compact support such that $\sum_{g\in Z^d}\chi(\cdot - g) = 1$. Let

$$\chi_g(\eta,t) = \chi(\eta - g)b(t,\eta/t^c)$$

so that

$$\int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, b(t,\xi)\, d\xi = \sum_{g\in Z^d} t^{-cd}\int e^{it^{1-2c}\Psi(t,x;\eta)}\, \chi_g(\eta,t)\, d\eta \tag{5.20}$$

and $|\partial_\eta^\gamma \chi_g(\eta,t)| \lesssim t^{-\frac{5}{4}}$ for any γ. Recall that $\Phi(t,x;\xi)$ has a unique critical point ξ_0 defined by $x - DS(t,\xi_0) = 0$. For those finitely many choices of $g \in \mathbb{Z}^d$ for which

$$|\xi_0 t^c - g| \leq 100$$

we estimate the corresponding terms in (5.20) by means of stationary phase, see Theorem 7.7.5 in [11]. This requires uniform derivative bounds on the phase Ψ and the integrand χ_g, which have been established with the exception of the first derivative $D\Psi(t,x;\eta)$. Close to a critical point, however, this first derivative is controlled by means of the second derivative, whereas the size of Ψ itself is irrelevant for stationary phase. Hence all the conditions of stationary phase are satisfied and one has

$$\Big|\int e^{it^{1-2c}\Psi(t,x;\eta)}\, \chi_g(\eta,t)\, d\eta\Big| \lesssim (t^{1-2c})^{-\frac{d}{2}} t^{-\frac{5}{4}}$$

so that the contribution to (5.20) is

$$\lesssim t^{-cd}(t^{1-2c})^{-\frac{d}{2}} t^{-\frac{5}{4}} \lesssim t^{-\frac{5}{4}-\frac{d}{2}}. \tag{5.21}$$

Now suppose that

$$|\xi_0 t^c - g| \asymp k \geq 100.$$

Since $D\Phi(t,x;\xi)/t$ is a normalized diffeomorphism with respect to the ξ variable this implies that

$$|\partial_\eta \Psi(t,x;\eta)| = t^{c-1}|D\Phi(t,x;\eta/t^c)| \gtrsim t^{c-1}\, t\, k\, t^{-c} = k.$$

Applying nonstationary phase (i.e., integration by parts) to the corresponding integral yields

$$\Big|\int e^{it^{1-2c}\Psi(t,x;\eta)}\, \chi_g(\eta,t)\, d\eta\Big| \lesssim (t^{1-2c}k)^{-N} t^{-\frac{5}{4}}.$$

The contribution to (5.20) is therefore

$$\lesssim t^{-cd}(t^{1-2c})^{-N} \sum_{k\geq 100} k^{-N+d-1} t^{-\frac{5}{4}} \lesssim t^{-2-\frac{d}{2}}$$

provided N is large. Combining this with (5.21) finishes the proof. $\square$

In the next lemma we show that the error term (5.9) can be treated by the same method.

Lemma 5.4. *Under the assumption* (5.14) *the contribution of* (5.9) *is no larger than* $t^{-2-\frac{d}{2}}$ *provided* N *is large.*

Proof. Fix some α with $|\alpha| = N+1$. Let $\Psi(t,x;\eta)$ be defined by (5.19) and set $G_\alpha(t,x;\eta) = F_\alpha(t,x;\eta/t^c)$ where

$$F_\alpha(t,x;\xi) = \int_0^1 D_x^\alpha V(\theta x + (1-\theta)\, DS(t,\xi))\, d\theta. \tag{5.22}$$

As in the previous proof one has

$$\begin{aligned} &\int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, F_\alpha(t,x;\xi)\, d\xi \\ &= \sum_{g\in Z^d} t^{-cd+(1-c)|\alpha|} \int e^{it^{1-2c}\Psi(t,x;\eta)}\, \chi(\eta-g,t)\, G_\alpha(t,x;\eta)\, (\partial_\eta \Psi(t,x;\eta))^\alpha\, d\eta. \end{aligned} \tag{5.23}$$

As before, one applies the method of stationary phase to those terms in (5.23) for which $|t\xi_0 - g| \leq 100$, where ξ_0 is the critical point of $\Phi(t,x;\xi)$. This requires derivative bounds on G_α and the phase Ψ. The latter ones were explained in the previous proof, whereas G_α satisfies

$$|\partial_\eta^\gamma G_\alpha(t,x;\eta)| \lesssim t^{-\frac{3}{4}-|\alpha|\beta} = t^{-\frac{3}{4}-|\alpha|(1-c)} \quad \text{for all multi-indices } \gamma$$

