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# Asymptotics of implied volatility to arbitrary order

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**Abstract** In a unified model-free framework that includes long-expiry, short-expiry, extreme-strike, and jointly-varying strike-expiry regimes, we generate implied volatility and implied variance approximations, with rigorous error estimates asymptotically smaller than any given power of $L$, where $L$ denotes the exogenously given absolute log of an option price that approaches zero. Our results, therefore, sharpen to *arbitrarily* high order of accuracy (and, moreover, extend to general extreme regimes) the model-free asymptotics of implied volatility. We then apply these general formulas to particular examples: Heston (using a previously known $L$ expansion) and Lévy (using saddlepoint methods to derive $L$ expansions).

**Keywords** Implied volatility · Asymptotics

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**JEL Classification** G13

## 1 Introduction

Asymptotic approximations of implied volatility serve several purposes. First, they reveal information contained in implied volatility observations, in the following sense: explicit formulas for a given model can connect, on the one hand, information about the model's parameters, and, on the other hand, key features

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(such as level/slope/convexity with respect to strike/expiry) of the implied volatility skew/smile. This leads to an understanding of which specific parameters influence which specific smile features, and it facilitates numerical calibration of those parameters to implied volatility data. Second, asymptotic formulas provide guidance for extrapolating or interpolating implied volatility to unobserved strikes and expiries, by suggesting the proper functional forms to use in parameterizing the volatility skew. Third, asymptotic formulas can serve as checks on, or as substitutes for, numerical integration or discrete Fourier schemes in regimes, such as deep-out-of-the-money, where naively implemented discretizations are prone to numerical error.

Pursuant to these background motivations (and complementary to previous works on asymptotic regimes of SDE parameters, such as [7] or [11]), a growing body of research explores asymptotic regimes of strikes and expiries: a typical result focuses on either long expiries, or short expiries, or extreme strikes. Taking a broader view in this paper, we exploit the similarities of extreme-strike and extreme-expiry asymptotics, to introduce a general framework that *unifies* all three extreme strike/expiry regimes, together with variants in which strike and expiry vary jointly.

Our approach encompasses not only general asymptotic regimes, but also general models. Our primary results express the implied volatility $V$ in a *model-free* way, not in terms of the parameters of a particular process, but rather in terms of $L$, the absolute log of the option price, and $k$, the log strike. This type of model-independent formula has precedents in the literature; the leading examples in each regime are as follows. Deferring precise definitions until the body of this paper, let us write $L_-$ and $L_+$ for the absolute logs of the prices of, respectively, an out-of-the-money call and a covered-call position (long one share, short one call). Then the following asymptotics are known: For short expiries with constant strike, Roper and Rutkowski [18] show that

$$V^2 \sim \frac{k^2}{2L_-}. \tag{1.1}$$

For long expiries, Tehranchi [20] shows that

$$V^2 = 8L_+ - 4\log L_+ + 4k - 4\log\pi + o(1). \tag{1.2}$$

For large strikes with constant expiry, Gulisashvili [9] shows that

$$V = G_-\Big(k, L_- \frac{1}{2}\log L_-\Big) + O\big(L_-^{-1/2}\big), \tag{1.3}$$

where

$$G_-(\kappa, u) := \sqrt{2}\big(\sqrt{u+\kappa} - \sqrt{u}\big),$$

and that (1.3) implies other model-free results, including the moment formula (Lee [13]) and tail-wing formula (Benaim and Friz [2]).

We sharpen all of the above formulas to *arbitrarily high* order of accuracy, in the following sense: We generate, for any given $J > 0$, implied volatility and implied variance formulas with rigorous error estimates of the type $O(1/L^J)$, where $L \to \infty$. Low-order special cases of our main theorem suffice to refine the previously known results. In particular, Corollary 6.1 sharpens (1.3), Corollary 7.2 sharpens (1.1), and Corollary 7.11 sharpens (1.2). Moreover, in each case, our extension relaxes previously imposed regime assumptions, such as bounded strikes or constant expiry.

In applications, our error estimates have the form $O(f(\theta))$ as $\theta \to \infty$, where $f$ is some specified function, and $\theta$ parameterizes the strikes and/or expiries of the option contracts. We do not estimate the best constant $\alpha$ such that $\alpha f(\theta)$ bounds the error for sufficiently large $\theta$, nor do we estimate how large a $\theta$ is sufficiently large. For those reasons, computing the numerical value of $f(\theta)$ for a particular contract would not approximate the numerical size of the error in our implied volatility formula; rather, the $f(\theta)$ indicates the rate (modulo a multiplicative constant) at which a bound on that error eventually approaches zero as strikes/expiries approach an extreme.

In Sect. 8 we apply our general results to specific models. For such applications, it is necessary to either generate an approximation of $L$, or else use a known approximation. We do generate $L$ approximations in Sect. 8.3, but that is not this paper's primary purpose. Rather, we focus mainly on converting exogenously given $L$ approximations into approximations of implied volatility $V$; or in other words, inverting the Black–Scholes formula asymptotically in general extreme regimes, with careful error estimates, *given* (approximate or exact) option prices in general models.

Consider for example the Heston model at large strikes, Lévy models at short expiries, and Lévy models at long expiries. In all three cases, there exist asymptotic expansions (according to, respectively, Friz et al. [8], Figueroa-Lopez and Forde [5], and our Lemma 8.5), which approximate $L$ in terms of the model's parameters. Inserting these $L$ approximations into our main theorem's corollaries, we obtain explicit parametric implied volatility formulas, again with careful error estimates showing that we sharpen the sharpest previously known implied volatility formulas for those models. In particular, Corollary 8.1 sharpens Friz et al. [8] in the Heston case, Corollary 8.3 sharpens Figueroa-Lopez and Forde [5] (hence Tankov [19] and Roper [17]) in the short-dated Lévy case, and Corollary 8.6 sharpens Tehranchi [20] in the long-dated Lévy case.

In a fourth application, distinct from the above fixed-strike or fixed-expiry regimes, we let strike and expiry grow jointly. We derive saddlepoint approximations for Lévy-driven option prices (and hence for $L$), which our implied volatility asymptotics then map into the smile formulas of Corollary 8.7. Numerical experiments show the resulting approximations' remarkable accuracy across a wide range of strikes and expiries.

## 2 Preliminaries

This section collects some definitions and notation.

For any differentiable function $f$, let $Df$ denote the partial derivative of $f$ with respect to its *last* argument; in particular, if $f$ is a function of a single variable, then $Df$ denotes its derivative. Likewise, for an $n$ times differentiable $f$, let $D^n f$ denote the $n$th partial derivative of $f$ with respect to its *last* argument.

Degenerate sums $\sum_1^0$ are understood to mean 0.

### 2.1 Asymptotics

For a function $a$ and a nonvanishing function $b$ on an interval $(\theta_0, \infty)$, write $a = o(b)$ if $a(\theta)/b(\theta) \to 0$ as $\theta \to \infty$, and $a = O(b)$ if $\limsup_{\theta\to\infty} |a(\theta)/b(\theta)| < \infty$. We write $a \asymp b$ if $a = O(b)$ and $b = O(a)$. We write $a \sim b$ if $a(\theta)/b(\theta) \to 1$ as $\theta \to \infty$.

### 2.2 Auxiliary functions

For $x \in [0, \infty)$, define

$$\begin{aligned} R_n^\pm(x) &:= 1 \pm \frac{1}{(1+x)^{n-1/2}}, \quad n \ge 1, \\ R_0^\pm(x) &:= 1, \\ A_n^\pm(x) &:= (-1)^n \frac{(2n-1)!!}{2^n} \frac{R_{n+1}^\pm(x)}{R_1^\pm(x)}, \quad n \ge 0, \end{aligned}$$

where $(-1)!! := 1$ and

$$(2n-1)!! := (2n-1)(2n-3)\cdots 3 \times 1, \quad n \ge 1.$$

For each $n \ge 1$, define the function $f_n : \mathbb{R}^n \to \mathbb{R}$ to satisfy, for all $N > 1$, and all real sequences $a_1, a_2, \ldots$,

$$\log\left(1 + \sum_{n=1}^{N-1} a_n \epsilon^n + O(\epsilon^N)\right) = \sum_{n=1}^{N-1} f_n(a_1, \ldots, a_n)\epsilon^n + O(\epsilon^N) \tag{2.1}$$

as $\epsilon \to 0$. The first three $f_n$ are $f_1(a_1) := a_1$ and $f_2(a_1, a_2) := a_2 - a_1^2/2$ and $f_3(a_1, a_2, a_3) := a_1^3/3 - a_1 a_2 + a_3$. Let

$$B_n^\pm(x) := f_n\big(A_1^\pm(x), \ldots, A_n^\pm(x)\big).$$

Each $f_n$ is a polynomial (expressible in terms of *Bell* polynomials, which we omit for brevity).

## 3 Option pricing formulas

All prices are understood to be denominated relative to a (possibly nonzero interest rate bearing) bank account, whose initial balance equals the initial underlying stock price. In effect, this normalizes the underlying to 1 initially, and interest rates to 0.

The relation between call price (in *any* arbitrage-free setting, not necessarily the Black–Scholes model) and implied volatility is by definition given by the Black–Scholes [3] formula. Specifically, let

$$F(\kappa, v) := \frac{\kappa^2}{2v^2} - \frac{\kappa}{2} + \frac{v^2}{8},$$

and define $C_- : [0, \infty) \times (0, \infty) \to (0, \infty)$ by

$$C_-(\kappa, v) := \mathcal{N}(-\kappa/v + v/2) - e^\kappa \mathcal{N}(-\kappa/v - v/2) = \frac{1}{\sqrt{2\pi}} \int_0^v e^{-F(\kappa,w)}\,\mathrm{d}w, \quad (3.1)$$

where the right-hand formula expresses $C_-$ as the integral of its $v$-derivative, and $\mathcal{N}$ denotes the standard normal CDF. Thus $C_-$ expresses the call price as a function of the log strike price (or "moneyness") $\kappa \geq 0$ and the dimensionless ("unannualized") implied volatility $v > 0$. Dividing $v$ by the square root of time to expiry would produce the usual annualized implied volatility. Define $C_+ : [0, \infty) \times (0, \infty) \to (0, \infty)$ by

$$C_+(\kappa, v) := 1 - C_-(\kappa, v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-F(\kappa,w)}\,\mathrm{d}w, \quad (3.2)$$

which expresses a *covered-call* (long underlying, short call) combination's price as a function of log strike $\kappa \geq 0$ and dimensionless implied volatility $v > 0$. Differentiating (3.1) and (3.2) in $v$, we have

$$DC_\pm(\kappa, v) = \mp \frac{e^{-F(\kappa,v)}}{\sqrt{2\pi}}. \quad (3.3)$$

Define $G_\pm : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$G_\pm(\kappa, u) := \sqrt{2}\big(\sqrt{u+\kappa} \pm \sqrt{u}\big).$$

One can verify that $G_\pm$ are inverses of $F$ in the sense that

$$\begin{aligned} v &= G_-\big(\kappa, F(\kappa, v)\big) \quad \text{if } 2\kappa > v^2 > 0, \\ v &= G_+\big(\kappa, F(\kappa, v)\big) \quad \text{if } 0 < 2\kappa < v^2, \end{aligned} \quad (3.4)$$

and

$$u = F\big(\kappa, G_\pm(\kappa, u)\big) \quad \text{if } \kappa, u > 0. \quad (3.5)$$

Recalling that $D^n G_\pm$ denotes the $n$th partial derivative of $G_\pm$ in its second argument, we have for $n \geq 1$ that

$$D^n G_\pm(\kappa, u) = \pm(-1)^{n-1} \frac{(2n-3)!!}{(2u)^{n-1/2}} R_n^\pm(\kappa/u) = DG_\pm(\kappa, u) \frac{A_{n-1}^\pm(\kappa/u)}{u^{n-1}}, \quad (3.6)$$

which is useful in the following $N$-term approximate formula for $C_\pm$, with error bound.

**Lemma 3.1** (Option price expansion) *If* $\kappa > 0$ *and* $v > 0$ *satisfy*

$$\pm(v^2/2 - \kappa) > 0, \quad (3.7)$$

*then, for any* $N \geq 1$, *we have* (*with each* $\pm$ *in accordance with* (3.7)) *that*

$$C_{\pm}(\kappa, v) = \pm\Big(\frac{e^{-F(\kappa,v)}}{\sqrt{2\pi}} \sum_{n=1}^{N} D^n G_{\pm}\big(\kappa, F(\kappa, v)\big) + \frac{1}{\sqrt{2\pi}} \int_{F(\kappa,v)}^{\infty} e^{-u} D^{N+1} G_{\pm}(\kappa, u)\, \mathrm{d}u\Big), \tag{3.8}$$

*where the remainder term satisfies*

$$\left|\frac{1}{\sqrt{2\pi}} \int_{F(\kappa,v)}^{\infty} e^{-u} D^{N+1} G_{\pm}(\kappa, u)\, \mathrm{d}u\right| \leq \frac{(2N-1)!!}{\sqrt{2\pi}} \frac{e^{-F(\kappa,v)} R^{\pm}_{N+1}(\kappa/F(\kappa, v))}{(2F(\kappa, v))^{N+1/2}}. \tag{3.9}$$

## 4 Asymptotic regimes

Henceforth let the log strike and implied volatility $(\kappa, v)$ vary in $[0, \infty) \times (0, \infty)$, along a *path* $(k(\theta), V(\theta))$ parameterized by $\theta \geq 0$.

