The Moment Formula for Implied Volatility at Extreme Strikes

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Abstract

Consider options on a nonnegative underlying random variable with arbitrary distribution. In the absence of arbitrage, we show that at any maturity T, the large-strike tail of the Black-Scholes implied volatility skew is bounded by the square root of $2|x|/T$, where x is log-moneyness. The smallest coefficient that can replace the 2 depends only on the number of finite moments in the underlying distribution. We prove the *moment formula*, which expresses explicitly this model-independent relationship. We prove also the reciprocal moment formula for the small-strike tail, and we exhibit the symmetry between the formulas. The moment formula, which evaluates readily in many cases of practical interest, has applications to skew extrapolation and model calibration.

Key words: Implied volatility, moment.

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1 Introduction

This section outlines briefly the contribution of this paper, deferring to sections 2 and 3 the explicit definitions of some terminology.

Let us write the squared Black-Scholes implied volatility I^2 as a coefficient times $|x|/T$, the ratio of absolute-log-moneyness to maturity. In Section 3, we show that as $x \to \infty$, the limsup of this coefficient is a number $\beta_R \in [0,2]$, which can be termed the right-hand or OTM-call or large-strike *tail slope*. Similarly, we show that as $x \to -\infty$, the limsup is a number $\beta_L \in [0,2]$, which can be termed the left-hand or OTM-put or small-strike *tail slope*.

Then we establish the explicit one-to-one correspondence

$$\text{large-strike tail slope} \quad \longleftrightarrow \quad \text{number of finite moments of the underlying } S_T.$$

In particular, $\beta_R = 2$ if and only if the underlying has *no* finite moments of order greater than 1; at the opposite extreme, $\beta_R = 0$ if and only if the underlying has finite moments of *all* positive orders. In general, we prove the *moment formula for implied volatility* at large strikes:

$$\frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2} = \sup\{p : \mathbb{E}S_T^{1+p} < \infty\}. \tag{1.1}$$

In the opposite tail, we establish the reciprocal relationship:

$$\text{small-strike tail slope} \quad \longleftrightarrow \quad \text{number of finite moments of } 1/S_T,$$

by proving the *moment formula for implied volatility* at small strikes:

$$\frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2} = \sup\{q : \mathbb{E}S_T^{-q} < \infty\}. \tag{1.2}$$

Note that the right-hand sides of (1.1) and (1.2) are real numbers or infinity; the phrase "number of finite moments" does not necessarily refer to an integer.

Section 4 exhibits the symmetry between the large-strike and small-strike moment formulas.

Section 5 presents some applications. For extrapolating the volatility skew with splines, the moment formula raises warnings against spline functions that grow faster than $|x|^{1/2}$, *and* against those that grow *slower* than $|x|^{1/2}$. For calibrating models to the volatility skew, note that each moment formula's left-hand side is, in principle, observable from options data; while the right-hand side is, in a wide class of models, easily calculated from the parameters. By building this direct link between data and parameters, the moment formula can facilitate the calibration procedure.

The moment formula assumes only the existence of a martingale measure. The underlying, a nonnegative random variable, is required to have positive and finite expectation; but this is no restriction because the concepts of "moneyness" and "implied volatility" each already entail this condition. Beyond this, we make *no assumptions* on the distribution of S_T.

1.1 Related work

Hodges (1996) and Gatheral (1999) provide arbitrage bounds on the slope $\partial I/\partial x$ of the implied volatility skew, bounds derived from the strike-monotonicity of call and put prices. These $\partial I/\partial x$ bounds depend on I, so bounds on I itself would follow by solving ODEs; and Lipton (2001) mentions that the resulting I bounds are $O(|x|^{1/2})$ for large $|x|$. Our approach and our results differ from theirs in significant ways. First, our direct concise proof of the $O(|x|^{1/2})$ bound avoids ODEs and indeed avoids differentiability assumptions altogether; second, we give the *best possible constant* in that bound; third, and most importantly, our moment formula shows explicitly how the best constant depends only on the *number of finite moments* in the underlying distribution; and fourth, our model-calibration application and part of our skew-extrapolation application will depend on the moment formula, not merely on the $O(|x|^{1/2})$ bound.