(recall that $\beta = 1-c$). These bounds exploit that near a critical point $DS(t,\xi)$ is close to x, which implies that the argument of $D_x^\alpha V$ in (5.22) is of size t so that $|D^\gamma F_\alpha(t,x;\xi)| \lesssim t^{-\frac{3}{4}-\beta|\alpha|}\, t^{c|\gamma|}$. Thus by the method of stationary phase those terms in (5.23) for which $|t\xi_0 - g| \leq 100$ are

$$\lesssim t^{-\frac{3}{4}-|\alpha|(1-c)}\, t^{-cd+(1-c)|\alpha|}\, (t^{1-2c})^{-\frac{d}{2}-\frac{1}{2}|\alpha|} \lesssim t^{-\frac{3}{4}-\frac{d}{2}+(c-\frac{1}{2})(N+1)}.$$

Since $c < \frac{1}{2}$, the right-hand side can be made less than any negative power of t, as claimed. Now consider any $g \in \mathbb{Z}^d$ for which $|t^c\xi_0 - g| \asymp k \geq 100$. Then

$$|\partial_\eta \Psi(t,x;\eta)| \asymp k$$

so that the corresponding term in (5.23) is

$$\lesssim (k\, t^{1-2c})^{-p}\, t^{1-c}\, k^{N+1} \lesssim t^{-2-\frac{d}{2}}\, k^{-d-1}$$

provided the parameter p is taken large enough. The term t^{1-c} is a somewhat wasteful bound on $\partial_\eta^\gamma G_\alpha(t,x;\eta)$ which is due to the fact that without knowing the relative position of $DS(t,\xi)$ and x in (5.22) one merely has $|D^\gamma F_\alpha(t,x;\xi)| \lesssim t^{1+c(|\gamma|-1)}$ for $|\gamma| \geq 1$, and thus $|\partial_\eta^\gamma G_\alpha(t,x;\eta)| \lesssim t^{1-c}$ uniformly in γ. Summing over $g \in \mathbb{Z}^d$ with $|t^c\xi_0 - g| \asymp k \geq 100$ finishes the proof. $\square$

Having dispensed of various error terms, we conclude from the preceding (see in particular (5.5), (5.10) and (5.11)) that we should choose a such that

$$
\begin{aligned}
ia_t \quad &= \sum_{1\le|\alpha|\le N} \frac{1}{\alpha!}\Big\{(-i)^{|\alpha|}D^\alpha\big[V(DS(t,\xi))a(t,\xi)\big] \\
&\quad + \sum_{\substack{m\ge1,|\nu_1|,\ldots,|\nu_m|\ge2\\ \nu_1+\ldots+\nu_m+\gamma=\alpha}} \mathrm{coeff}\, D^\gamma\big[D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}SD_x^\alpha V(DS(t,\xi))a(t,\xi)\big]\Big\} \qquad (5.24)
\end{aligned}
$$

where coeff are the coefficients given by Lemma 5.1. More precisely, we will satisfy this equality up to error terms that give time integrable contributions to the $L^2(|x| \asymp t)$ norm. A simple application of product and chain rules shows that (at least formally) (5.24) with $N = \infty$ is of the form

$$
ia_t = \sum_{\substack{|\alpha|\ge1,|\nu_1|,\ldots,|\nu_m|\ge2\\ \nu_1+\ldots+\nu_m+\gamma=\alpha}} \mathrm{coeff}\, D_x^\alpha V(DS(t,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}SD_x^\alpha V(DS(t,\xi))D^\gamma a(t,\xi) \qquad (5.25)
$$

where now $m = 0$ is also allowed and coeff are suitable complex coefficients. The relation (5.25) is the starting point for our construction of the amplitude a. Recall that $c = 1-\beta$. In what follows we fix some $\delta > 0$ such that $\beta > \frac{1}{2} + \delta$.

Proposition 5.5. *Given N there exists, with probability one, a C^∞ function $a(t,\xi)$ on the same subset of $(0,\infty)\times\mathbb{R}^d\setminus\{0\}$ on which S is defined so that for large t equality holds in (5.25) for all $|\alpha| \le N$ and up to certain negligible error terms. Moreover, for some large M depending only on N and $\beta > \frac{1}{2}$ the function $a(t,\xi)$ is of the form $a = \sum_{j=0}^M a_j$ where $a_0 = 1$ and for all $1 \le j \le M$, all multi-indices γ, and all large t*

$$
|D^\gamma a_j(t,\xi)| \lesssim t^{-j\frac{\delta}{2}+c|\gamma|}. \qquad (5.26)
$$

The constants implicit in this notation are random but uniform for ξ in compact subsets of $\mathbb{R}^d\setminus\{0\}$. Finally,

$$
|D^\gamma a(t,\xi)| \lesssim t^{-\frac{\delta}{2}+c|\gamma|} \quad \text{and} \quad |D^\gamma a_t(t,\xi)| \lesssim t^{-\frac{3}{4}-\delta+c|\gamma|} \qquad (5.27)
$$

for all γ, which proves (5.14).