Unless otherwise stated, all lim, lim sup, lim inf, and asymptotic relations are as $\theta \to \infty$. The word *eventually* preceding a statement means that there exists $\theta_0$ such that the statement holds for all $\theta > \theta_0$.

Again, $k$ and $V$ always denote functions of $\theta$. In order to avoid introducing new notation for $C_{\pm}$, $F$, $D^n G_{\pm}$ regarded as functions of $\theta$, let us agree that expressions in the context of a $\theta$-quantifier (such as "for all $\theta$" or "as $\theta \to \infty$" or "eventually") should be read as functions of $\theta$. For example, in such contexts, $F$ or $F(k, V)$ will be understood to mean $F(k(\theta), V(\theta))$. Likewise, $C_{\pm}$ or $C_{\pm}(k, V)$ will be understood to mean $C_{\pm}(k(\theta), V(\theta))$.

### 4.1 The + and − asymptotic regimes

Unless otherwise specified, assume that eventually $(k, V) \in (0, \infty) \times (0, \infty)$. Assume that either

$$\text{Case } (+): \quad C_+ \longrightarrow 0, \quad \text{or equivalently} \quad L_+ := \log(1/C_+) \longrightarrow \infty \tag{4.1}$$

or

$$\text{Case } (-): \quad C_- \longrightarrow 0, \quad \text{or equivalently} \quad L_- := \log(1/C_-) \longrightarrow \infty$$

as $\theta \to \infty$. In Case $(-)$, moreover, assume that

$$0 \vee \log(1/k) = o(L_-). \tag{4.2}$$

An equivalent formulation of condition (4.2) is that for some (equivalently: for every) constant $\epsilon > 0$, we have

$$\log(k \wedge \epsilon) = o(L_-). \tag{4.3}$$

A sufficient condition for (4.2) or equivalently (4.3) is that $\liminf k > 0$.

Although we have assumed $k$ eventually positive, the case of $k$ eventually negative follows from standard reflection arguments of the type in [13] or [9]. Remark 8.9 gives an application of this principle.

Unless otherwise stated, *in Sect. 5 through Sect. 7*, we assume the conditions of this Sect. 4.1; and all expressions involving $\pm$ are to be read as a pair of equations, one for Case $(+)$ and the other for Case $(-)$. In Sects. 4.2 and 8 (the "application" sections), however, this section's conditions are not a priori assumptions, but rather consequences of the setups in those particular sections.

The $(\pm)$ cases bifurcate the positive quadrant of the $(\kappa, v)$-plane, in the following sense.

**Lemma 4.1** (Path classification) *If we have Case* $(+)$, *then eventually*

$$0 < 2k < V^2.$$

*If we have Case* $(-)$, *then eventually*

$$0 < V^2 < 2k.$$

By (3.4), therefore, $F$ has inverse $G_\pm$ in the sense that

$$V = G_\pm\bigl(k, F(k, V)\bigr) \tag{4.4}$$

eventually, in Cases $(\pm)$, respectively.

## 4.2 Examples of asymptotic regimes

The hypotheses of Sect. 4.1 can be verified in typical applications.

*Example 4.2* (Large strikes) Let $T > 0$ be constant and $k(\theta) := \theta$. If there exist a probability measure and a nonnegative random variable $S_T$ such that $\mathbb{E}S_T = 1$ and $C_-(k, V) = \mathbb{E}(S_T - e^k)^+ > 0$ for all $\theta$, then by dominated convergence $C_- \to 0$, so we have Case $(-)$.

*Example 4.3* (Short expiries) Let $k > 0$ be constant and $T(\theta)$ some function such that $T \downarrow 0$ as $\theta \to \infty$. If there exist a probability measure and a random variable $S_T \geq 0$ such that $C_-(k, V) = \mathbb{E}(S_T - e^k)^+ > 0$ for all $\theta$ and $\mathbb{E}S_T = 1$, and the paths of $S$ are right-continuous a.s., then by dominated convergence $C_- \to 0$, so we have Case $(-)$.

*Example 4.4* (Long expiries) Let $k > 0$ be constant and $T(\theta)$ some function such that $T \uparrow \infty$ as $\theta \to \infty$. If there exist a probability measure and a supermartingale $S \geq 0$ (which therefore has an a.s. limit $S_\infty$) such that $C_+(k, V) = \mathbb{E}(S_T \wedge e^k) > 0$ for all $\theta$, and such that $\lim_{\theta\to\infty} \mathbb{E}(S_T \wedge e^k) \to 0$ (or equivalently, by dominated convergence, $S_\infty = 0$), then by the definition (4.1) we have Case $(+)$. This is essentially the setup of ([20], Sect. 2).

Jointly varying strike-expiry regimes could fall into either Case $(+)$ or $(-)$; see Sect. 7.2.

## 5 Asymptotic solution

### 5.1 Overview

Fixing an arbitrary path $(k, V)$ satisfying the Sect. 4.1 assumptions, we intend to extract $V$ explicitly from $C$ or from $L = \log(1/C)$. In the extreme regimes we have $V \to 0$ or $V \to \infty$, but $C(k, \cdot)$ is singular at $\infty$ and 0 (unless $k = 0$), so one cannot expect to have a solution purely in powers of $C$. Instead, our plan is to proceed from (3.8) and solve iteratively for $F$, thence $V$.

Section 5.1 gives the intuition of the approach and motivates the notation. The precise versions of this section's statements are in Sects. 5.2–5.5, which link to complete rigorous proofs. The outline is as follows.

*Step 1:* Approximate $F(k, V)$ using $\phi(k, L)$, where the function $\phi$ is constructed by applying (3.8) iteratively.

*Step 2:* Insert the $F$ approximation $\phi$ into the (4.4) relation $V = G(k, F)$, producing

$$V \approx G\big(k, \phi(k, L)\big). \tag{5.1}$$

Proposition 5.6 (FAT) analyzes the error in this approximation.

*Step 3:* Estimate the error in replacing the "input" $L$ by an approximation $\hat{L}$, in order to apply (5.1) in cases when the exact $L$ is unavailable. Accordingly, we write $\phi$ as a function of $(\kappa, \lambda)$, placeholders for (log strike, $-\log$ option price), where eventually $L$ or $\hat{L}$ is plugged into the $\lambda$ slot.

*Step 4:* Estimate the error in replacing the $G$ "output" by a series expansion of $G$, in order to simplify the formulas.

Here we give further details on the more difficult Steps 1 and 2.

The Step 1 expansion of $F$ is by taking logs in (3.8), expanding out to $N$ terms, and rearranging to get

$$F = L + h_N^*(k, F) + O(1/F^N), \tag{5.2}$$

for an explicit function $h_N^*$ involving an $N$-term sum, specified in (5.13), and derived from the explicit form of $D^n G_\pm$ for $n = 1, \dots, N-1$.

In order to solve approximately for $F$ in (5.2), first truncate the $O(1/F^N)$ remainder, and then iteratively apply (5.2) to approximate $F$ successively by $\varphi_{N,0}$, $\varphi_{N,1}$, $\varphi_{N,2}, \dots$, where $\varphi_{N,0} := L$ and

$$\varphi_{N,p+1} := L + h_N^*(k, \varphi_{N,p}), \quad p = 0, 1, 2, \dots \tag{5.3}$$

The idea is that each successive iterate improves, by a factor of $O(1/L)$, the iterative error (as distinct from the truncation error in (5.2), which is not improved by iteration). To see this intuitively, combine (5.2) and (5.3), to relate the error in iteration $p+1$ to the error in iteration $p$ via

$$\begin{aligned} |\varphi_{N,p+1} - F| &= |h_N^*(k, \varphi_{N,p}) - h_N^*(k, F)| + O(1/F^N) \\ &= O(1/L)|\varphi_{N,p} - F| + O(1/L^N), \end{aligned} \tag{5.4}$$

where the last expression comes from showing that

$$\varphi_{N,p} \sim L \sim F \tag{5.5}$$

for all $p$, and that the derivative of $h_N^*$ in its second argument satisfies, for all $\Lambda \sim L$,

$$Dh_N^*(k,\Lambda) = O(1/L). \tag{5.6}$$

By applying induction (or related arguments) to (5.4), doing $P$ iterations approximates $F$ with error

$$|F - \varphi_{N,P}| = O(1/L^N) + O(1/L^{P-1}),$$

and converting this in Step 2 from an $F$-estimation error into a $V$-estimation error gives the conclusion

$$|V - G(k,\varphi_{N,P})| = \frac{1}{L^{1/2}} O\Big(\frac{1}{L^N} + \frac{1}{L^{P-1}}\Big), \tag{5.7}$$

where the factor of $L^{-1/2}$ (which can be improved in the $(-)$ case) comes from estimating the derivative of $G_\pm$ in its second argument.

This basic version of our result has the drawback that (5.3), and hence $G(k,\varphi_{N,P})$, inherit the messiness of $h_N^*$. Revisiting Step 1, we should like to be able to replace $h_N^*$ with a simpler function $h$ (which will still depend on $N$). Moreover, we should like the freedom to modify each individual iterate $p = 1, 2, \ldots$ by adding some $\eta_p$ that does not depend on the previous iterate, but rather is chosen, in typical applications, to cancel out any messy minor terms that arise in the $p$th iterate. So let us generalize the iteration (5.3) to

$$\begin{aligned} \phi_{H,p+1}(\kappa,\lambda) &:= \lambda + h\big(\kappa, \phi_{H,p}(\kappa,\lambda)\big) + \eta_{p+1}(\kappa,\lambda), \\ \varphi_{H,p} &:= \phi_{H,p}\big(k, L(k,V)\big), \end{aligned} \tag{5.8}$$

which is no longer indexed just by $N$ but, more generally, by the *iteration scheme* $H := (h; \eta_1, \ldots, \eta_P)$.

In order to retain the error improvement of a factor of $O(1/L)$ with each iteration, we require the functions $h$ and each $\eta$ to satisfy (in place of $h_N^*$) the property (5.6), which we describe as a *sublog* condition because it stipulates that the function's derivative be asymptotically no larger than the log function's derivative.

Because the $\eta_p$ are allowed to differ from each other, and $h$ is allowed to differ from $h_N^*$, the $\varphi_H$ has additional error unaccounted in the (5.4) analysis of $\varphi_N$; so let us define a *residual* $\psi$ that depends on

$$|h - h_N^*| \quad \text{and on} \quad |\eta_{p+1} - \eta_p| \quad \text{for } p = 1, \ldots, P-1. \tag{5.9}$$

For a *regular* iteration scheme $H$, this residual $\psi$ is small in the sense of Definition 5.4; for such $H$ our Proposition 5.6 (FAT) proves that $\varphi_{H,P}$ can replace $\varphi_{N,P}$ in the conclusion (5.7), provided that the error estimate in (5.7) is modified by replacing the $O(1/L^{P-1})$ term by $O(\psi/L^P)$, which captures the error due to halting the iteration after the $P$th iterate, together with the error due to extending from (5.3) which uses $h_N^*$, to (5.8) which uses a more general (but in practice simpler) function $h+\eta_p$.

This closes the expository overview. The following sections give the precise statements and proofs.

### 5.2 Step 1: Approximate $F(k, V)$ using $\phi(k, L)$

We fix an arbitrary path $(k, V)$ satisfying the Sect. 4.1 assumptions.

Motivated by (5.6), we build the iteration scheme using functions, which grow like log or slower, in the following sense.

**Definition 5.1** (Sublog function) We say that $\eta : (0,\infty)\times(0,\infty)\to\mathbb{R}$ is a *sublog function* (in Case ($\pm$)) if $D\eta : (0,\infty)\times(0,\infty)\to\mathbb{R}$ exists, and if for all functions $\Lambda(\theta)$ such that $\Lambda\sim L_\pm$ as $\theta\to\infty$, we have

$$\eta(k,\Lambda)=o(L_\pm), \tag{5.10}$$

$$D\eta(k,\Lambda)=O(1/L_\pm), \tag{5.11}$$

as $\theta\to\infty$. Recall that $D\eta$ denotes the partial derivative of $\eta$ in its second argument.

Our main example of a sublog function is the following, motivated by (5.2).