We take note of two other papers, which have used steepest-descent/saddle-point methods (instead of moment analysis) to calculate implied volatility asymptotics far-from-the-money. Each assumes a specific diffusion model for volatility: Avellaneda-Zhu (1998) use a non-mean-reverting model for instantaneous volatility, and Gatheral-Matytsin-Youssfi (2000) use the Heston square-root model. In contrast, our formula is distinguished by its full *model-independent* generality and its explication of the fundamental correspondence between *moments* and implied volatility tails.

2 Call and Put Prices

Let V_t be the time-t price of a claim paying at some fixed time $T > 0$ the random variable V_T.

Let B_t be the time-t price of a discount bond maturing at T.

Assuming that the prices (of B, V, and any other assets under consideration) admit no arbitrage, there must exist a probability measure $\mathbb{P}$, called the (T-)*forward* measure, under which all B_t-discounted asset prices are martingales. For definitions of "admit no arbitrage" that make this statement true, see for example Delbaen and Schachermayer (1994).

Let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Then the price of the claim satisfies

$$V_0 = B_0 \mathbb{E} V_T.$$

Under deterministic interest rates, the forward measure is identical to the usual risk-neutral measure; but under stochastic interest rates, the forward measure has the advantage that the discounting takes place outside the expectation.

In the cases of interest here, V_T is the payoff of a call or put on a nonnegative underlying randomness S_T. Write C and P for the time-0 call and put prices as a function of strike:

$$C(K) = B_0\mathbb{E}(S_T - K)^+$$
$$P(K) = B_0\mathbb{E}(K - S_T)^+$$

for $K > 0$.

2.1 Upper Bounds

The standard conventions about ∞ are in force, so each of the following bounds holds automatically if the expectation on the right-hand side is infinite. Also, for $q > 0$, the random variable S_T^{-q} is understood to take the value ∞ in the event that $S_T = 0$.

The following theorem is nearly identical to results obtained in Broadie-Cvitanic-Soner (1998). The differences, though minor, make it appropriate to present briefly a full proof. Then, in the subsequent remarks, we investigate its implications for the extreme-strike behavior of option prices.

Theorem 2.1. *For each $p > 0$ we have for all $K > 0$ the call price bound*

$$C(K) \leqslant \frac{B_0\mathbb{E}S_T^{p+1}}{p+1}\left(\frac{p}{p+1}\right)^p \frac{1}{K^p}. \tag{2.1}$$

For each $q > 0$ we have for all $K > 0$ the put price bound

$$P(K) \leqslant \frac{B_0\mathbb{E}S_T^{-q}}{1+q}\left(\frac{q}{1+q}\right)^q K^{1+q}. \tag{2.2}$$

Proof. For all $s \geqslant 0$ we have

$$s - K \leqslant \frac{s^{p+1}}{p+1}\left(\frac{p}{p+1}\right)^p \frac{1}{K^p},$$

because the left-hand side and right-hand side, as functions of s, have equal values and first derivatives at $s = (p+1)K/p$, but the right-hand side has positive second derivative. Moreover, since the right-hand side is nonnegative, the left-hand side can be improved to $(s-K)^+$. Now substitute S_T for s, take expectations, and multiply by B_0 to obtain (2.1).

Similarly, a convexity argument shows that for all $s > 0$,

$$(K-s)^+ \leqslant \frac{s^{-q}}{1+q}\left(\frac{q}{1+q}\right)^q K^{1+q}$$

which implies (2.2). □

Remark 2.1. Although both (2.1) and (2.2) are true for all K, the bound (2.1) is useful mainly for K large, while (2.2) is useful mainly for K small.

Remark 2.2. By put-call parity, the call price bound (2.1) leads to a large-K put price bound, and the put price bound (2.2) leads to a small-K call price bound.

Remark 2.3. Taking $p \downarrow 0$ in (2.1) recovers the familiar bound $C(K) \leqslant B_0 \mathbb{E} S_T$. Taking $q \downarrow 0$ in (2.2) recovers the familiar bound $P(K) \leqslant B_0 K$.

Remark 2.4. In a wide class of specifications for the underlying dynamics, $\log S_T$ has a distribution whose characteristic function f is explicitly known. In such cases, one calculates $\mathbb{E} S_T^{p+1}$ simply by extending f analytically to a strip in $\mathbb{C}$ containing $-i(p+1)$, and evaluating f there; if no such extension exists, then $\mathbb{E} S_T^{p+1} = \infty$.

Remark 2.5. Extreme-strike option price bounds are useful in controlling the error, at *all* strikes, in discrete Fourier transform methods for option pricing. See Lee (2001).