Proof. By induction. Set $a_0 = 1$ and suppose that $a_0, a_1, \ldots, a_{j-1}$ have been constructed satisfying (5.26). Then define

$$a_j(t,\xi) := -i \sum_{\substack{j=k-1+4(|\alpha|-m)\\ k\le j-1}} \text{coeff} \int_t^\infty D_x^\alpha V(DS(\tau,\xi)) D^{\nu_1}S \cdot \ldots \cdot D^{\nu_m} S(\tau,\xi)\, D^\gamma a_k(\tau,\xi)\, d\tau \tag{5.28}$$

where coeff are precisely those coefficients appearing in (5.25). The sum here extends over all $|\alpha| \ge 1$, $k, m, \nu_1, \ldots, \nu_m, \gamma$ satisfying the conditions stated in (5.28) as well as

$$|\nu_1|, \ldots, |\nu_m| \ge 2 \;\text{ and }\; \nu_1 + \ldots + \nu_m + \gamma = \alpha,$$

see (5.25). The meaning of the condition $j = k - 1 + 4(|\alpha| - m)$ will become clear in the proof. For the moment we only point out that it insures that the sum in (5.28) is finite, since $|\alpha| \ge 2m$. To verify (5.26) for j we begin with an integral as in (5.28) that does not contain a term $D^\gamma a_k$, in other words $k = 0$ and $\sum_j \nu_j = \alpha$. As usual, we take T and T^* so that $|DS(T,\xi)| = R$ and $|DS(T^*,\xi)| = 2R$ for some large R. The annulus $R \le |x| \le 2R$ is split up into $R^{1-\beta} = R^c$ many shells $\mathcal{S}_{\ell,R}$ of thickness R^β, see (3.1). Notice that $R \asymp T \asymp T^*$ and define $\tau_\ell, \widetilde{\tau}_\ell$ as in (4.24). Thus, with

$$z_\ell(t) := \partial_\xi S(\tau_\ell,\xi) + (t - \tau_\ell)\partial_\xi S_t(\tau_\ell,\xi)$$

one has

$$\int_T^{T^*} D_x^\alpha V(DS(t,\xi)) D^{\nu_1}S \cdot \ldots \cdot D^{\nu_m}S(t,\xi)\, dt$$

$$= \sum_\ell \int_{\tau_\ell}^{\widetilde{\tau}_{\ell+1}} D_x^\alpha V(z_\ell(t))\, D^{\nu_1}S(\tau_\ell,\xi) \cdot \ldots \cdot D^{\nu_m}S(\tau_\ell,\xi)\, dt \tag{5.29}$$

$$+ \sum_\ell O\Big(|\tau_{\ell+1} - \widetilde{\tau}_{\ell+1}| \sup_{R\le|x|\le 2R} |D_x^\alpha V(x)| \sup_{t\ge T} |D^{\nu_1}S(t,\xi) \cdot \ldots \cdot D^{\nu_m}S(t,\xi)| \Big) \tag{5.30}$$

$$+ \sum_\ell O\Big(|\tau_{\ell+1} - \tau_\ell|^3 \sup_{\substack{R\le|x|\le 2R\\ |\alpha'|=|\alpha|+1}} |D_x^{\alpha'} V(x)| \sup_{t\ge T} |DS_{tt}(t,\xi)| \sup_{t\ge T} |D^{\nu_1}S(t,\xi) \cdot \ldots \cdot D^{\nu_m}S(t,\xi)| \Big) \tag{5.31}$$

$$+ \sum_\ell O\Big(|\tau_{\ell+1} - \tau_\ell|^2 \sup_{R\le|x|\le 2R} |D_x^\alpha V(x)| \sup_{t\ge T} |\partial_t [D^{\nu_1}S(t,\xi) \cdot \ldots \cdot D^{\nu_m}S(t,\xi)]| \Big). \tag{5.32}$$

By (4.30) one has $|\tau_{\ell+1} - \widetilde{\tau}_{\ell+1}| \lesssim T^{\frac14}$. Another estimate that we shall use is

$$\big| D^{\nu_1}S(t,\xi) \cdot \ldots \cdot D^{\nu_m}S(t,\xi) \big| \lesssim t^{m + c\sum_{j=1}^m (|\nu_j|-2)} = t^{(1-2c)m + c|\alpha|}\, t^{-c|\gamma|}, \tag{5.33}$$

which follows immediately from (5.12), as well as

$$|DS_{tt}(t,\xi)| \lesssim t^{\frac14 - 2\beta} \;\text{ and }\; |D^\nu[\partial_t S(t,\xi)]| \lesssim t^{c|\nu| - \frac34} \;\text{ for }\; |\nu| \ge 2. \tag{5.34}$$