**Lemma 5.2** *Define the function $h^*_{N,\pm}$, or more briefly $h^*_N$, by*

$$h_0^*(\kappa,\lambda) := -\frac{1}{2}\log\lambda, \tag{5.12}$$

$$h_N^*(\kappa,\lambda) := -\frac{1}{2}\log\lambda+\log\frac{1}{2\sqrt{\pi}}+\log R_1^\pm(\kappa/\lambda)+\sum_{n=1}^{N-1}\frac{B_n^\pm(\kappa/\lambda)}{\lambda^n}, \tag{5.13}$$

*for $N\geq 1$. Then $h_N^*$ is a sublog function for each $N\geq 0$.*

Motivated by (5.3) and (5.8), our iteration scheme can use $h_N^*$, or, more generally, the sum of a sublog function $h$ (applied recursively) and a sublog perturbation $\eta_p$ in the $p$th iterate (applied non-recursively):

**Definition 5.3** Let $P\geq 1$ and $H:=(h;\eta_1,\dots,\eta_P)$, where $h$ and each $\eta$ are sublog functions. Define

$$\phi_{H,0}(\kappa,\lambda):=\lambda.$$

For $p=0,\dots,P-1$, define the function $\phi_{H,p+1}$ recursively by

$$\phi_{H,p+1}(\kappa,\lambda):=\lambda+h\big(\kappa,\phi_{H,p}(\kappa,\lambda)\big)+\eta_{p+1}(\kappa,\lambda)$$

at all $(\kappa,\lambda)$ such that $\phi_{H,p}(\kappa,\lambda)>0$. For $p=0,\dots,P$, let

$$\varphi^\pm_{H,p}(\theta):=\phi_{H,p}\Big(k(\theta),L_\pm\big(k(\theta),V(\theta)\big)\Big)$$

for all $\theta$ large enough. An omitted second subscript is understood to be $P$. Thus $\varphi_H^\pm := \varphi_{H,P}^\pm$ and $\phi_H := \phi_{H,P}$.

Motivated by the discussion of (5.9), we define a *regular* iteration scheme to be one whose *residual* $\psi$ is not too large.

**Definition 5.4** (Regular iteration scheme) Let $N \geq 0$ and $P \geq 1$, and define $H := (h; \eta_1, \dots, \eta_P)$, where $h$ and each $\eta$ are sublog functions. Let

$$\eta_0 := -h, \qquad \eta^* := h_N^* - h. \tag{5.14}$$

If there exists a function $\psi(\theta)$ such that $\psi/L^P = O(1)$ and

$$\sum_{p=0}^{P-1} \frac{|\eta_{p+1}(k,L) - \eta_p(k,L)|}{L^{P-p}} + |\eta^*(k, \varphi_{H,P-1}) - \eta_P(k,L)| = O\Big(\frac{\psi}{L^P}\Big), \tag{5.15}$$

as $\theta \to \infty$, then we say that $H$ is a *regular iteration scheme (RISC),* or, more specifically, a *$P$-ply RISC with $N$-residual $\psi$.*

Residuals $\psi$ are not unique; for example, multiplying $\psi$ by a constant preserves (5.15).

*Example 5.5* For any $N \geq 0$ and $P \geq 1$, letting

$$h := h_N^* \quad \text{and} \quad \eta_1 := \eta_2 := \cdots := \eta_P := 0$$

produces a $P$-ply RISC, with $N$-residual $\psi = L$, because

$$\eta_1(k,L) - \eta_0(k,L) = O(L)$$

by (5.10).

### 5.3 Step 2: Approximate $V$ by inserting the $F$ approximation $\phi$ into $G$

To arbitrarily high order of accuracy, our main theorem approximates $V$ by a function of $k$ and $L$, obtained by consummating Steps 1 and 2 of the approach outlined in the Sect. 5.1 overview.

**Proposition 5.6** (Full asymptotic theorem, FAT) *Let $P \geq 1$ and $N \geq 0$. Suppose that $H := (h; \eta_1, \dots, \eta_P)$ is a $P$-ply RISC with $N$-residual $\psi$. Define $V_{H,\pm}(\theta)$ by*

$$V_{H,\pm} := G_\pm\big(k, \phi_H(k, L_\pm)\big). \tag{5.16}$$

*Then as $\theta \to \infty$,*

$$|V_{H,-} - V| = O\bigg(\frac{1 \wedge (k/L_-)^{N\wedge 1}}{L_-^{1/2}}\Big(\frac{\psi}{L_-^P} + \frac{1}{L_-^N}\Big)\bigg) \tag{5.17}$$

*and*

$$|V_{H,+} - V| = O\left(\frac{1}{L_+^{1/2}}\left(\frac{\psi}{L_+^P} + \frac{1}{L_+^N}\right)\right). \tag{5.18}$$

*In particular, accuracy to order arbitrarily high in powers of* $1/L$ *can be obtained by choosing* $N$ *and* $P$ *sufficiently large and* $h := h_N^*$ *and* $\eta_1 := \cdots := \eta_P := 0$.

Converting from implied volatility $V$ into *implied variance* $V^2$, we have the following.

**Corollary 5.7** (FAT for implied variance) *Let* $P \geq 1$ *and* $N \geq 0$. *Let* $H$ *be a* $P$*-ply RISC with* $N$*-residual* $\psi$. *Then as* $\theta \to \infty$,

$$|V_{H,-}^2 - V^2| = O\left(\left(\frac{k}{L_-} \wedge \frac{k^2}{L_-^2}\right)\left(\frac{\psi}{L_-^P} + \frac{1}{L_-^N}\right)\right), \quad N > 0,$$

$$|V_{H,-}^2 - V^2| = O\left(\frac{1+k}{L_-}\right), \quad N = 0,$$

$$|V_{H,+}^2 - V^2| = O\left(\left(1 + \frac{k^{1/2}}{L_+^{1/2}}\right)\left(\frac{\psi}{L_+^P} + \frac{1}{L_+^N}\right)\right), \quad N \geq 0.$$

### 5.4 Step 3: Replace the input of $G$ or $G^2$

In (5.16), the function $G$ is evaluated at $(k, \phi_H(k, L))$, but we may replace $L$ with some $\hat{L} \sim L$.

This completes Step 3 of the argument outlined in the Sect. 5.1 overview.

**Lemma 5.8** *Let* $H$ *be a RISC and let* $\hat{L} \sim L_\pm$. *Define*

$$\hat{V}_{H,\pm} = G_\pm\big(k, \phi_H(k, \hat{L}_\pm)\big).$$

*Then, as* $\theta \to \infty$, *we have* $\phi_H(k, \hat{L}_\pm) \sim L_\pm$; *and in Case* $(-)$,

$$|\hat{V}_{H,-} - V_H| = O\left(\frac{1 \wedge (k/L_-)}{L_-^{1/2}}|\hat{L} - L_-|\right),$$

$$|\hat{V}_{H,-}^2 - V_H^2| = O\left(\left(\frac{k}{L_-} \wedge \frac{k^2}{L_-^2}\right)|\hat{L} - L_-|\right),$$

*and in Case* $(+)$,

$$|\hat{V}_{H,+} - V_H| = O\left(\frac{1}{L_+^{1/2}}|\hat{L} - L_+|\right),$$

$$|\hat{V}_{H,+}^2 - V_H^2| = O\left(\left(1 + \frac{k^{1/2}}{L_+^{1/2}}\right)|\hat{L} - L_+|\right).$$

*Example 5.9* In particular, if we have some $\hat{C}_\pm \asymp C_\pm$ and $\hat{L}_\pm := \log(1/\hat{C}_\pm)$, then $L_\pm - \hat{L}_\pm = O(1)$ and $\hat{L}_\pm \sim L_\pm$, and the additional errors from Lemma 5.8 become $O((k \wedge L_-)/L_-^{3/2})$ and $O(1/L_+^{1/2})$. In the case $N = 0$, these additional errors are absorbed into the error estimates in (5.17), (5.18). This special case of Lemma 5.8 therefore extends, to general extreme regimes, an extreme-strike result of [9].

*Remark 5.10* Although our applications (Sect. 8) will work directly with option price asymptotics, our results above are also applicable given transition density asymptotics, because density asymptotics imply option price asymptotics, by results of the type in Gulisashvili (under regular variation conditions; see [9], Theorem 7.1), and therefore they yield asymptotic approximations $\hat{L}$ to $L$.

### 5.5 Step 4: Replace the output of $G$ or $G^2$

Replacing the output of $G^2$ by a truncated series simplifies the expression but produces an additional error term.

**Lemma 5.11** *Define* $a_m^\pm := 0$ *for even* $m \geq 0$, *and* $a_{-1}^\pm := 4 \pm 4$, *and*

$$a_m^\pm := \frac{\pm 2\pi}{\Gamma(1-m/2)\Gamma(-m/2)\Gamma(2+m)}, \quad \textit{for odd } m > 0.$$

*If* $k = O(L)$, *then, for any odd* $M \geq -1$ *and any* $\Lambda \sim L$, *as* $\theta \to \infty$,

$$\left| \sum_{m=-1}^{M} \frac{a_m^\pm k^{m+1}}{(\Lambda + k/2)^m} - G_\pm^2(k, \Lambda) \right| = O\left( \frac{k^{M+3}}{L^{M+2}} \right).$$

A similar expansion exists for $G$, with an error estimate by a similar proof, which we omit for brevity.

Alternatively, we may simply take the square root of a $V^2$ approximation, or square a $V$ approximation, to obtain an approximation of $V$ or $V^2$, respectively. The next lemma controls the resulting error.

**Lemma 5.12** *If some functions* $\hat{a}(\theta)$ *and* $a(\theta)$ *satisfy* $\hat{a} \sim a$ *as* $\theta \to \infty$, *then*

$$|\hat{a}^2 - a^2| = O\big(|\hat{a} - a|\hat{a}\big) \quad \textit{and} \quad |\hat{a} - a| = O\big(|\hat{a}^2 - a^2|/\hat{a}\big).$$

*Example 5.13* If $k = O(L)$, then by Lemmas 5.11 and 5.12,

$$G_-^2(k, \Lambda) = \frac{k^2}{2\Lambda + k} + O\left(\frac{k^4}{L^3}\right), \tag{5.19}$$

$$G_-(k, \Lambda) = \frac{k}{\sqrt{2\Lambda + k}} + O\left(\frac{k^3}{L^{5/2}}\right), \tag{5.20}$$

$$G_+^2(k, \Lambda) = 8\Lambda + 4k - \frac{k^2}{2\Lambda + k} + O\left(\frac{k^4}{L^3}\right) = 8\Lambda + 4k + O\left(\frac{k^2}{L}\right), \tag{5.21}$$

$$G_+(k,\Lambda) = \Big(8\Lambda + 4k - \frac{k^2}{2\Lambda+k}\Big)^{1/2} + O\Big(\frac{k^4}{L^{7/2}}\Big)$$
$$= (8\Lambda+4k)^{1/2} + O\Big(\frac{k^2}{L^{3/2}}\Big)$$

for any $\Lambda \sim L$.

## 6 Corollaries for general $(k, L)$

In Sects. 6 and 7, we generate corollaries specializing Proposition 5.6 in various directions. The statement of each corollary will be nonetheless self-contained, in the sense that the reader does not need to refer to Proposition 5.6. Each corollary's statement requires only the standing assumptions of Sects. 3 and 4. (The corollaries' *proofs* may refer to Proposition 5.6, but the additional hypotheses of Proposition 5.6 will be proved, not assumed.)

With no further assumptions on $(k, L)$, the FAT in Proposition 5.6 generates implied volatility formulas such as Corollaries 6.1 and 6.3. For simplifications in the case that $k/L$ converges, see Sect. 7.

### 6.1 Case $(-)$

This corollary is applicable in the fixed-expiry large-strike regime, the fixed-strike short-expiry regime, and some cases of jointly varying strike-expiry regimes. In particular, (6.1) applied at a fixed expiry recovers (1.3), an important theorem of Gulisashvili. We can extend to arbitrarily high order of accuracy in $L$ by keeping each $\eta = 0$ (or sufficiently small), but taking $N$ and $P$ sufficiently large. For example, the refinement (6.2) comes from incrementing $N$ to 1.

**Corollary 6.1** (Implied volatility formulas in Case $(-)$) *Write $L := L_-$. If Case $(-)$ holds, then, as $\theta \to \infty$, the dimensionless implied volatility $V$ has expansions*

$$\Big|G_-\Big(k, L - \frac{\log L}{2}\Big) - V\Big| = O\Big(\frac{1}{L^{1/2}}\Big), \tag{6.1}$$

$$\Big|G_-\Big(k, L - \log\frac{\sqrt{4\pi L}}{1-(1+k/L)^{-1/2}}\Big) - V\Big| = O\Big(\frac{\log L}{L^{3/2}}\Big). \tag{6.2}$$

*Remark 6.2* If $T \to 0$, then convergence $|\hat{V} - V| \to 0$ of the error in dimensionless implied volatility does not necessarily imply convergence of the error $|\hat{V} - V|/\sqrt{T}$ in annualized implied volatility.