Corollary 2.2. *If* $\mathbb{E} S_T^{p+1} < \infty$, *then* $C(K) = O(K^{-p})$ *as* $K \to \infty$.

If $\mathbb{E} S_T^{-q} < \infty$, *then* $P(K) = O(K^{1+q})$ *as* $K \to 0$.

Proof. This follows immediately. □

3 Implied Volatility and the Moment Formula

Define

$$F_0 = \mathbb{E} S_T,$$

which one interprets as today's T-forward price of the payoff S_T. For example, if S is a stock paying no dividends, then $F_0 = S_0/B_0$.

Since we intend to study how implied volatility relates to moneyness, we assume $0 < F_0 < \infty$. This assumption is innocuous, because the very concepts of "implied volatility" and "moneyness" would both degenerate, if the forward price were to be zero or infinite. Moreover, if S is the price of a traded asset, then no-arbitrage dictates that $\mathbb{E} S_T < \infty$ must hold.

Fixing F_0, the (log-)*moneyness* x is related to strike by the definition

$$x \equiv \log(K/F_0), \tag{3.1}$$

so let $K(x) := F_0 e^x$ be the strike at moneyness x. Note that we have chosen the sign convention such that K increases as x increases.

The Black-Scholes *implied volatility* at moneyness x is defined as the $I(x)$ that uniquely solves

$$C(K(x)) = C^{BS}(x, I(x)), \tag{3.2}$$

where

$$C^{BS}(x,\sigma) := B_0(F_0\Phi(d_+) - K(x)\Phi(d_-)), \qquad d_\pm := \frac{-x}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2},$$

and Φ is the normal cumulative distribution function.

Equivalently, it is the $I(x)$ that uniquely solves

$$P(K(x)) = P^{BS}(x, I(x)), \tag{3.3}$$

where

$$P^{BS}(x,\sigma) := B_0(K(x)\Phi(-d_-) - F_0\Phi(-d_+)).$$

So, using bond and forward prices, we have defined implied volatility in a general way which allows stochastic interest rates and dividends. This definition of $I(x)$ could also be described as a "Black (1976)" implied volatility. In the special case of constant interest rates and dividends, $I(x)$ coincides with the usual Black-Scholes (1973) implied volatility.

The terminology "[implied] volatility *skew*" will refer to the function I.

The following pair of identities will be useful. For any $\beta > 0$ and $x > 0$,

$$C^{BS}(x, \sqrt{\beta|x|/T}) = B_0F_0\Phi(-\sqrt{f_-(\beta)|x|}) - B_0F_0e^x\Phi(-\sqrt{f_+(\beta)|x|}),$$

and for any $\beta > 0$ and $x < 0$,

$$P^{BS}(x, \sqrt{\beta|x|/T}) = B_0F_0e^x\Phi(-\sqrt{f_-(\beta)|x|}) - B_0F_0\Phi(-\sqrt{f_+(\beta)|x|}),$$

where

$$f_\pm(\beta) := \frac{1}{\beta} + \frac{\beta}{4} \pm 1.$$

3.1 The Large-Strike Tail

Consider the right-hand (or large-K or positive-x or OTM-call) tail of the square of implied volatility. First we show that this tail slope is no larger than 2.

Lemma 3.1. *There exists $x^* > 0$ such that for all $x > x^*$,*

$$I(x) < \sqrt{2|x|/T}.$$

Proof. By the strict monotonicity of C^{BS} in its second argument, we need only establish that

$$C^{BS}(x, I(x)) < C^{BS}(x, \sqrt{2|x|/T}), \tag{3.4}$$

whenever $x > x^*$. On the left-hand side of (3.4), we have

$$\lim_{x\to\infty} C(K(x)) = \lim_{K\to\infty} B_0 \mathbb{E}(S_T - K)^+ = 0$$

by dominated convergence, because $\mathbb{E}S_T < \infty$. On the right-hand side,

$$\lim_{x\to\infty} C^{BS}(x, \sqrt{2|x|/T}) = B_0F_0[\Phi(0) - \lim_{x\to\infty} e^x \Phi(-\sqrt{2|x|})] = B_0F_0/2$$

by L'Hôpital's rule. This proves (3.4). □

Now we prove the explicit formula relating the right-hand tail slope to how many finite moments the underlying possesses.