The first inequality is (4.29) whereas the second follows easily from (5.12) and the decay of derivatives of V. Observe that we have included γ in (5.33). Despite the fact that here $\gamma = 0$ since we are assuming that $k = 0$, it will be convenient to include it in the estimates for later applications. Finally, recall that there are $T^{1-\beta} = T^c$ many terms in the sums over ℓ above. Consequently the error terms above can be estimated as follows:

$$\begin{aligned}
(5.30) &\lesssim T^c\, T^{\frac14}\, T^{-\frac34-|\alpha|\beta}\, T^{m+c\sum_j(|\nu_j|-2)} \\
&= T^{\frac12-(|\alpha|+1)(1-c)}\, T^{(1-2c)m+c|\alpha|}\, T^{-c|\gamma|} = T^{c-\frac12}\, T^{-(1-2c)(|\alpha|-m)}\, T^{-c|\gamma|} \\
(5.31) &\lesssim T^c\, T^{3\beta}\, T^{-\frac34-(|\alpha|+1)\beta}\, T^{\frac14-2\beta} T^{(1-2c)m+c|\alpha|}\, T^{-c|\gamma|} \\
&= T^{c-\frac12}\, T^{-|\alpha|(1-c)}\, T^{(1-2c)m+c|\alpha|}\, T^{-c|\gamma|} = T^{c-\frac12}\, T^{-(1-2c)(|\alpha|-m)}\, T^{-c|\gamma|} \\
(5.32) &\lesssim T^c\, T^{2\beta}\, T^{-\frac34-|\alpha|\beta}\, T^{(1-2c)m+c|\alpha|}\, T^{-\frac34-1+2c}\, T^{-c|\gamma|} \\
&= T^{c-\frac12}\, T^{-(1-2c)(|\alpha|-m)}\, T^{-c|\gamma|}.
\end{aligned} \tag{5.35}$$

To pass to (5.35) we used that

$$\Big|\partial_t[D^{\nu_1}S(t,\xi)\cdot\ldots\cdot D^{\nu_m}S(t,\xi)]\Big| \lesssim \sum_{j=1}^m |D^{\nu_j}[\partial_t S(t,\xi)]| \prod_{\substack{s=1\\ s\neq j}}^m t^{1+c(|\nu_s|-2)} \lesssim t^{m+c(|\alpha|-|\gamma|-2m)}\; t^{-\frac34} t^{-1+2c},$$

see (5.34). Now let Y_ℓ be one of the summands in (5.29). By the measurability considerations in the proof of Lemma 4.5 the $\{Y_\ell\}$ form a martingale difference sequence with respect to the usual family of increasing σ-algebras. Hence Lemma 2.1 implies that up to probability at most e^{-T^ε}

$$\begin{aligned}
|\sum_\ell Y_\ell| &\le T^\varepsilon\Big(\sum_\ell \|Y_\ell\|_\infty^2\Big)^{\frac12} \lesssim T^\varepsilon T^{\frac{c}{2}+1-c}\, T^{-\frac34-|\alpha|(1-c)}\, T^{(1-2c)m+c(|\alpha|-|\gamma|)} \\
&\lesssim T^{\frac14-\frac{c}{2}+\varepsilon}\, T^{-(1-2c)(|\alpha|-m)}\, T^{-c|\gamma|} \lesssim T^{-\frac{\delta}{2}[-1+4(|\alpha|-m)]}\, T^{-c|\gamma|} \lesssim T^{-j\frac{\delta}{2}}.
\end{aligned} \tag{5.36}$$

To pass to the final inequality in (5.36) one uses that $j = k-1+4(|\alpha|-m)$ with $k=0$ and $\gamma = 0$. Moreover, the second inequality in (5.36) follows from

$$\frac14 - \frac{c}{2} + \varepsilon - (1-2c)(|\alpha|-m) \le \frac{\delta}{2} - 2\delta(|\alpha|-m)$$

which is equivalent to

$$(1-2c-2\delta)(|\alpha|-m) \ge \frac14(1-2c-2\delta)+\varepsilon \quad\text{or}\quad 0<\varepsilon\le\frac14(1-2c-2\delta). \tag{5.37}$$

The final statement holds as $|\alpha|-m \ge |\alpha|/2 \ge 1/2$ and $1-2c-2\delta = 2(\beta-\frac12-\delta) > 0$. We now compare the bound on $|\sum_\ell Y_\ell|$ from (5.36) with the bound on the error terms obtained in (5.35), which is $T^{c-\frac{1}{2}}\,T^{-(1-2c)(|\alpha|-m)}$. This latter quantity is indeed smaller than the first bound in (5.36), as $\frac{1}{4}-\frac{c}{2}>c-\frac{1}{2}$. We have arrived at the conclusion that

$$\Big|\int_T^{T^*} D_x^\alpha V(DS(\tau,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(t,\xi)\,dt\Big|\lesssim T^{-j\frac{\delta}{2}}$$

up to probability at most e^{-T^ε}. In fact, we have shown the stronger statement

$$\begin{aligned}&\sup_{T\le\tau\le T^*}\Big|\int_T^{\tau} D_x^\alpha V(DS(\tau,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(t,\xi)\,dt\Big|\\ &\lesssim T^{\frac{1}{4}-\frac{c}{2}+\varepsilon}\,T^{-(1-2c)(|\alpha|-m)}\,T^{c(\sum_j|\nu_j|-|\alpha|)}\lesssim T^{-j\frac{\delta}{2}}.\end{aligned}\tag{5.38}$$