For example, if $\hat{V} := G_-(k, L - \frac{\log L}{2})$ as in (6.1), then the dimensionless error $O(L^{-1/2})$ implies an annualized error $O((TL)^{-1/2})$. If $L = O(1/T)$, which is typical of diffusions as $T \to 0$, then the annualized error is $O(1)$, so convergence does not follow. A benefit of our approach is that it also generates refined approximations; for instance if $\hat{V} := G_-(k, L - \log\frac{\sqrt{4\pi L}}{1-(1+k/L)^{-1/2}})$ as in (6.2), then indeed we have the annualized convergence $|\hat{V} - V|/\sqrt{T} \to 0$.

6.2 Case (+)

This corollary is applicable in the fixed-strike long-expiry regime, and some cases of jointly varying strike-expiry regimes. For illustrative purposes, it takes $(N, P) = (1, 2)$ and $\eta$ nonzero in the FAT in Proposition 5.6, but extensions to arbitrarily high order come from taking $N$ and $P$ larger, and $\eta$ smaller or zero.

**Corollary 6.3** (Implied volatility formulas in Case (+)) *Write $L := L_+$. If Case (+) holds, then, as $\theta \to \infty$, the dimensionless implied variance $V^2$ has expansions*

$$\left|G_+^2\Big(k, L - \frac{\log L}{2} - \frac{\log \pi}{2}\Big) - V^2\right| = O\bigg(\Big(1 + \frac{k^{1/2}}{L^{1/2}}\Big)\Big(\frac{\log L}{L} + \frac{k \wedge L}{L}\Big)\bigg), \tag{6.3}$$

$$\left|G_+^2\Big(k, L - \frac{\log L}{2} - \frac{\log \pi}{2} + \frac{\log L}{4L}\Big) - V^2\right| = O\bigg(\Big(1 + \frac{k^{1/2}}{L^{1/2}}\Big)\Big(\frac{1}{L} + \frac{k \wedge L}{L}\Big)\bigg), \tag{6.4}$$

*and the dimensionless implied volatility $V$ has expansions*

$$\left|G_+\Big(k, L - \frac{\log L}{2} - \frac{\log \pi}{2}\Big) - V\right| = O\Big(\frac{\log L}{L^{3/2}} + \frac{k \wedge L}{L^{3/2}}\Big), \tag{6.5}$$

$$\left|G_+\Big(k, L - \frac{\log L}{2} - \frac{\log \pi}{2} + \frac{\log L}{4L}\Big) - V\right| = O\Big(\frac{1}{L^{3/2}} + \frac{k \wedge L}{L^{3/2}}\Big). \tag{6.6}$$

## 7 Corollaries for convergent $k/L$

In practice, $k/L$ will typically converge to some limit in $[0, \infty)$ as $\theta \to \infty$. Two common examples are as follows. First, if $k$ is constant or bounded, then $k/L \to 0$. Second, if

$$L = \alpha_1 \theta + o(\theta) \tag{7.1}$$

for some constant $\alpha_1 > 0$, along some path where $k(\theta) = b\theta$ for some $b > 0$, then $k/L \to b/\alpha_1$.

In such cases where $k/L$ converges, our implied volatility asymptotics admit simplifications; for example, the results of Corollaries 7.2 and 7.8 are simplifications, in the sense that they facilitate truncation—including, if desired, the truncation of the $V$ expansion (while making clear what impact such a truncation has on the error estimate), and also including the truncation of $L$ expansions (such as (7.4) or (7.16)) that plug into the $V$ expansion. Regarding the latter point, truncation of the $L$ expansion (which is usually required, given the unavailability of exact $L$) becomes simplified because the $V$ approximations in this section's corollaries make clear how the error in approximating $L$ translates into an error in approximating $V$, and thereby indicate how many terms of an $L$ expansion need to be retained.

In some cases, the simplifications (assuming convergent $k/L$) of this section are, moreover, sharper than the corresponding results of the previous section. For example, (7.13) both simplifies and sharpens (6.4) in the case $\limsup k < \infty$.

### 7.1 Short-expiry and related regimes

Section 7.1 assumes, unless otherwise stated, that the path belongs to Case $(-)$ and $k/L \to 0$. This includes the *short-expiry* bounded-strike regime, meaning the paths such that $k(\theta) \asymp 1$ (equivalently, $0 < \liminf k \le \limsup k < \infty$) and $V(\theta) \to 0$. It also includes the fixed-expiry *extreme-strike* regime under "thin-tailed" distributions, because at a fixed expiry, $k/L \to 0$ is equivalent to the underlying share price having finite moments of all positive orders. (The case of extreme strikes under general distributions, regardless of tail behavior, belongs to Sect. 6.1.)

**Corollary 7.1** (Case $(-)$ with $k/L \to 0$) *Write $L := L_-$. If $k/L \to 0$ as $\theta \to \infty$, then for the dimensionless implied variance*,

$$\left|G_-^2\Big(k, L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}}\Big) - V^2\right| = O\Big(\frac{k^2}{L^2}\frac{\log L + |\log k| + k}{L}\Big)$$

$$= o\Big(\frac{k^2}{L^2}\Big), \tag{7.2}$$

$$\left|G_-^2\Big(k, L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}} + \frac{9\log L}{4L}\Big) - V^2\right| = O\Big(\frac{k^2}{L^2}\frac{|\log k| + k}{L}\Big)$$

$$= o\Big(\frac{k^2}{L^2}\Big), \tag{7.3}$$

*and for the dimensionless implied volatility*,

$$\left|G_-\Big(k, L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}}\Big) - V\right| = O\Big(\frac{k}{L^{3/2}}\frac{\log L + |\log k| + k}{L}\Big)$$

$$= o\Big(\frac{k}{L^{3/2}}\Big), \tag{7.4}$$

$$\left|G_-\Big(k, L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}} + \frac{9\log L}{4L}\Big) - V\right| = O\Big(\frac{k}{L^{3/2}}\frac{|\log k| + k}{L}\Big)$$

$$= o\Big(\frac{k}{L^{3/2}}\Big). \tag{7.5}$$

Substitutions of the type in Sect. 5.5 reduce the Corollary 7.1 formulas to the following form.

**Corollary 7.2** (Case $(-)$ expanded, with $k/L \to 0$ and proxies for $G$ and $L$) *Write $L = L_-$. Let*

$$W(\kappa, \lambda) := \frac{\kappa^2}{2\lambda}\bigg(1 + \frac{3\log\lambda}{2\lambda} - \frac{\kappa + \log(\kappa^2/(16\pi))}{2\lambda} + \frac{9(\log\lambda)^2}{4\lambda^2}$$

$$- \Big(9 + 6\kappa + 6\log\big(\kappa^2/(16\pi)\big)\Big)\frac{\log\lambda}{4\lambda^2}\bigg). \tag{7.6}$$

*Then the dimensionless implied variance and volatility have expansions*

$$\big|W(k,\hat{L})-V^2\big| = O\Big(\frac{k^2}{L}\Big(\frac{k^2+(\log k)^2}{L^2}+\frac{|L-\hat{L}|}{L}\Big)\Big), \tag{7.7}$$

$$\big|\big(W(k,\hat{L})\big)^{1/2}-V\big| = O\Big(\frac{k}{L^{1/2}}\Big(\frac{k^2+(\log k)^2}{L^2}+\frac{|L-\hat{L}|}{L}\Big)\Big) \tag{7.8}$$

*for any function* $\hat{L}\sim L$.

*Remark 7.3* Ignoring any terms of the Corollary 7.2 expansion adds their absolute value to the error estimate. For example, dropping all sub-leading terms of $W$ yields

$$\left|\frac{k^2}{2\hat{L}}-V^2\right| = O\Big(\frac{k^2}{L}\frac{k+|\log k|+|L-\hat{L}|+\log L}{L}\Big) = o\Big(\frac{k^2}{L}\Big).$$

Therefore $k^2/(2L)\sim V^2$, which implies the constant-$k$ result of Roper and Rutkowski [18], and a constant-expiry result of Gulisashvili [10]. Corollary 7.2 sharpens both previous results.

We formulate the next corollary with a view towards diffusion examples.

**Corollary 7.4** *Write* $L := L_-$. *Assume that* $k\asymp 1$ *and* $T(\theta)\to 0$, *and that for all* $\theta>0$,

$$L = \frac{\alpha_{-1}}{T}-\frac{3}{2}\log T+\alpha_0+\varepsilon(\theta),$$

*where the coefficients* $\alpha_{-1},\alpha_0$ *may depend on* $\theta$, *provided that* $0<\liminf\alpha_{-1}$ *and* $\limsup(|\alpha_{-1}|+|\alpha_0|)<\infty$, *and where* $\varepsilon(\theta)=o(T^{-1})$ *as* $\theta\to\infty$. *Then the implied variance has the expansion*

$$\frac{V^2}{T} = \frac{k^2}{2\alpha_{-1}}-\frac{k^2}{4\alpha_{-1}^2}\Big(k+\log\frac{k^2}{16\pi}+2\alpha_0-3\log\alpha_{-1}\Big)T+O(T^2+\varepsilon T).$$

*In particular, if* $\varepsilon=O(T)$ *then the remainder is* $O(T^2)$.

Finally, let us include the $k=0$ regime—a much simpler case, because $C_-(0,\cdot)$ is analytic.

**Proposition 7.5** (Short-expiry at-the-money implied volatility) *Instead of the condition in Sect. 4.1, assume that* $k=0$ *and* $C_-\to 0$. *Then for all sufficiently large* $\theta$, *we have the convergent power series*

$$\begin{aligned} V &= \sqrt{8}\sum_{j=1}^{\infty}\frac{D^j\,\mathrm{inverf}(0)}{j!}C_-^j \\ &= \sqrt{2\pi}\Big(C_-+\frac{\pi}{12}C_-^3+\frac{7\pi^2}{480}C_-^5+\frac{127\pi^3}{40320}C_-^7+\cdots\Big). \end{aligned} \tag{7.9}$$

Expansions of $C_-$ in $T$ can be substituted into (7.9). For concreteness, let us take an expansion with remainder $O(T^{5/2})$; but the same principle applies to expansions of any order.

**Corollary 7.6** (Short-expiry $T$-expansion for at-the-money implied volatility) *Instead of the Sect.* 4.1 *conditions, assume that* $k=0$ *and* $T(\theta)\to 0$ *and*

$$C_- = \alpha_{1/2}T^{1/2} + \alpha_{3/2}T^{3/2} + O(T^{5/2}),$$

*where the coefficients* $\alpha_{1/2}, \alpha_{3/2}$ *may depend on* $\theta$, *provided that they satisfy* $\limsup(|\alpha_{1/2}|+|\alpha_{3/2}|)<\infty$ *as* $\theta\to\infty$. *Then*

$$\frac{V}{\sqrt{T}} = \sqrt{2\pi}\Big(\alpha_{1/2} + \Big(\frac{\pi}{12}\alpha_{1/2}^3 + \alpha_{3/2}\Big)T\Big) + O(T^2),$$

$$\frac{V^2}{T} = 2\pi\alpha_{1/2}^2 + \Big(\frac{\pi^2}{3}\alpha_{1/2}^4 + 4\pi\alpha_{1/2}\alpha_{3/2}\Big)T + O(T^2).$$

## 7.2 Large-strike and/or long-expiry regimes

This section assumes we have either Case $(+)$ with $\lim(k/L)\in[0,\infty)$, or Case $(-)$ with $\lim(k/L)\in(0,\infty)$. (Paths in Case $(-)$ with $\lim(k/L)=0$ belong to Sect. 7.1.) This section therefore includes the bounded-strike *long-expiry* regime, because $k/L\to 0$. It also includes the fixed-expiry *large-strike* regime and hybrid *large-strike long-expiry* regimes in cases where $k/L$ has a limit, which will be verifiable in our applications, via expansions of the type in (7.1). (Even in cases lacking such an expansion, the convergence of $k/L$ in the fixed-expiry large-strike regime, which is equivalent to the moment formula's limsup being a limit, can still be verified, by sufficient conditions in Benaim and Friz [1], or necessary and sufficient conditions in Gulisashvili [10].)