Theorem 3.2 (The Moment Formula, part 1). *Let*

$$\tilde{p} := \sup\{p : \mathbb{E}S_T^{1+p} < \infty\} \qquad\qquad \beta_R := \limsup_{x\to\infty} \frac{I^2(x)}{|x|/T}.$$

Then $\beta_R \in [0,2]$ *and*

$$\tilde{p} = \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2},$$

where $1/0 := \infty$*. Equivalently,*

$$\beta_R = 2 - 4(\sqrt{\tilde{p}^2 + \tilde{p}} - \tilde{p}),$$

where the right-hand expression is to be read as zero, in the case $\tilde{p} = \infty$*.*

Proof. Lemma 3.1 implies $\beta_R \in [0,2]$. We need to show that $\tilde{p} = f_-(\beta_R)/2$.

For any $\beta \in (0,2)$, L'Hôpital's rule implies that

$$\lim_{x\to\infty} \frac{e^{-cx}}{C^{BS}(x, \sqrt{\beta|x|/T})} = \begin{cases} 0 & \text{for } c > f_-(\beta)/2 \\ \infty & \text{for } c \leqslant f_-(\beta)/2, \end{cases}$$

which we will use in both stages of the proof.

To prove $\tilde{p} \leqslant f_-(\beta_R)/2$, note that $f_- : (0,2) \xrightarrow{\text{onto}} (0,\infty)$ is strictly decreasing. So it suffices to show that for any $\beta \in (0,2)$ with $f_-(\beta)/2 < \tilde{p}$, we have $\beta_R \leqslant \beta$. Choose $p \in (f_-(\beta)/2, \tilde{p})$. By Corollary 2.2, as $x \to \infty$,

$$\frac{C^{BS}(x, I(x))}{C^{BS}(x, \sqrt{\beta|x|/T})} = \frac{O(e^{-px})}{C^{BS}(x, \sqrt{\beta|x|/T})} \longrightarrow 0.$$

The result follows from the monotonicity of C^{BS} in its second argument.

To prove $\tilde{p} \geqslant f_-(\beta_R)/2$, it suffices to show that for any $p \in (0, f_-(\beta_R)/2)$, we have $\mathbb{E}S_T^{1+p} < \infty$. Choose β such that $Q := f_-(\beta)/2 \in (p, f_-(\beta_R)/2)$. For x sufficiently large,

$$\frac{C(K(x))}{e^{-Qx}} \leqslant \frac{C^{BS}(x, \sqrt{\beta|x|/T})}{e^{-Qx}} \longrightarrow 0 \qquad \text{as } x \to \infty,$$

so there exists K_* such that for all $K > K_*$, we have $C(K) < K^{-Q}$. Then, as claimed,

$$\begin{aligned}\mathbb{E}S_T^{1+p} &= \mathbb{E}\int_0^\infty (p+1)pK^{p-1}(S_T - K)^+ dK \\ &\leqslant p(p+1)B_0^{-1}\left[\int_0^{K_*} K^{p-1}C(K)dK + \int_{K_*}^\infty K^{p-Q-1}dK\right] < \infty,\end{aligned}$$

where the first step uses a mixture of calls to span the twice-differentiable payoff S^{1+p}; see the appendix of Carr-Madan (1998). □

3.2 The Small-Strike Tail

Consider the left-hand (or small-K or negative-x or OTM-put) tail of the square of implied volatility. First we show that this tail slope is no larger than 2.

Lemma 3.3. *For any $\beta > 2$ there exists x^* such that for all $x < x^*$,*

$$I(x) < \sqrt{\beta|x|/T}.$$

For $\beta = 2$, the same conclusion holds, if and only if *S_T satisfies* $\mathbb{P}(S_T = 0) < 1/2$.

Proof. For case $\beta > 2$ and the "if" part of case $\beta = 2$: There exists x^* such that for all $x < x^*$,

$$\mathbb{P}(S_T < F_0e^x) < \Phi(-\sqrt{f_-(\beta)|x|}) - e^{-x}\Phi(-\sqrt{f_+(\beta)|x|})$$

because as $x \to -\infty$, the left-hand side approaches $\mathbb{P}(S_T = 0)$, while the right-hand side approaches either 1 (in case $\beta > 2$) or $1/2$ (in case $\beta = 2$). So

$$P^{BS}(x, I(x)) = B_0\mathbb{E}(K(x) - S_T)^+ \leqslant B_0K(x)\mathbb{P}(S_T < F_0e^x) < P^{BS}(x, \sqrt{\beta|x|/T})$$

for all $x < x^*$. The result follows from strict monotonicity of P^{BS} in its second argument.