By Borel-Cantelli one now obtains that, with probability one and some random constant C_ρ,

$$\sup_{\rho<|\xi|<\rho^{-1}}\Big|\int_t^\infty D_x^\alpha V(DS(\tau,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(\tau,\xi)\,d\tau\Big|\le C_\rho\,t^{-j\frac{\delta}{2}}$$

for all t and arbitrary $\rho>0$, as desired. The same argument also shows that

$$\sup_{\rho<|\xi|<\rho^{-1}}\Big|D^\gamma\int_t^\infty D_x^\alpha V(DS(\tau,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(\tau,\xi)\,d\tau\Big|\lesssim t^{-j\frac{\delta}{2}+c|\gamma|}$$

for any multi-index γ. Indeed, consider the case when all the derivatives fall on $D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(\tau,\xi)$. In view of (5.12) this gives an increase by $t^{|\gamma|c}$, as claimed. All other cases are of the same order of magnitude.

Next, we turn to those terms in (5.28) that contain $a_k(t,\xi)$ for some $0<k\le j-1$. The martingale argument that was used for $k=0$ does not immediately apply to the case $k\ne0$ as $a_k(t,\xi)$ depends on all random variables. This issue, however, can be circumvented by inductively reducing oneself to the case $k=0$. More precisely, consider

$$\int_T^{T^*} D_x^\alpha V(DS(t,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(t,\xi)\,D^\gamma a_k(t,\xi)\,dt=$$

$$= D^\gamma a_k(T,\xi)\int_T^{T^*} D_x^\alpha V(DS(t,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(t,\xi)\,dt\tag{5.39}$$

$$-\int_T^{T^*} D_x^\alpha V(DS(t,\xi))D^{\nu_1}S\cdot\ldots\cdot D^{\nu_m}S(t,\xi)\,D^\gamma[a_k(T,\xi)-a_k(t,\xi)]\,dt.\tag{5.40}$$

The first expression (5.39) is controlled by means of the induction hypothesis and the previous calculation with $k=0$. Indeed, using (5.26) with $k<j$ and the one before last estimate in (5.36) shows that

$$\begin{aligned}(5.39)\;&\lesssim\; T^{-k\frac{\delta}{2}+c|\gamma|}\,T^{-\frac{\delta}{2}[-1+4(|\alpha|-m)]}\,T^{-c|\gamma|}\\ &\lesssim\; T^{-\frac{\delta}{2}[-1+k+4(|\alpha|-m)]}=T^{-j\frac{\delta}{2}},\end{aligned}$$

since $j = -1 + k + 4(|\alpha| - m)$. Clearly, the same calculation also shows that

$$|D^\nu(5.39)| \lesssim T^{-j\frac{\delta}{2}+|\nu|c}$$

for any multi-index ν. By definition (5.28) the integral in (5.40) is the sum of terms of the form

(5.41)
$$\int_T^{T^*} D_x^\alpha V(DS(t,\xi)) D^{\nu_1}S \cdot \ldots \cdot D^{\nu_m}S(t,\xi) \int_T^t D_x^{\alpha'} V(DS(\tau,\xi)) D^{\nu_1'}S \cdot \ldots \cdot D^{\nu'_{m'}}S(\tau,\xi)\, D^{\gamma'} a_{k'}(\tau,\xi)\, d\tau\, dt$$

where the various parameters satisfy the relations

$$\begin{aligned} \alpha &= \nu_1 + \ldots + \nu_m + \gamma, \quad |\alpha| \ge 1,\ |\nu_j| \ge 2 \\ \alpha' + \gamma &= \nu_1' + \ldots + \nu'_{m'} + \gamma', \quad |\alpha'| \ge 1,\ |\nu_j'| \ge 2 \\ k &= -1 + k' + 4(|\alpha'| - m'), \quad k' \le k - 1 \\ j &= -1 + k + 4(|\alpha| - m), \quad k \le j - 1. \end{aligned} \tag{5.42}$$

The last two lines imply that

$$j = -2 + k' + 4(|\alpha| + |\alpha'| - m - m') \ \text{ and } \ k' \le j - 2. \tag{5.43}$$