**Corollary 7.7** (Case $(\pm)$ for convergent $k/L$) *In Case* $(+)$, *assume* $k/L$ *has a limit in* $[0,\infty)$ *and write* $L:=L_+$. *In Case* $(-)$, *assume* $k/L$ *has a limit in* $(0,\infty)$ *and write* $L:=L_-$. *Let*

$$\delta := |k/L-\lim(k/L)|, \qquad \varrho_1^{\pm} := \lim R_1^{\pm}(k/L), \qquad \varrho_2^{\pm} := \lim R_2^{\pm}(k/L).$$

*Then the dimensionless implied variance* $V^2$ *has expansions*

$$\Big|G_{\pm}^2\Big(k, L+\log\frac{\varrho_1^{\pm}}{\sqrt{4\pi L}}\Big) - V^2\Big| = O\Big(\delta + \frac{\log L}{L}\Big) \tag{7.10}$$

*and*

$$\Big|G_{\pm}^2\Big(k, L+\log\frac{R_1^{\pm}(k/\varphi_1^{\pm})}{\sqrt{4\pi L}} - \frac{1}{2L}\log\frac{\varrho_1^{\pm}}{\sqrt{4\pi L}} - \frac{\varrho_2^{\pm}}{2\varrho_1^{\pm}L}\Big) - V^2\Big|$$
$$= O\Big(\frac{\delta}{L} + \frac{(\log L)^2}{L^2}\Big) \tag{7.11}$$

*as* $\theta\to\infty$, *where* $\varphi_1^{\pm} := L + \log(\varrho_1^{\pm}/\sqrt{4\pi L})$.

*Similarly, the dimensionless implied volatility $V$ has the following expansions: In* (7.10) *and* (7.11), *replacing each $G^2$ by $G$ and each $V^2$ by $V$ causes each right-hand-side error estimate to be multiplied by $O(1/L^{1/2})$.*

In the $(+)$ case with $k/L \to 0$, the Corollary 7.7 formulas reduce to the following forms.

**Corollary 7.8** (Case $(+)$ with $k/L \to 0$) *Write $L := L_+$. If $k/L \to 0$ as $\theta \to \infty$, then the dimensionless implied variance has expansions*

$$\left|8L - 4\log L + 4k - 4\log\pi - V^2\right| = O\left(\varepsilon_1 + \frac{\log L}{L}\right) \tag{7.12}$$

*and*

$$\left|8L - 4\log L + 4k - 4\log\pi + \frac{2\log L}{L} - \frac{k^2 + 4k + 8 - 4\log\pi}{2L} - V^2\right|$$
$$= O\left(\varepsilon_2 + \frac{(\log L)^2}{L^2}\right), \tag{7.13}$$

*where $\varepsilon_1(\theta) := (k^2 + k)/L$ and $\varepsilon_2(\theta) := k^4/L^3 + (k^3 + k^2 \log L)/L^2$. In particular, if* $\limsup k < \infty$, *then $\varepsilon_1$ and $\varepsilon_2$ drop out of* (7.12) *and* (7.13).

*Similarly, the dimensionless implied volatility $V$ has the following expansions: In* (7.12) *and* (7.13), *replacing each expansion by its square root and each $V^2$ by $V$ causes each right-hand-side error estimate to be multiplied by* $O(1/L^{1/2})$.

*Remark 7.9* The $8L - 4\log L + 4k - 4\log\pi$ in (7.12) recovers an important asymptotic approximation by Tehranchi for long-expiry dimensionless implied variance. More generally, Corollaries 6.3 and 7.7 allow $k$ to vary without bound; and more precisely, our extended asymptotics generate the higher-order approximation (7.13), and further approximations accurate to arbitrarily high powers of $1/L$ (by increasing $N$ and $P$ in the FAT in Proposition 5.6).

Expansions of $L$ induce expansions of $V$. For example:

**Corollary 7.10** *Suppose that $k(\theta) = \theta$, and as $\theta \to \infty$, we have*

$$L_- = \alpha_1 k + \alpha_{1/2} k^{1/2} + \alpha_\ell \log k + \alpha_0 + O(k^{-r}) \tag{7.14}$$

*for some $r \in (0, 1/2)$, some constant $\alpha_1 > 0$, and some $\alpha_{1/2}, \alpha_\ell, \alpha_0$ which may depend on $\theta$, provided that* $\limsup(|\alpha_{1/2}| + |\alpha_\ell| + |\alpha_0|) < \infty$. *Then the dimensionless implied volatility has the expansion*

$$V = \beta_{1/2} k^{1/2} + \beta_0 + \beta_{\ell-1/2} \frac{\log k}{k^{1/2}} + \frac{\beta_{-1/2}}{k^{1/2}} + O\left(\frac{1}{k^{r+1/2}}\right),$$

*where*

$$\begin{aligned}
\beta_{1/2} &:= \sqrt{2\alpha_1+2}-\sqrt{2\alpha_1},\\
\beta_0 &:= \Big(\frac{1}{\sqrt{2\alpha_1+2}}-\frac{1}{\sqrt{2\alpha_1}}\Big)\alpha_{1/2},\\
\beta_{\ell-1/2} &:= \Big(\frac{1}{\sqrt{2\alpha_1+2}}-\frac{1}{\sqrt{2\alpha_1}}\Big)\Big(\alpha_\ell-\frac{1}{2}\Big),\\
\beta_{-1/2} &:= \Big(\frac{1}{\sqrt{2\alpha_1+2}}-\frac{1}{\sqrt{2\alpha_1}}\Big)\Big(\alpha_0+\log\frac{R_1^-(1/\alpha_1)}{\sqrt{4\pi\alpha_1}}\Big)\\
&\quad+\Big(\frac{1}{2(2\alpha_1)^{3/2}}-\frac{1}{2(2\alpha_1+2)^{3/2}}\Big)\alpha_{1/2}^2.
\end{aligned} \tag{7.15}$$

Corollary 7.10 targets large-strike Heston asymptotics, while Corollary 7.11 targets large-strike and/or long-expiry Lévy asymptotics. The two could be unified, but the formulas are separately less cumbersome.

**Corollary 7.11** *Suppose that for either $L_+$ or $L_-$, we have, as $\theta\to\infty$,*

$$L_\pm=\alpha_1\theta+\frac{1}{2}\log\theta+\alpha_0+\frac{\alpha_{-1}}{\theta}+O\Big(\frac{1}{\theta^2}\Big), \tag{7.16}$$

*for some constant $\alpha_1>0$, and some $\alpha_0,\alpha_{-1}$ which may depend on $\theta$, provided that $\limsup(|\alpha_0|+|\alpha_{-1}|)<\infty$. If we have (7.16) for $L_+$ and $\limsup k(\theta)<\infty$, then*

$$\begin{aligned}
V^2 &= 8\alpha_1\theta+(4k+8\alpha_0-4\log\alpha_1-4\log\pi)\\
&\quad+\Big(8\alpha_{-1}-\frac{k^2+4k+8-4\log\alpha_1\pi+8\alpha_0}{2\alpha_1}\Big)\frac{1}{\theta}+O\Big(\frac{(\log\theta)^2}{\theta^2}\Big).
\end{aligned}$$

*If we have (7.16) for either $L_+$ or $L_-$, and $k=b\theta$ for some constant $b>0$, then*

$$V=\beta_{1/2}\theta^{1/2}+\frac{\beta_{-1/2}}{\theta^{1/2}}+\frac{\beta_{-3/2}}{\theta^{3/2}}+O\Big(\frac{(\log\theta)^2}{\theta^{5/2}}\Big), \tag{7.17}$$

*where*

$$\begin{aligned}
\beta_{1/2} &:= \sqrt{2\alpha_1+2b}\pm\sqrt{2\alpha_1},\\
\beta_{-1/2} &:= \Big(\frac{1}{\sqrt{2\alpha_1+2b}}\pm\frac{1}{\sqrt{2\alpha_1}}\Big)\gamma,\\
\beta_{-3/2} &:= \frac{\alpha^*}{(2\alpha_1+2b)^{1/2}}-\frac{\gamma^2}{2(2\alpha_1+2b)^{3/2}}\pm\Big(\frac{\alpha^*}{(2\alpha_1)^{1/2}}-\frac{\gamma^2}{2(2\alpha_1)^{3/2}}\Big)
\end{aligned} \tag{7.18}$$

*and*

$$\gamma := \alpha_0 + \log \frac{R_1^{\pm}(b/\alpha_1)}{\sqrt{4\pi\alpha_1}},$$
$$\alpha^* := \alpha_{-1} - \frac{R_2^{\pm}(b/\alpha_1)}{2\alpha_1 R_1^{\pm}(b/\alpha_1)} - \gamma \frac{(\alpha_1+b)^{3/2} \pm \alpha_1^{3/2}}{2\alpha_1(\alpha_1+b)(\sqrt{\alpha_1+b} \pm \sqrt{\alpha_1})}, \tag{7.19}$$

*where each* $\pm$ *in* (7.18) *and* (7.19) *is determined by the* $\pm$ *in* (7.16).

## 8 Applications

In the large-strike Heston and short-expiry Lévy cases, we recall the call-price asymptotics due to, respectively, Friz et al. ([8]; FGGS henceforth) and Figueroa-Lopez and Forde [5]. In the long-expiry Lévy case, we carry out a saddlepoint expansion beyond leading order, and without bounding the strikes, thereby refining and extending the option-price asymptotics of Tehranchi [20].

In all of the above cases, we then input the option-price asymptotics into our general implied volatility formulas, which then sharpen the sharpest previously known approximations of $V$ or $V^2$. In all cases, the Sect. 8 formulas (specializing our model-free results of previous sections, which were expressed in terms of $(k, L)$) are expressed here in terms of strike (but not $L$), expiry, and the parameters (or cumulant functions) of the particular models.

### 8.1 Large-strike Heston

As an application of Corollary 7.10, consider the case of Heston [12] dynamics.

**Corollary 8.1** (Heston model large-strike asymptotics) *Let* $T > 0$ *be any constant. Let* $k(\theta) := \theta$ *and assume that there exists a probability measure such that for all* $\theta > 0$,

$$C_-(k, V) = \mathbb{E}(S_T - e^k)^+,$$

*where*

$$\mathrm{d}S_t = S_t\sqrt{v_t}\,\mathrm{d}W_t, \quad S_0 = 1,$$
$$\mathrm{d}v_t = (a + bv_t)\,\mathrm{d}t + c\sqrt{v_t}\,\mathrm{d}Z_t, \quad v_0 > 0,$$

*where* $a \geq 0, b \leq 0, c > 0$ *and the correlated Brownian motions* $W$ *and* $Z$ *satisfy* $\mathrm{d}\langle W, Z\rangle_t = \rho\,\mathrm{d}t$ *with* $\rho \in (-1, 0]$. *Let*

$$s_+ := \sup\{p \geq 1 : \mathbb{E}S_T^p < \infty\},$$
$$\alpha_1 := s_+ - 1,$$
$$\alpha_{1/2} := -\frac{2}{c}\left(\frac{2v_0}{\sigma(s_+)}\right)^{1/2},$$

$$\alpha_\ell := \frac{3}{4} - \frac{a}{c^2},$$

$$\alpha_0 := \log \frac{s_+(s_+-1)}{A},$$

*where* $\chi(s) := s\rho c + b$ *and* $\Delta(s) := \chi(s)^2 - c^2(s^2 - s)$ *and*

$$\sigma(s) := T^*(s)\frac{2\rho c\chi(s) - c^2(2s-1)}{2\Delta(s)} + \frac{(c^2(2s-1) - 2\rho c\chi(s))\chi(s) + 2\rho c\Delta(s)}{\Delta(s)(\chi(s)^2 - \Delta(s))},$$

$$T^*(s) := \frac{2}{\sqrt{-\Delta(s)}}\Big(\arctan\frac{\sqrt{-\Delta(s)}}{\chi(s)} + \pi\Big),$$

$$A := \frac{1}{2\sqrt{\pi}}\Big(\frac{2v_0}{\sigma(s_+)c^2}\Big)^{1/4}\Big(\frac{\sigma(s_+)v_0 s_+(s_+-1)}{2}\Big)^{-a/c^2}$$
$$\times \exp\bigg(-v_0\Big(\frac{\chi(s_+)}{c^2} - \frac{\sigma'(s_+)}{c^2\sigma(s_+)^2}\Big) - \frac{aT}{c^2}\chi(s_+)\bigg).$$

*Then*

$$V = \beta_{1/2}k^{1/2} + \beta_0 + \beta_{\ell-1/2}\frac{\log k}{k^{1/2}} + \frac{\beta_{-1/2}}{k^{1/2}} + O\big(k^{-3/4}\big), \tag{8.1}$$

*where each* $\beta$ *is defined in* (7.15).

The leading term of (8.1) is given by the moment formula (without error estimates). FGGS find the next two terms explicitly in the Heston model, with error estimate $O(k^{-1/2})$. Here we have found the fourth term $\beta_{-1/2}/k^{1/2}$ explicitly, reducing the error to $O(k^{-3/4})$.

*Example 8.2* (Heston) In Fig. 1, we plot the refined approximation (8.1) against the true implied volatility generated by the Heston model, with the same parameters as in FGGS, namely

$$(a, b, c, \rho, v_0) = (0.0429, -0.6067, 0.2928, -0.7571, 0.0654).$$

We also plot the FGGS approximation, which does not include the $O(k^{-1/2})$ term.