For the "only if" part of case $\beta = 2$: By monotonicity of P^{BS}, we have

$$B_0K(x)/2 > P^{BS}(x, \sqrt{2|x|/T}) > B_0\mathbb{E}(K(x) - S_T)^+ \geqslant B_0K(x)\mathbb{P}(S_T = 0)$$

for arbitrary $x > x^*$. Divide by $B_0K(x)$ to obtain the result. □

Now we prove the explicit formula relating the left-hand tail slope to how many finite moments the underlying's reciprocal possesses.

Theorem 3.4 (The Moment Formula, part 2). *Let*

$$\tilde{q} := \sup\{q : \mathbb{E}S_T^{-q} < \infty\} \qquad \beta_L := \limsup_{x\to-\infty} \frac{I^2(x)}{|x|/T}.$$

Then $\beta_L \in [0,2]$ *and*

$$\tilde{q} = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2},$$

where $1/0 := \infty$*. Equivalently,*

$$\beta_L = 2 - 4(\sqrt{\tilde{q}^2 + \tilde{q}} - \tilde{q}),$$

where the right-hand expression is to be read as zero, in the case $\tilde{q} = \infty$*.*

Proof. Lemma 3.3 implies $\beta_L \in [0,2]$. We need to show that $\tilde{q} = f_-(\beta_L)/2$.

For any $\beta \in (0,2)$, L'Hôpital's rule implies that

$$\lim_{x\to-\infty} \frac{e^{(1+c)x}}{P^{BS}(x,\sqrt{\beta|x|/T})} = \begin{cases} 0 & \text{for } c > f_-(\beta_L)/2 \\ \infty & \text{for } c \leqslant f_-(\beta_L)/2, \end{cases}$$

which we will use in both stages of the proof.

To prove $\tilde{q} \leqslant f_-(\beta_L)/2$, note that $f_- : (0,2) \xrightarrow{\text{onto}} (0,\infty)$ is strictly decreasing. So it suffices to show that for any $\beta \in (0,2)$ with $f_-(\beta)/2 < \tilde{q}$, we have $\beta_L \leqslant \beta$. Choose $q \in (f_-(\beta)/2, \tilde{q})$. By Corollary 2.2, as $x \to -\infty$,

$$\frac{P^{BS}(x,I(x))}{P^{BS}(x,\sqrt{\beta|x|/T})} = \frac{O(e^{(1+q)x})}{P^{BS}(x,\sqrt{\beta|x|/T})} \longrightarrow 0.$$

The result follows from the monotonicity of P^{BS} in its second argument.

To prove $\tilde{q} \geqslant f_-(\beta_L)/2$, it suffices to show that for any $q \in (0, f_-(\beta_L)/2)$, we have $\mathbb{E}S_T^{-q} < \infty$. Choose β such that $Q := f_-(\beta)/2 \in (q, f_-(\beta_L)/2)$. For $|x|$ sufficiently large,

$$\frac{P(K(x))}{e^{(1+Q)x}} \leqslant \frac{P^{BS}(x,\sqrt{\beta|x|/T})}{e^{(1+Q)x}} \longrightarrow 0 \qquad \text{as } x \to -\infty,$$

so there exists K_* such that for all $K < K_*$, we have $P(K) < K^{1+Q}$. Then, as claimed,

$$\begin{aligned} \mathbb{E}S_T^{-q} &= \mathbb{E}\int_0^\infty -q(-q-1)K^{-q-2}(K-S_T)^+ dK \\ &\leqslant q(q+1)B_0^{-1}\left[\int_0^{K_*} K^{Q-q-1}dK + \int_{K_*}^\infty K^{-q-2}P(K)dK\right] < \infty, \end{aligned}$$

where the first step uses a mixture of puts to span the twice-differentiable payoff S^{-q}; see the appendix of Carr-Madan (1998). □

Remark 3.1. The proofs of Theorems 3.2 and 3.4 make rigorous the following idea. By the Black-Scholes formula, the tail behavior of the implied volatility skew carries the same information as the tail behavior of option prices. In turn, the tail behavior of option prices carries the same information as the number of finite moments – intuitively, option prices are bounded by moments, according to Theorem 2.1; on the other hand, moments are bounded by option prices, because power payoffs are mixtures, across a continuum of strikes, of call or put payoffs.