The inductive reduction should now be clear: If $k = 0$ in (5.41), then one uses the same martingale argument as above in combination with the induction hypothesis, which controls the size of the inner integral. If $k' > 0$, then the previous reduction needs to be repeated, i.e., pull $a_{k'}(T,\xi)$ outside of the entire expression and leave $a_{k'}(t,\xi) - a_{k'}(T,\xi)$ inside. The expression with $a_{k'}(T,\xi)$ on the outside is controlled by the the induction hypothesis and the argument that we shall elaborate below for the $k' = 0$ case. On the other hand, there are two possibilities for the remaining part with $a_{k'}(t,\xi) - a_{k'}(T,\xi)$ inside. Writing this difference out as we did above by means of (5.28) leads to expressions that are either "a-free" or not. In the latter case, one repeats, whereas in the former case the martingale argument applies again. Since the index of the amplitude goes down by at least one at each step, the process terminates. We shall not supply the complete details of this reduction, as they are straightforward. Rather, we shall discuss the aforementioned case of $k' = 0$ in (5.41) in full detail. We start with a slightly informal calculation that will explain the mechanism behind the numerology that appears in (5.43). The point is that integrals such as (5.41) are by a factor of $(T^{\frac{1-\beta}{2}-\varepsilon})^{-2} = T^{-c+2\varepsilon}$ smaller than the corresponding integral with absolute values inside. The square here is due to the fact that two integrations are being performed, each averaging gaining a factor of $T^{\frac{1-\beta}{2}-\varepsilon}$. In other words,

$$(5.41) \lesssim T^{-\frac{3}{4}-|\alpha|\beta}\, T^{m+c(|\alpha|-|\gamma|-2m)}\, T^{-\frac{3}{4}-|\alpha'|\beta}\, T^{m'+c(|\alpha'|+|\gamma|-2m')}\, T^{2-c+2\varepsilon} \tag{5.44}$$

$$\lesssim T^{\frac{1}{2}-c+2\varepsilon}\, T^{-(1-2c)(|\alpha|+|\alpha'|-m-m')} \lesssim T^{-\frac{\delta}{2}[-2+4(|\alpha|+|\alpha'|-m-m')]} \lesssim T^{-j\frac{\delta}{2}}. \tag{5.45}$$

Here the bound in (5.44) is obtained by using the usual point-wise bound on the derivatives of V, whereas the products are controlled by means of (5.33) in conjunction with (5.42). The final inequality in (5.45) follows from (5.43), whereas the first can be seen to follow from

$$\frac{1}{2}(1-2c-2\delta+2\varepsilon) \leq (1-2c-2\delta)(|\alpha|+|\alpha'|-m-m').$$

Since $|\alpha|-m \geq |\alpha|/2$ and similarly for α', this reduces to

$$\varepsilon \leq \frac{1}{4}(1-2c-2\delta).$$

Observe that this is the same condition as in (5.37). This argument shows that the presence of -2 in (5.43) matches exactly with the fact that *two* averages are being performed in (5.41), and similarly for any other number of averages. For the rigorous martingale version of this informal argument, we will approximate (5.41) by a sum of martingale differences Z_ℓ where

$$Z_\ell := \int_{\tau_\ell}^{\widetilde{\tau}_{\ell+1}} D_x^\alpha V(z_\ell(t))\, dt\; D^{\nu_1} S\cdot \ldots \cdot D^{\nu_m} S(\tau_\ell,\xi) \int_T^{\tau_\ell} D_x^{\alpha'} V(DS(\tau,\xi)) D^{\nu_1'} S\cdot \ldots \cdot D^{\nu'_{m'}} S(\tau,\xi)\, d\tau$$

where $\tau_\ell, \widetilde{\tau}_{\ell+1}, z_\ell(t)$ are as above. To estimate $\|Z_\ell\|_\infty$, one first uses (5.38) to obtain a bound on the second integral:

$$\begin{aligned}
&\Big|\int_T^{\tau_\ell} D_x^{\alpha'} V(DS(\tau,\xi)) D^{\nu_1'} S\cdot \ldots \cdot D^{\nu'_{m'}} S(\tau,\xi)\, d\tau\Big| \\
&\lesssim T^{\frac14-\frac{c}{2}+\varepsilon}\, T^{-(1-2c)(|\alpha'|-m')}\, T^{c(\sum_{j=1}^{m'}|\nu_j'|-|\alpha'|-|\gamma'|)} \lesssim T^{\frac14-\frac{c}{2}+\varepsilon}\, T^{-(1-2c)(|\alpha'|-m')}\, T^{c|\gamma|}
\end{aligned}$$

where we have used that $\gamma'=0$ and the second relation in (5.42). Therefore,

$$\begin{aligned}
\|Z_\ell\|_\infty &\lesssim T^\beta\, T^{-\frac34-|\alpha|\beta}\, T^{m+c(|\alpha|-2m)}\, T^{\frac14-\frac{c}{2}+\varepsilon}\, T^{-(1-2c)(|\alpha'|-m')}\, T^{c|\gamma|} \\
&\lesssim T^{\frac12-\frac{3c}{2}+\varepsilon}\, T^{-(1-2c)(|\alpha|+|\alpha'|-m-m')}.
\end{aligned}$$