### 8.2 Short-expiry Lévy

As an application of Corollary 7.2, consider the case of Lévy dynamics.

**Corollary 8.3** (Small-time asymptotics for implied variance of exponential Lévy processes) *Let* $k > 0$ *be any constant. Let* $T(\theta) \to 0$ *and assume that there exists a probability measure such that for all* $\theta > 0$,

$$C_-(k, V) = \mathbb{E}(e^{X_T} - e^k)^+,$$

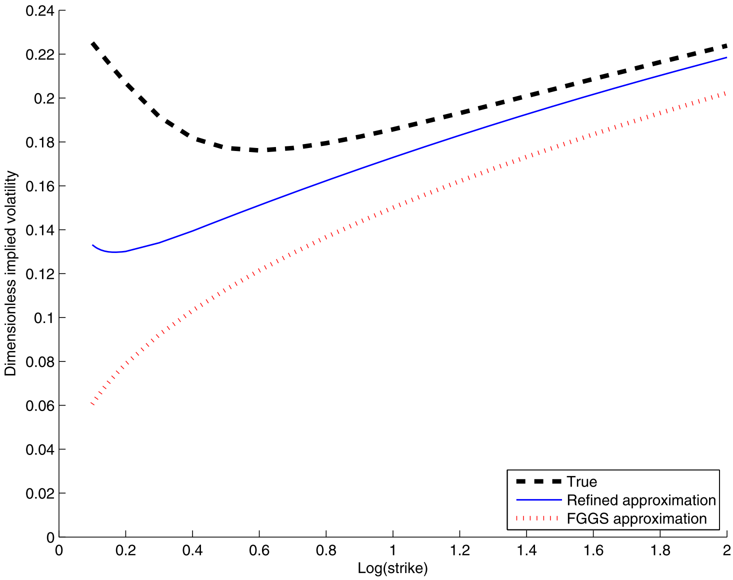


**Fig. 1** Implied volatility in the Heston model at large strikes: $T = 1$ year.
In Fig. 1, the FGGS [8] Heston smile approximation consists of the first three terms of (8.1), and our refined approximation includes all four terms of (8.1)

*where $X$ is a Lévy process with generating triplet $(\sigma^2, b, \nu)$ such that $e^X$ is a martingale and $\nu$ has a positive $C^1$ density $p_\nu$ such that $\sup_{|x|>\epsilon}(e^x \vee 1)p_\nu(x) < \infty$ for every $\epsilon > 0$. Let*

$$a := \int_{-\infty}^{\infty} (e^x - e^k)^+ p_\nu(x)\,\mathrm{d}x.$$

*Then, with $W$ defined in (7.6),*

$$V^2 = W\Big(k, \log\frac{1}{aT}\Big) + O\big(|\log T|^{-3}\big). \tag{8.2}$$

The approximation (8.2) sharpens the $o(|\log T|^{-2})$ approximation of Figueroa-Lopez and Forde [5].

*Remark 8.4* The error of $O(1/|\log T|^2)$ in the dimensionless implied variance implies that the *annualized* implied variance has error $O(1/(|\log T|^2 T))$.

The blow-up $1/(|\log T|^2 T) \to \infty$ of the error bound as $T \to 0$ should be regarded in the context that the true annualized implied variance $V^2/T$ *also* blows up as $T \to 0$. If instead of the absolute annualized error $W/T - V^2/T$, we consider the *relative* error $W/V^2 - 1$ (which is the same regardless of the annualized vs. dimensionless convention), then we have convergence to 0; in particular, by (7.6) and (8.2), the relative error is $O(1/|\log T|^2)$ as $T \to 0$.

This rate of convergence, however, is slow; for example, in order to reduce the estimate of the relative error by a factor of 4, it is necessary to *square* the $T$. We therefore do not recommend this approximation.

### 8.3 Long-expiry and/or large-strike Lévy

As an application of Corollary 7.11, consider the case of Lévy dynamics.

First we carry out, beyond leading order, a saddlepoint expansion for option prices, valid for log strikes that are constant or linear or affine in expiry. Figueroa-Lopez et al. [6], in a preprint on large-time Lévy asymptotics, obtain (contemporaneously) a result similar to Lemma 8.5.

**Lemma 8.5** (Refined saddlepoint expansion of Lévy-driven option prices) *Let* $T(\theta) := \theta$ *and*

$$k(\theta) := \kappa_0 + b\theta$$

*for some constants* $\kappa_0 \geq 0$ *and* $b \geq 0$. *Assume that there exists a probability measure* $\mathbb{P}$ *such that*

$$C_+(k, V) = \mathbb{E}(e^{X_T} \wedge e^k),$$
$$C_-(k, V) = \mathbb{E}(e^{X_T} - e^k)^+$$

*for all* $\theta > 0$, *where* $X$ *is a nonconstant Lévy process such that* $e^X$ *is a martingale. Let* $X_1$ *have cumulant generating function*

$$\mathcal{L}(u) := \log \mathbb{E} e^{uX_1}$$

*and let*

$$\mathcal{L}_b(u) := \mathcal{L}(u) + b(1 - u).$$

*Assume that* $\mathcal{L}_b'$ *has a real root* $u_* \in (0, 1)$ *or* $u_* \in (1, \sup\{u \geq 1 : \mathbb{E}e^{uX_1} < \infty\})$ *such that*

$$\mathcal{L}_b''(u_*) > 0, \tag{8.3}$$

$$\operatorname{Re} \mathcal{L}_b(u_* + iy) < \mathcal{L}_b(u_*) \quad \text{for all real } y \neq 0, \tag{8.4}$$

$$\limsup_{y \to \pm\infty} \operatorname{Re} \mathcal{L}_b(u_* + iy) < \mathcal{L}_b(u_*). \tag{8.5}$$

*A sufficient condition for* (8.3)–(8.5) *is that* $X_1$ *admits a density.*

*Let* $L := L_+$ *if* $u_* < 1$, *or* $L := L_-$ *if* $u_* > 1$. *Let*

$$Q(z) := \frac{e^{\kappa_0(1-z)}}{2\pi z(1-z)}.$$

*Then as* $\theta \to \infty$, *we have*

$$L = \alpha_1 \theta + \frac{1}{2}\log\theta + \alpha_0 + \frac{\alpha_{-1}}{\theta} + O\Big(\frac{1}{\theta^2}\Big), \tag{8.6}$$

*where*

$$\alpha_1 := -\mathcal{L}_b(u_*),$$
$$\alpha_0 := -\log|\gamma_0|,$$
$$\alpha_{-1} := -\gamma_{-1}/\gamma_0,$$
$$\gamma_0 := \frac{\sqrt{2\pi}\,Q}{(\mathcal{L}_b'')^{1/2}},$$
$$\gamma_{-1} := -\bigg(2Q'' - \frac{2\mathcal{L}_b'''Q'}{\mathcal{L}_b''} + \Big(\frac{5\mathcal{L}_b'''^2}{6\mathcal{L}_b''^2} - \frac{\mathcal{L}_b''''}{2\mathcal{L}_b''}\Big)Q\bigg)\frac{\sqrt{\pi/8}}{(\mathcal{L}_b'')^{3/2}},$$

*where $Q$ and $\mathcal{L}_b$ and their derivatives are evaluated at $u_*$, and the powers $1/2$ and $3/2$ refer to the principal branch.*

Implied volatility formulas follow in the next two corollaries. In the bounded-$k$ case, the leading-order (affine) terms shown in (8.7) agree with the affine approximation due to [20].

**Corollary 8.6** (Large-$T$ asymptotics for Lévy processes) *Under the assumptions of Lemma 8.5, let $\kappa_0 > 0$ and $b := 0$; hence $k$ is constant and $T = \theta$. Then as $\theta \to \infty$, the dimensionless implied variance has expansion*

$$\begin{aligned} V^2 &= 8\alpha_1 T + (4k + 8\alpha_0 - 4\log\alpha_1 - 4\log\pi) \qquad (8.7)\\ &\quad + \Big(8\alpha_{-1} - \frac{k^2 + 4k + 8 - 4\log\alpha_1\pi + 8\alpha_0}{2\alpha_1}\Big)\frac{1}{T} + O\Big(\frac{(\log T)^2}{T^2}\Big)\\ &= (\text{affine in } k \text{ and } T) + \frac{\text{quadratic in } k}{T} + O\Big(\frac{(\log T)^2}{T^2}\Big). \qquad (8.8)\end{aligned}$$

**Corollary 8.7** (Joint-$KT$ asymptotics for Lévy processes) *Under the assumptions of Lemma 8.5, let $\kappa_0 := 0$ and $b > 0$; hence $k = bT = b\theta$. Then as $\theta \to \infty$, the dimensionless implied volatility has the third-order expansion*

$$V = \beta^{\pm}_{1/2}\theta^{1/2} + \frac{\beta^{\pm}_{-1/2}}{\theta^{1/2}} + \frac{\beta^{\pm}_{-3/2}}{\theta^{3/2}} + O\Big(\frac{(\log\theta)^2}{\theta^{5/2}}\Big), \qquad (8.9)$$

*using the $\beta^-$ coefficients in the case $u_* > 1$, and the $\beta^+$ coefficients in the case $u_* < 1$, where all coefficients are defined in (7.18). In particular, restating just the first* two *terms of (8.9) explicitly,*

$$\begin{aligned} V &= \big(\sqrt{2\alpha_1 T + 2k} \pm \sqrt{2\alpha_1 T}\big)\\ &\quad + \Big(\frac{1}{\sqrt{2\alpha_1 T + 2k}} \pm \frac{1}{\sqrt{2\alpha_1 T}}\Big)\log\frac{u_*|1-u_*|\sqrt{\mathcal{L}''(u_*)}R_1^{\pm}(k/(\alpha_1 T))}{\sqrt{2\alpha_1}}\\ &\quad + O\Big(\frac{1}{\theta^{3/2}}\Big), \qquad (8.10)\end{aligned}$$

*where $\alpha_1 := -\mathcal{L}(u_*) + (u_* - 1)(k/T)$ and $u_*$ solves $\mathcal{L}'(\cdot) = k/T$.*

*Example 8.8* (Variance gamma) Let $X$ be a *variance gamma* (VG) process defined by the cumulant function

$$\mathcal{L}(u) = \frac{u \log(1-(m+\sigma^2/2)\nu) - \log(1-(um+u^2\sigma^2/2)\nu)}{\nu},$$

with parameters $\nu > 0$, $\sigma > 0$ and $m < 0$. Let $\ell := \log(1-\nu(m+\sigma^2/2))$. Then

$$u_* = \frac{\nu\sigma^2 - (\ell - \nu k/T)m\nu - \sqrt{\nu^2\sigma^4 + (\ell-\nu k/T)^2(m^2\nu^2 + 2\nu\sigma^2)}}{\nu(\ell-\nu k/T)\sigma^2},$$

and $X$ can be shown to satisfy the hypotheses of Corollary 8.6 and also (unless $u_* = 1$) Corollary 8.7. Moreover, those corollaries' conclusions become fully explicit formulas.

Figures 2, 3 and 4 display the true implied variance generated by the variance gamma model, with parameters from Madan et al. [15] given by

$$(m, \sigma, \nu) = (-0.1436, 0.1213, 0.1686),$$

at expiries $T \in \{5, 1, 0.25\}$. At the longer expiries $\{5, 1\}$, the figures include also Tehranchi's [20] large-$T$ affine approximation, which is the first line of (8.7). In the same figures, we plot also our approximations.

At expiry 5, the refined large-$T$ formula from (8.7) and (8.8) adds to the affine formula a (quadratic in $k)/T$ term, introducing the curvature shown in Fig. 2.

At all expiries $\{5, 1, 0.25\}$, our jointly varying strike-expiry regime's formula (8.10), or equivalently just the first two terms of (8.9), has remarkable accuracy; in Figs. 2–4, the joint-$KT$ approximation is, to the naked eye, indistinguishable from the exact volatility smile across *all* displayed strikes and expiries.

In Fig. 3, truncating the joint-$KT$ formula (8.9) to just a single term introduces a visible (but well-behaved) error, while the error in our two-term formula remains invisible on this scale.

*Remark 8.9* (Extension to negative log strikes) Our $k > 0$ results apply to negative log strikes, by first changing to *share measure*, then computing implied volatility asymptotics using our $k > 0$ formulas, and then reflecting each strike to obtain the asymptotics for negative log strikes, as justified in [13], Theorem 4.1. In the Lévy case, the effect of the measure change is simply to replace $\mathcal{L}$ with the share-measure cumulant generating function $\tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}(u) = \mathcal{L}(1-u)$. We use this device to plot the left-hand half of each figure.