Remark 3.2. In a wide class of specifications for the state dynamics, as discussed in Remark 2.4, the maximal moment exponents $\tilde{p}$ and $\tilde{q}$ are readily computable functions of the model's parameters. Examples include Lévy processes and affine jump-diffusion processes popular in financial modelling.

4 The Symmetry of the Small-Strike and Large-Strike Formulas

Section 3.2 gave a stand-alone proof of the small-strike moment formula. An alternative approach is to deduce the small-strike formula from the large-strike formula, proved in Section 3.1, together with a symmetry argument, which will be proved in this section. The symmetry argument explains, moreover, why the definition of $\tilde{q}$ is "missing a factor of S_T" compared to $\tilde{p}$. We give this alternative proof in three steps.

First we exhibit the symmetry.

Theorem 4.1. *Assume $\mathbb{P}(S_T = 0) = 0$. Define the probability measure $\mathbb{S}$ by the likelihood ratio $d\mathbb{S}/d\mathbb{P} = S_T/F_0$. Let $\mathbb{E}_\mathbb{S}$ denote expectation with respect to $\mathbb{S}$. Then for each moneyness x,*

$$\mathbb{E}_\mathbb{S}(S_T^{-1} - K^{-1})^+ = \mathbb{E}_\mathbb{S}(S_T^{-1})\Phi\left(\frac{\log(\mathbb{E}_\mathbb{S}(S_T^{-1})/K^{-1})}{I\sqrt{T}} + \frac{I\sqrt{T}}{2}\right) - K^{-1}\Phi\left(\frac{\log(\mathbb{E}_\mathbb{S}(S_T^{-1})/K^{-1})}{I\sqrt{T}} - \frac{I\sqrt{T}}{2}\right),$$

where $K := K(x)$ and $I := I(x)$ are defined as in (3.1)–(3.3).

Proof. By the definition (3.3) of I, and then the definition of $\mathbb{S}$, we have

$$B_0\left[K\Phi\left(\frac{\log(K/F_0)}{I\sqrt{T}} + \frac{I\sqrt{T}}{2}\right) - F_0\Phi\left(\frac{\log(K/F_0)}{I\sqrt{T}} - \frac{I\sqrt{T}}{2}\right)\right]$$
$$= B_0\mathbb{E}(K - S_T)^+ = B_0K\mathbb{E}S_T(S_T^{-1} - K^{-1})^+ = B_0F_0K\mathbb{E}_\mathbb{S}(S_T^{-1} - K^{-1})^+.$$

Dividing both sides by B_0F_0K shows that

$$\mathbb{E}_\mathbb{S}(S_T^{-1} - K^{-1})^+ = \frac{1}{F_0}\Phi\left(\frac{\log(F_0^{-1}/K^{-1})}{I\sqrt{T}} + \frac{I\sqrt{T}}{2}\right) - \frac{1}{K}\Phi\left(\frac{\log(F_0^{-1}/K^{-1})}{I\sqrt{T}} - \frac{I\sqrt{T}}{2}\right).$$

Since $\mathbb{E}_\mathbb{S}(S_T^{-1}) = F_0^{-1}$, this completes the proof. □

Remark 4.1. This proves that a Dollar-denominated K-strike option on the Euro has the same implied volatility as a Euro-denominated $1/K$-strike option on the Dollar.

Remark 4.2. Depending on the context, $\mathbb{S}$ is sometimes called "foreign" risk-neutral measure or "share" measure; it is the appropriate pricing measure if one takes as numeraire a claim on S_T. See El Karoui, Geman, Rochet (1995).

Now we restate Theorem 3.2 in a way which makes explicit the role of the probability measure.

Theorem 4.2. *Let $T > 0$. Let U be a nonnegative random variable with finite positive expectation $\mathbb{E}_{\mathbb{Q}}U$ with respect to some probability measure $\mathbb{Q}$. For each $z \in \mathbb{R}$, let $k(z) := (\mathbb{E}_{\mathbb{Q}}U)e^z$, and assume there exists a unique $I_{\mathbb{Q}}(z)$ satisfying*

$$\mathbb{E}_{\mathbb{Q}}(U - k(z))^+ = (\mathbb{E}_{\mathbb{Q}}U)\Phi\left(\frac{-z}{I_{\mathbb{Q}}(z)\sqrt{T}} + \frac{I_{\mathbb{Q}}(z)\sqrt{T}}{2}\right) - k(z)\Phi\left(\frac{-z}{I_{\mathbb{Q}}(z)\sqrt{T}} - \frac{I_{\mathbb{Q}}(z)\sqrt{T}}{2}\right).$$