Hence, up to probability at most e^{-T^ε},

$$\Big(\sum_\ell \|Z_\ell\|_\infty\Big)^{\frac12} \lesssim T^{\frac12-c+2\varepsilon}\, T^{-(1-2c)(|\alpha|+|\alpha'|-m-m')} \lesssim T^{-j\frac{\delta}{2}}, \tag{5.46}$$

which agrees with the informal calculation in (5.45). There are three error terms involved in the approximation of (5.41) by $\sum_\ell Z_\ell$. One derives from replacing $\tau_{\ell+1}$ with $\widetilde{\tau}_{\ell+1}$, another from the linearization of $DS(t,\xi)$, and the final one from freezing t at τ_ℓ. The first two error terms can be treated in the same way as those in (5.30) and (5.31), respectively, and we therefore skip the details. On the other hand, freezing the second integral in (5.41) at time $t=\tau_l$ introduces a new type of error that we now estimate. Clearly, the error in question is bounded by

$$\begin{aligned}
&T\, T^{-\frac34-|\alpha|\beta}\, T^{m+c(\sum_{j=1}^m|\nu_j|-2m)}\, T^\beta\, T^{-\frac34-|\alpha'|\beta}\, T^{m'+c(\sum_{j=1}^{m'}|\nu_j'|-2m')} \\
&= T^{-\frac12+\beta}\, T^{-(1-2c)(|\alpha|+|\alpha'|-m-m')}
\end{aligned} \tag{5.47}$$

since $\sum_{j=1}^{m} |\nu_j| = |\alpha| - |\gamma|$ and $\sum_{j=1}^{m'} |\nu'_j| = |\alpha'| + |\gamma|$. Comparing this with the desired bound (5.46), one sees that (5.47) is smaller than (5.46) by $T^{-2\varepsilon}$, so that it can be ignored. This concludes the inductive proof of (5.26).
Now fix some large N and let $a = \sum_{j=0}^{M} a_j$ with another large integer M that we need to specify. The way to choose M is to cancel as many orders of magnitude in t as possible for each of the terms in (5.25) with $1 \le |\alpha| \le N$. The terms with $|\alpha| > N$ are negligible by Lemma 5.4. It is clearly impossible to achieve

$$i\partial_t \sum_{j=0}^{M} a_j(t,\xi) - \sum \operatorname{coeff} D_x^\alpha V(DS(t,\xi)) D^{\nu_1} S \cdot \ldots \cdot D^{\nu_m} S D_x^\alpha V(DS(t,\xi)) D^\gamma \sum_{j=0}^{M} a_j(t,\xi) = 0. \tag{5.48}$$

(with the sum as in (5.25)) by means of our construction based on (5.28). On the other hand, it is also clear that by taking M sufficiently large one can achieve that this difference is only going to involve $a_j(t,\xi)$ and $D_x^\alpha V(DS(t,\xi)$ with indices j and multi-indices α so that

$$j + 4|\alpha| \ge j + 4(|\alpha| - m) \ge j_0,$$

see (5.28) with some fixed j_0. Taking j_0 large, it follows from (5.26) that the resulting expression will again be negligible in the usual sense. This can be seen by putting absolute values inside the integrals, since either $D_x^\alpha V(DS(t,\xi)$ or $a_j(t,\xi)$ can be made less than any given large negative power of t.
Finally, we check that (5.27) holds. The first inequality follows immediately from (5.26). For the second, take derivatives in (5.28). With the sum being over all admissible choices of parameters in (5.28) one has

$$\begin{aligned} |D^\nu \partial_t a_j(t,\xi)| &\lesssim \sum t^{-\frac{3}{4} - |\alpha|(1-c)}\, t^{m + c(|\alpha| - |\gamma| - 2m)}\, t^{c|\nu|}\, t^{-k\frac{\delta}{2} + |\gamma| c} \\ &\lesssim t^{-\frac{3}{4} - k\frac{\delta}{2}}\, t^{-2\delta(|\alpha| - m)}\, t^{c|\nu|} = t^{-\frac{3}{4} - (j+1)\frac{\delta}{2}}\, t^{c|\nu|} \end{aligned}$$

where the second inequality follows from $\beta \ge \frac{1}{2} + \delta$, and the final equality holds since $j = -1 + k + 4(|\alpha| - m)$. Summing over $j \ge 1$ proves (5.27) and thus the proposition. $\square$

It should be clear from the previous proof that the usual definition of modified wave operators with $a(t,\xi) = 1$ does not apply in our situation. Indeed, the first significant term in the stationary phase expansion of

$$\int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, [V(x) - V(\partial S(t,\xi))] \hat{\phi}(\xi)\, d\xi$$

is

$$\int_{\mathbb{R}^d} e^{i\Phi(t,x;\xi)}\, \langle D_x^2 V(x)(x - \partial S(t,\xi)), x - \partial S(t,\xi)\rangle \hat{\phi}(\xi)\, d\xi. \tag{5.49}$$