*Remark 8.10* (Intuition of high accuracy) Figure 4 plots also the four-term approximation to the (log) call price $L$ from (8.6), but expressed as a volatility, by *exact* inversion of the Black–Scholes formula. It shows that this $L$ approximation loses accuracy as $k \to 0$, which is not surprising, given that (8.6) is asymptotic for *large* strikes.

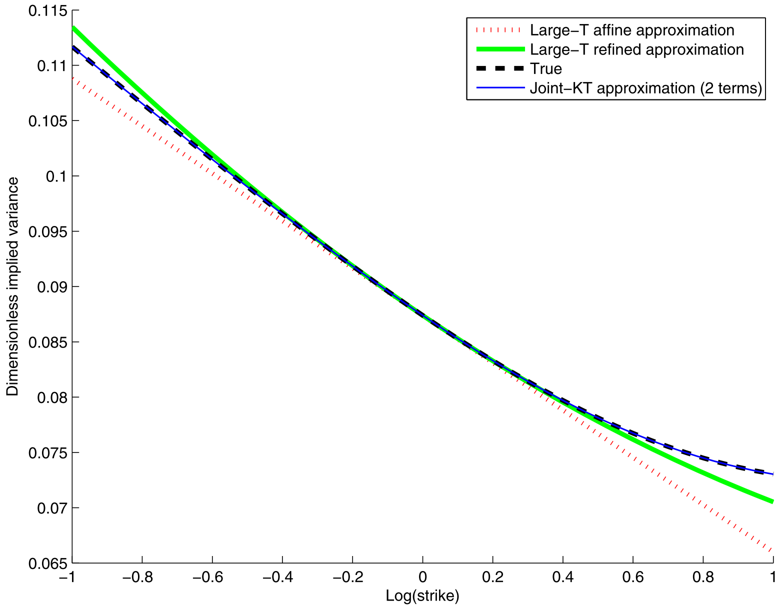


**Fig. 2** Implied variance in the VG model: $T = 5$ years.
In Figs. 2 and 3, the "large-$T$ affine" approximation due to [20] consists of the first line of (8.7). In Fig. 2, our "large-$T$ refined" approximation is the full formula (8.7), including up to the $1/T$ term. In Figs. 2–4, our "joint-$KT$ (2 terms)" approximation is (8.10), or equivalently the first 2 terms of (8.9). The joint-$KT$ approximation is visually *indistinguishable* from the true implied variance in all three figures

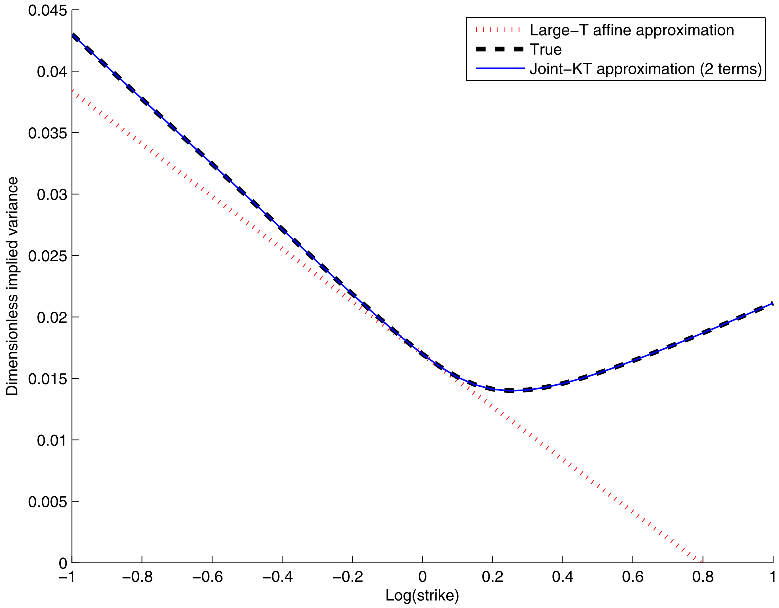


**Fig. 3** Implied variance in the VG model: $T = 1$ year

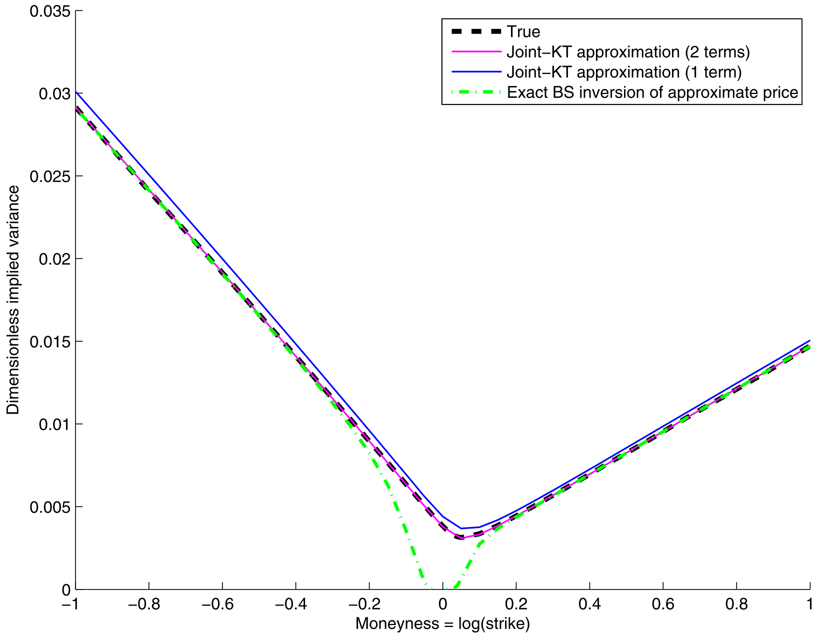


**Fig. 4** Implied variance in the VG model: $T = 0.25$ year.
In Fig. 4, the "joint-$KT$ (1 term)" approximation is the first line of (8.10) (or equivalently, the first term of (8.9)). The "joint-$KT$ (2 terms)" approximation remains visually indistinguishable from the true $V^2$

The more unexpected phenomenon is not the inaccuracy of (the at-the-money part of) that $L$ proxy (expressed exactly as a volatility), but rather the accuracy of the $V$ approximations that are plotted in the same figure. In particular, the 1-term joint-$KT$ approximation of $V$ in the plot is the leading term of (8.10)—a compound approximation, which comes from the leading-term $L$ approximation in (8.6), converted into an implied volatility, by a second approximation, namely the leading term of (7.17). The high accuracy of this compound approximation indicates that the errors in the two constituent approximations exhibit some cancellation, which can be understood as follows.

The intuition is that the error in applying (8.6) to approximate $L$ in the Lévy model can be largely canceled by applying the *same* approximation *also* in the Black–Scholes model, to solve the inverse problem of finding implied volatility. This error-canceling re-application of (8.6) produces a formula which agrees with (and thus gives insight into the accuracy of) the leading term of our Corollary 8.7.

Specifically, consider a general Lévy model of the type in Lemma 8.5 and fix a contract $(k, T)$, with particular attention to small $k$ where the error cancellation manifests most significantly. Let $\sigma_{\text{imp}}$ be that contract's exact implied volatility under the Lévy dynamics. Then apply the saddlepoint approximation (8.6) with $b = k/T = k/\theta$ to the Lévy log call price *and* to the Black–Scholes log call price with constant volatility $\sigma_{\text{imp}}$. We have, respectively,

$$L^{\text{Lévy}} = \alpha_1^{\text{Lévy}}\theta + \mathcal{E}^{\text{Lévy}},$$
$$L^{\text{BS}} = \alpha_1^{\text{BS}}\theta + \mathcal{E}^{\text{BS}}, \tag{8.11}$$

where $\mathcal{E}$ denotes the error of the leading-order saddlepoint approximation (indeed (8.11) defines $\mathcal{E}$), and the superscripts "Lévy" and "BS" denote the general Lévy and Black–Scholes cases, respectively. The left-hand sides of (8.11) are equal at $(k, T)$, by choice of $\sigma_{\text{imp}}$.

Moreover, it makes sense intuitively that $\mathcal{E}^{\text{Lévy}}$ and $\mathcal{E}^{\text{BS}}$ would roughly *cancel* each other, as both errors are the residuals of saddlepoint expansions, and both the general and the BS models are Lévy models, and the two models' distributions have comparable variances, because the BS variance was chosen as the near-the-money (small $k$) implied variance of the general Lévy model.

Under the assumption (not generally true, but defensible as discussed above) that $\mathcal{E}^{\text{Lévy}} = \mathcal{E}^{\text{BS}}$, we have

$$\alpha_1 T = \alpha_1^{\text{BS}} T = -\mathcal{L}_b^{\text{BS}}(u_*^{\text{BS}})T = \frac{\sigma_{\text{imp}}^2}{2}\left(\frac{k/T}{\sigma_{\text{imp}}^2} - \frac{1}{2}\right)^2 T, \tag{8.12}$$

where the first equality is by (8.11), and the second is by the explicit form of the Gaussian cumulant generating function, and we abbreviate $\alpha_1^{\text{Lévy}}$ as $\alpha_1$. Solving for $\sigma_{\text{imp}}$, we have

$$\sigma_{\text{imp}} = \sqrt{2\alpha_1 T + 2k} \pm \sqrt{2\alpha_1 T}, \tag{8.13}$$

which agrees with—and thus heuristically explains the accuracy of—the leading term of our joint-$KT$ approximation (8.10) plotted in Figs. 2–4.

Likewise, the ATM accuracy of the large-$T$ approximations (8.7) plotted in Fig. 2 can be understood by taking $k = 0$ in (8.13), to obtain $\sigma_{\text{imp}}^2 = 8\alpha_1 T$, which agrees with—and heuristically explains the ATM accuracy of—the leading term of (8.7). (The plotted large-$T$ approximations include more terms of (8.7) beyond the $8\alpha_1 T$, but their ATM effect is small; the "affine" proxy adds only $-0.0006$ to the ATM implied variance, and the "refined" proxy adds only 0.000003 more. The ATM accuracy of the plotted large-$T$ approximations is largely attributable to the accuracy of the leading term, for reasons discussed above.)

## 9 Concluding remarks

Our methods generate nearly universal asymptotic approximations for implied volatility—universal in two senses: across all models, and also across general extreme regimes in strike and/or expiry, provided that $L$ (the exogenously given absolute log price of the call or covered-call combination) approaches zero.

Our approximation formulas include rigorous error estimates. By recursive refinement, they attain arbitrary order of accuracy, in the sense of having asymptotic errors smaller than any given power of $L$.

Moreover, in some applications such as in Figs. 2–4, these approximations, in concert with saddlepoint methods, have remarkable accuracy not just in extreme regimes, but also across a full range of strikes (from a fraction of the spot to a multiple of the spot) and a full range of expiries (from months to years).

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## Appendix: Proofs

The following properties are easy to verify. Here we use $\leftarrow$ to mean limit as $x \to 0+$ and $\to$ to mean limit as $x \to \infty$. We have then

$$\begin{array}{ccccc} 1 \pm 1 & \leftarrow & R_n^{\pm}(x) & \to 1, \\ 2n+1 & \leftarrow & R_{n+1}^{-}(x)/R_1^{-}(x) & \to 1, \\ 1 & \leftarrow & R_{n+1}^{+}(x)/R_1^{+}(x) & \to 1, \\ (1 \mp 1)/2 & \leftarrow & xDR_1^{\pm}(x)/R_1^{\pm}(x) & \to 0, \\ 0 & \leftarrow & xDA_n^{\pm}(x) & \to 0. \end{array} \tag{A.1}$$

So $R_n^{\pm}(x)$, $1/R_1^{+}(x)$, $xDR_1^{\pm}(x)/R_1^{\pm}(x)$, $A_n^{\pm}(x)$, $xDA_n^{\pm}(x)$, $B_n^{\pm}(x)$ and $xDB_n^{\pm}(x)$ are all bounded on $[0, \infty)$. Moreover,

$$\begin{gathered} \frac{1}{4} \wedge \frac{x}{4} \le R_1^{-}(x) \le 1 \wedge \frac{x}{2}, \\ 1 \le R_1^{+}(x) \le 2. \end{gathered} \tag{A.2}$$

The functions $y \mapsto \log R_1^{\pm}(e^y)$ are Lipschitz, so for all positive functions $a(\theta) \sim b(\theta)$ as $\theta \to \infty$, we have

$$R_1^{\pm}(a) \sim R_1^{\pm}(b), \quad \theta \to \infty. \tag{A.3}$$

*Proof of Lemma* 3.1 By changing variables in (3.1) and (3.2),

$$C_{\pm}(\kappa, v) = \pm \frac{1}{\sqrt{2\pi}} \int_{F(\kappa,v)}^{\infty} e^{-u} DG_{\pm}(\kappa, u)\, \mathrm{d}u.$$

Integrating by parts $N$ times produces (3.8). Moreover, (3.9) follows from

$$\left| \int_{F(\kappa,v)}^{\infty} e^{-u} D^{N+1} G_{\pm}(\kappa, u)\, \mathrm{d}u \right| \le \left| D^{N+1} G_{\pm}\big(\kappa, F(\kappa, v)\big) \right| \int_{F(\kappa,v)}^{\infty} e^{-u}\, \mathrm{d}u,$$

which holds because $(D^{N+1}G_{\pm})(D^{N+2}G_{\pm}) < 0$, hence $|D^{N+1}G_{\pm}(\kappa, \cdot)|$ is decreasing. □

*Proof of Lemma 4.1* Note that $C_\pm(\kappa, \sqrt{2\kappa}) \to 1/2$ as $\kappa \to \infty$ and $C_+(\kappa, \sqrt{2\kappa}) \to 1$ as $\kappa \to 0$. Therefore

$$0 < \inf\{C_+(\kappa, \sqrt{2\kappa}) : \kappa > 0\} = \inf\{C_+(\kappa, v) : \kappa > 0, v \le \sqrt{2\kappa}\}.$$

So if $C_+(k, V) \to 0$, then eventually $V > \sqrt{2k}$ as claimed. In the $(-)$ case, note that for $\kappa > 0$,

$$C_-\big(\kappa, \sqrt{2\kappa}\big) = \frac{1}{2} - e^\kappa \mathcal{N}\big(-\sqrt{2\kappa}\big) = \sqrt{\kappa/\pi}\big(1 + o(1)\big) \quad \text{as } \kappa \to 0.$$

Therefore $|\log C_-(\kappa, \sqrt{2\kappa})| \sim |\log \kappa|/2$ as $\kappa \to 0$. Then for some $\kappa_0 > 0$ and all $\kappa \in (0, \kappa_0)$, we have $|\log \kappa| > |\log C_-(\kappa, \sqrt{2\kappa})|$.