Then the moment formula holds for large z. Specifically, with $\tilde{p}_{\mathbb{Q}} := \sup\{p : \mathbb{E}_{\mathbb{Q}}U^{1+p} < \infty\}$,

$$\limsup_{z\to\infty} \frac{I^2_{\mathbb{Q}}(z)}{|z|/T} = 2 - 4\left(\sqrt{\tilde{p}^2_{\mathbb{Q}} + \tilde{p}_{\mathbb{Q}}} - \tilde{p}_{\mathbb{Q}}\right),$$

where the right-hand expression is to be read as zero, in the case $\tilde{p}_{\mathbb{Q}} = \infty$.

Proof. This just restates what was already established by the proof of Theorem 3.2. □

Finally, we combine these two results to produce the alternative proof of Theorem 3.4.

Theorem 4.3. *The small-strike moment formula holds. Specifically, with $\tilde{q} := \sup\{q : \mathbb{E}S_T^{-q} < \infty\}$,*

$$\beta_L := \limsup_{x\to-\infty} \frac{I^2(x)}{|x|/T} = 2 - 4(\sqrt{\tilde{q}^2 + \tilde{q}} - \tilde{q}),$$

where the right-hand expression is to be read as zero, in the case $\tilde{q} = \infty$.

Proof. If $\mathbb{P}(S_T = 0) = 0$, then let $\mathbb{Q} := \mathbb{S}$ and $U := S_T^{-1}$. We have $\tilde{p}_{\mathbb{Q}} = \tilde{q}$, because for each p,

$$\mathbb{E}_{\mathbb{Q}}U^{1+p} = \mathbb{E}_{\mathbb{S}}S_T^{-1-p} = \frac{1}{F_0}\mathbb{E}S_T^{-p}.$$

By Theorem 4.1, taking $I_{\mathbb{Q}}(z) := I(-z)$ satisfies the hypotheses of Theorem 4.2. Hence, as claimed,

$$\limsup_{x\to-\infty} \frac{I^2(x)}{|x|/T} = \limsup_{z\to\infty} \frac{I^2_{\mathbb{Q}}(z)}{|z|/T} = 2 - 4\left(\sqrt{\tilde{p}^2_{\mathbb{Q}} + \tilde{p}_{\mathbb{Q}}} - \tilde{p}_{\mathbb{Q}}\right) = 2 - 4(\sqrt{\tilde{q}^2 + \tilde{q}} - \tilde{q}).$$

If $\mathbb{P}(S_T = 0) > 0$, then $\tilde{q} = 0$, so we need to prove $\beta_L = 2$. Lemma 3.1 implies $\beta_L \leqslant 2$. To show that $\beta_L \geqslant 2$, note that for any $\beta < 2$, there exists x^* such that for all $x < x^*$,

$$\begin{aligned} P^{BS}(x, I(x)) &= B_0\mathbb{E}(K(x) - S_T)^+ \geqslant B_0K(x)\mathbb{P}(S_T = 0) \\ &> B_0K(x)[\Phi(-\sqrt{f_-(\beta)|x|}) - e^{-x}\Phi(-\sqrt{f_+(\beta)|x|})] = P^{BS}(x, \sqrt{\beta|x|/T}), \end{aligned}$$

because the second line approaches 0 as $x \to -\infty$. By monotonicity of P^{BS}, we are done. □

Remark 4.3. Therefore the "factor of S_T missing from $\tilde{q}$" compared to $\tilde{p}$ can be explained as follows: It was consumed in the measure change from "domestic" measure $\mathbb{P}$ to "foreign" measure $\mathbb{S}$.

5 Applications

Applications of the moment formula include skew extrapolation and model calibration.

5.1 Skew extrapolation

By the $O(|x|^{1/2})$ bound, the linear or convex skews sometimes observed in near-the-money implied volatility cannot persist into the away-from-the-money tails. Likewise, any approximation of near-the-money implied volatility as linear or quadratic in x (such as in Fouque-Papanicolaou-Sircar (2000)) may be accurate in its intended domain, but must fail for K sufficiently large or small. So, when using splines to parametrically extrapolate volatility skews beyond the actively traded strikes, we do *not* recommend functional forms which allow either tail to grow *faster* than $|x|^{1/2}$. Moreover, unless the underlying has finite moments of all orders, we do *not* recommend functional forms which allow either tail to grow *slower* than $|x|^{1/2}$, because the moment formula rejects such functions.