By the method of stationary phase this integral is of the order $D_x^2 V(x) t\, t^{-\frac{d}{2}}$. Therefore,

$$\|(5.49)\|_{L^2(|x|\asymp t)} \lesssim \sup_{|x|\asymp t} |D_x^2 V(x)|\, t.$$

In the context of standard scattering theory [6], which requires $t^{-2-\varepsilon}$ of the second derivatives, this latter expression is integrable in time. In our case, however, $|D_x^2 V(x)t|$ is typically on the order of $t^{-\frac{3}{4}-\varepsilon}$ and therefore not integrable. We circumvent this difficulty here by means of the amplitude $a(t,\xi)$. In fact, it is easy to see from (5.28) and also natural in view of the aforementioned problem with $D^2 V(x)t$ that the first nonzero term after $a_0 = 1$ is

$$a_3(t,\xi) = \mathrm{const} \int_t^\infty D_x^2 V(DS(\tau,\xi))\, D^2 S(\tau,\xi)\, d\tau$$

with a suitable constant const (this allows one to slightly strengthen (5.27), but we do not exploit this fact). Generally speaking, the patient reader will easily find the first few a_j by means of explicit integration by parts. It turns out that $a_1 = a_2 = a_4 = a_5 = a_8 = 0$, whereas the first few nonzero terms decay like $t^{-\frac{3\delta}{2}}, t^{-3\delta}$ and $t^{-\frac{7\delta}{2}}$, respectively. In the context of (5.28) the zero terms a_j are identified by means of non-divisibility by four of $j+1-k$.

We have now arrived at our main result.

Theorem 5.6. *With probability one the following holds: For any $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\mathrm{supp}(\hat{\phi}) \subset \mathbb{R}^2 \setminus \{0\}$ let*

$$(U(t)\phi)(x) := \int_{\mathbb{R}^d} e^{i[x\cdot\xi - S(t,\xi)]}\, a(t,\xi)\, \hat{\phi}(\xi)\, d\xi,$$

where $S(t,\xi)$ and $a(t,\xi)$ are the C^∞ functions constructed in Corollary 4.8 and Proposition 5.5, respectively. For every such ϕ there exists $f \in L^2(\mathbb{R}^d)$ such that

$$\lim_{t\to\infty} \big\| e^{itH} f - U(t)\phi \big\| = 0. \tag{5.50}$$

In other words, $W^+ := s-\lim_{t\to\infty} e^{-itH} U(t)$ exists, W^+ is an isometry, and W^+ intertwines H and $\frac{1}{2}\triangle$, i.e., for all $t \geq 0$

$$e^{itH} W^+ = W^+ e^{i\frac{t}{2}\triangle}. \tag{5.51}$$

Finally, this implies that $\sigma_{a.c}(H) = [0,\infty)$.

Proof. The amplitude a was constructed in such a way that (5.2) converges, which implies that (5.50) holds. As $\|\phi - U(t)\phi\|_2 \lesssim t^{-\frac{\delta}{2}}$ by (5.26), it follows that W^+ is an isometry on $L^2(\mathbb{R}^d)$. For the intertwining property one computes

$$\begin{aligned} e^{itH}\, W^+\phi &= \lim_{s\to\infty} e^{-i(s-t)H}\, U(s)\phi \\ &= \lim_{s\to\infty} e^{-i(s-t)H}\, U(s-t)\, e^{i\frac{t}{2}\triangle}\phi + \lim_{s\to\infty} e^{-i(s-t)H}\, (U(s) - U(s-t)\, e^{i\frac{t}{2}\triangle})\phi. \end{aligned} \tag{5.52}$$

The first expression on the right-hand side is $W^+ e^{i\frac{t}{2}\triangle}\phi$, whereas the second is (up to the unitary factor $e^{-i(s-t)H}$) equal to

$$(U(s) - U(s-t)\, e^{i\frac{t}{2}\triangle})\phi(x) = \int e^{ix\cdot\xi} \Big[e^{-iS(s,\xi)}\, a(s,\xi) - e^{-i\frac{t}{2}|\xi|^2 - iS(s-t,\xi)}\, a(s-t,\xi) \Big]\, \hat{\phi}(\xi)\, d\xi.$$

Since

$$S(s,\xi) - S(s-t,\xi) = \frac{t}{2}|\xi|^2 + \int_{s-t}^{s} V(DS(\tau,\xi)\, d\tau \to \frac{t}{2}|\xi|^2 \quad \text{as} \quad s \to \infty$$

and $|a(s,\xi) - a(s-t,\xi)| \lesssim s^{-\frac{3}{4}-\delta}\, t$ by (5.27), with both estimates holding uniformly for ξ in compact subsets of $\mathbb{R}^d \setminus \{0\}$, one concludes that the term in brackets goes to zero as $s \to \infty$. Thus the second term in (5.52) goes to zero by Plancherel and the dominated convergence theorem. Finally, the Weyl criteria implies that the deterministic essential spectrum $\sigma_{ess}(H)$ of the Schrödinger operator with potential (1.2) coincides with the essential spectrum of the unperturbed operator $-\Delta$, $\sigma_{ess}(H) = [0,\infty)$. Therefore we conclude, from the intertwining properties of the wave operator W^+, that with probability one $\sigma_{a.c.}(H) = [0,\infty)$. □

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