5.2 Model calibration

The moment formula facilitates the calibration of model parameters to observed volatility skews. By observing the tail slopes of the skew, and applying the moment formula, one obtains $\tilde{p}$ and $\tilde{q}$. Combined with analysis of the characteristic function, as discussed in Remarks 2.4 and 3.2, this produces two identifying restrictions on the model's parameters. Indeed, in models such as the examples below, the $\tilde{p}$ and $\tilde{q}$ values *determine* two of the parameters. We do not propose that the moment formula alone should replace a full optimization algorithm, but it can facilitate the process by providing justifiable initial "guesses" for some or all of the parameters.

Example 5.1. In the double-exponential jump-diffusion model of Kou (2002), the asset price follows a geometric Brownian motion between jumps, which occur at event times of a Poisson process. The sizes of the up-jumps and down-jumps in returns are exponentially distributed with the parameters η_1 and η_2 respectively, and hence the means $1/\eta_1$ and $1/\eta_2$ respectively. Using the explicitly known characteristic function, one finds that

$$\tilde{q} = \eta_2 \qquad \tilde{p} = \eta_1 - 1. \tag{5.1}$$

So η_1 and η_2 can be inferred from $\tilde{p}$ and $\tilde{q}$, and hence (via the moment formula) from the tail slopes β_L and β_R of the volatility skew, which are in principle observable.

The intuition of (5.1) is as follows: the larger the expected size of an up-jump, the fatter the S_T distribution's right-hand tail, and the fewer the finite moments of positive order; similar intuition relates down-jumps and moments of negative order. Note that the jump *frequency* parameters have no effect on the tail slopes, by (5.1) and the moment formula.

Example 5.2. In the normal inverse Gaussian model of Barndorff-Nielsen (1998), returns have the NIG distribution, which can be described as follows. Consider two dimensional Brownian motion starting at $(a,0)$, with drift vector (b,c) where $c > 0$. The NIG(a,b,c,d) law is the distribution of the first coordinate of the Brownian motion at the stopping time when the second coordinate hits a barrier $d > 0$. Using the explicitly known characteristic function, one finds that

$$\tilde{q} = \sqrt{b^2+c^2}+b \qquad \tilde{p} = \sqrt{b^2+c^2}-b-1. \tag{5.2}$$

So b and c can be inferred from $\tilde{p}$ and $\tilde{q}$, and hence (via the moment formula) from the tail slopes β_L and β_R.

The intuition of (5.2) is as follows: increasing the c brings earlier stopping, hence thinner tails and more moments (of both positive and negative order) in the distribution of S_T; increasing the drift b fattens the right-hand tail and thins the left-hand tail, decreasing the number of positive-order moments and increasing the number of negative-order moments. Note that the parameters a and d have no effect on the tail slopes, by (5.2) and the moment formula.

6 Conclusions

Lemmas 3.1 and 3.3 give an $O(|x|^{1/2})$ bound on the tails of the volatility skew. Then the *moment formula*, established in Theorems 3.2 and 3.4, makes precise how small the constant in that bound can be chosen for each T: in one tail, it depends only on $\sup\{q : \mathbb{E}S_T^{-q} < \infty\}$; in the other, it depends only on $\sup\{p : \mathbb{E}S_T^{1+p} < \infty\}$. This fundamental linkage between moments and tail slopes is model-independent: it assumes only that a martingale measure exists, and that $0 < \mathbb{E}S_T < \infty$, a condition already implicit in the concepts of moneyness and implied volatility. Then Theorems 4.1-4.3 show that a "domestic/foreign" symmetry relates the large-strike and the small-strike formulas.

The moment formula has implications for skew extrapolation: it rejects functions that grow faster than $|x|^{1/2}$, and unless S_T has finite moments of all orders, it rejects those that grow slower than $|x|^{1/2}$. The moment formula also has application to model calibration: given a tractable characteristic function and sufficient options data, it relates explicitly the observable tail slopes to known functions of the model's parameters, yielding two identifying restrictions on those parameters; moreover, in models such as Examples 5.1 and 5.2, these restrictions indeed determine two of the parameters, with minimal computational effort.

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