Moreover, eventually $|\log(k \wedge \kappa_0)| < |\log C_-(k, V)|$ by (4.3), so for some $\theta_0 > 0$ and all $\theta > \theta_0$, we have $V < \sqrt{2k}$ or $k > \kappa_0$. On the other hand,

$$0 < \inf\{C_-(\kappa, \sqrt{2\kappa}) : \kappa \ge \kappa_0\} = \inf\{C_-(\kappa, v) : \kappa \ge \kappa_0, v \ge \sqrt{2\kappa}\}.$$

So for some $\theta_1 > 0$ and all $\theta > \theta_1$, we have $V < \sqrt{2k}$ or $k < \kappa_0$.

It follows that for all $\theta > \theta_0 \vee \theta_1$, we have $V < \sqrt{2k}$. □

*Proof of Lemma 5.2* The $N = 0$ case is clear, so let $N > 0$. Consider any $\Lambda \sim L_\pm$ as $\theta \to \infty$. To verify (5.11), we have

$$\begin{aligned} Dh_N^*(k, \Lambda) &= -\frac{1}{\Lambda}\left(\frac{1}{2} + \frac{kDR_1(k/\Lambda)}{\Lambda R_1(k/\Lambda)} + \sum_{n=1}^{N-1} \frac{(k/\Lambda)DB_n(k/\Lambda) + nB_n(k/\Lambda)}{\Lambda^n}\right) \\ &= O(1/L), \end{aligned}$$

using via (A.1) the boundedness of $xDB_n(x)$ and $B_n(x)$ and $xDR_1(x)/R_1(x)$.

To verify (5.10), consider the right-hand side of (5.13) evaluated along $\lambda = \Lambda(\theta)$. The first two terms are then $O(\log L)$. So is the last, because each $B_j$ is bounded. So is the third term in the $(+)$ case, because $R_1^+ \in [1, 2]$. Finally, for the third term in the $(-)$ case, (A.2) implies that as $\theta \to \infty$,

$$|\log R_1^-(k/\Lambda)| \le \log 4 + \left(0 \vee \log \frac{\Lambda}{k}\right) = O\left(\log \Lambda + \left(0 \vee \log \frac{1}{k}\right)\right). \tag{A.4}$$

So $h_{N_+}^*(k, \Lambda) = O(\log L_+)$, and $h_{N_-}^*(k, \Lambda) = O(\log L_- + (0 \vee \log(1/k))) = o(L_-)$ by (4.2). □

We need the next four lemmas to prove the FAT in Proposition 5.6. The first one, Lemma A.1, establishes (5.5).

**Lemma A.1** *Let $H := (h; \eta_1, \dots, \eta_P)$ as in Definition 5.3. For any function $\Lambda \sim L_\pm$ and any $p = 0, \dots, P$, we have*

$$\phi_{H,p}(k, \Lambda) \sim L_\pm. \tag{A.5}$$

*In particular,* $\varphi^{\pm}_{H,p} \sim L_{\pm}$. *Moreover,*

$$D\phi_{H,p}(k, \Lambda) \to 1 \tag{A.6}$$

*as* $\theta \to \infty$. *Recall that* $D\phi_{H,p}$ *denotes the partial derivative of* $\phi_{H,p}(\kappa, \lambda)$ *in its second argument.*

*Proof* Induct on $p$. The case $p = 0$ is clear. If (A.5) holds for some $p \geq 0$, then by (5.10),

$$\phi_{H,p+1}(k, \Lambda) = \Lambda + h\big(k, \phi_{H,p}(k, \Lambda)\big) + \eta_{p+1}(k, \Lambda) = \Lambda + o(L) + o(L) \sim L.$$

If (A.6) holds for some $p \geq 0$, then

$$\begin{aligned} D\phi_{H,p+1}(k, \Lambda) &= 1 + Dh\big(k, \phi_{H,p}(k, \Lambda)\big) D\phi_{H,p}(k, \Lambda) + D\eta_{p+1}(k, \Lambda) \\ &\longrightarrow 1 + 0 \times 1 + 0 \end{aligned}$$

by (5.11). □

Lemma A.2 estimates the difference between the $(p-1)$th and $p$th iterative approximations to $F$. We use it in the proof of Lemma A.3.

**Lemma A.2** *Let* $H := (h; \eta_1, \ldots, \eta_P)$ *be a* $P$*-ply RISC with* $N$*-residual* $\psi$. *Then for all* $p = 1, \ldots, P$, *we have*

$$\varphi^{\pm}_{H,p} - \varphi^{\pm}_{H,p-1} = O\Big(\frac{\psi}{L^{p-1}_{\pm}}\Big) \tag{A.7}$$

*as* $\theta \to \infty$.

*Proof* Without ambiguity, we suppress the $H$ subscript of $\varphi$. By (5.15), we have

$$\eta_p(k, L_{\pm}) - \eta_{p-1}(k, L_{\pm}) = O\Big(\frac{\psi}{L^{p-1}_{\pm}}\Big). \tag{A.8}$$

For $p = 1$, by definition of $\varphi$ and (A.8),

$$\varphi_1 - \varphi_0 = h(k, L) + \eta_1(k, L) = \eta_1(k, L) - \eta_0(k, L) = O(\psi),$$

so (A.7) holds for $p = 1$. If $P = 1$ we are done. Otherwise, fixing $P$ and inducting on $p$, assume that (A.7) holds for some $p \in \{1, \ldots, P-1\}$. By Definition 5.3 and (A.8) and the mean value theorem,

$$\begin{aligned} \varphi_{p+1} - \varphi_p &= h(k, \varphi_p) - h(k, \varphi_{p-1}) + \eta_{p+1}(k, L) - \eta_p(k, L) \\ &= O\Big(\frac{\psi}{L^p}\Big) + Dh(k, \Lambda)(\varphi_p - \varphi_{p-1}), \end{aligned}$$

where $\Lambda(\theta)$ is some point between $\varphi_p(\theta)$ and $\varphi_{p-1}(\theta)$. So $\Lambda \sim \varphi_p \sim \varphi_{p-1} \sim L$ by (A.5), and

$$Dh(k, \Lambda)(\varphi_p - \varphi_{p-1}) = O\Big(\frac{\varphi_p - \varphi_{p-1}}{L}\Big) = O\Big(\frac{\psi}{L^p}\Big), \quad \theta \to \infty$$

by (5.11) and the inductive hypothesis. Hence (A.7) holds with $p+1$ in place of $p$. □

In Lemma A.3, the first conclusion (A.9) estimates $|\eta^*(k, \varphi_{H,P-1}) - \eta_P|$, or equivalently (by (5.14)) the error $|h^*_N(k, \varphi_{H,P-1}) - (h + \eta_P)|$ introduced by using $h + \eta_P$ in place of $h^*_N(k, \varphi_{H,P-1})$ in the $P$th iterate. The second conclusion (A.10) helps to account for the combined error introduced in *all* iterates, by facilitating an argument given in the proof of the FAT in Proposition 5.6, which *augments* the scheme with an extra iteration—the $(P+1)$th—applied to the $P$th iterate $\varphi_{H,P}$. This argument's approach is analogous to the standard elementary technique whereby the sum of the first $P$ terms of a geometric series is analyzed by relating it to a summation augmented to include the $(P+1)$th term.

**Lemma A.3** *Let $H := (h; \eta_1, \ldots, \eta_P)$ be a $P$-ply RISC. Then*

$$\eta^*(k, \varphi_{H,P-1}) - \eta_P(k, L) = O\Big(\frac{\psi}{L^P}\Big), \tag{A.9}$$

$$\eta^*(k, \varphi_{H,P}) - \eta_P(k, L) = O\Big(\frac{\psi}{L^P}\Big). \tag{A.10}$$

*Proof* Equation (A.9) is by (5.15). Then (A.9) implies (A.10) because by the mean value theorem, there exists $\Lambda \sim \varphi_{H,P-1} \sim \varphi_{H,P}$ such that

$$\begin{aligned}|\eta^*(k, \varphi_{H,P-1}) - \eta^*(k, \varphi_{H,P})| &= |D\eta^*(\Lambda)|\,|\varphi_{H,P} - \varphi_{H,P-1}| \\ &= O(1/L)O(\psi/L^{P-1})\end{aligned}$$

by (5.11) and (A.7). □

The proof of the FAT in Proposition 5.6 will estimate $|C(k, V_H) - C(k, V)|$ as an intermediate step towards ultimately estimating the error $|V_H - V|$ of our implied volatility formula. In order to pass from the former to the latter, a part of the argument will estimate $|DC(k, V_H)|^{-1}$ in comparison to $e^{\varphi_H}$, which behaves as follows.

**Lemma A.4** *Let $H = (h; \eta_1, \ldots, \eta_P)$ be a $P$-ply RISC. Then*

$$e^{\varphi_H} = O\Big(\frac{R^{\pm}_{1\wedge N}(k/L)}{CL^{1/2}}\Big). \tag{A.11}$$

*Proof* We have

$$\begin{aligned}Ce^{\varphi_H} &= e^{h(k,\varphi_{H,P-1}) + \eta_P(k,L)} \\ &= e^{h^*_N(k,\varphi_{H,P-1})} e^{h(k,\varphi_{H,P-1}) + \eta_P(k,L) - h^*_N(k,\varphi_{H,P-1})}.\end{aligned} \tag{A.12}$$

The first factor in (A.12) is

$$e^{h_N^*(k,\varphi_{H,P-1})} = \frac{O(R^{\pm}_{1\wedge N}(k/\varphi_{H,P-1}))}{2\sqrt{\pi}\varphi_{H,P-1}^{1/2}} \exp\left(\sum_{n=1}^{N-1} \frac{B_n(k/\varphi_{H,P-1})}{\varphi_{H,P-1}^n}\right)$$
$$= O\left(\frac{R^{\pm}_{1\wedge N}(k/L)}{L^{1/2}}\right),$$

by (A.3) and boundedness of the $B_n$. The second factor is $O(1)$ by (A.9). □

Using the previous four lemmas, the proof of the FAT in Proposition 5.6 carries out Steps 1 and 2 of the argument outlined in the Sect. 5.1 overview.

*Proof of the FAT in Proposition 5.6* Suppressing the $\pm$ notation on $\{V_H, \varphi_H, L\}$, we have by (3.5) and (A.5),

$$F(k, V_H) = \varphi_H \sim L. \tag{A.13}$$

Let $N' := N \vee 1$. By (3.9)—which applies because (5.16) implies that $(k, V_H)$ satisfies (3.7)—we have

$$C(k, V_H) = \pm\frac{e^{-\varphi_H}}{\sqrt{2\pi}} \sum_{n=1}^{N'} D^n G(k, \varphi_H) + O\left(e^{-\varphi_H} \frac{R_{N'+1}(k/\varphi_H)}{\varphi_H^{N'+1/2}}\right)$$
$$= e^{-\varphi_H} \frac{R_1(k/\varphi_H)}{2\sqrt{\pi}\varphi_H^{1/2}} \left(1 + \sum_{n=1}^{N'-1} \frac{A_n(k/\varphi_H)}{\varphi_H^n} + O\left(\frac{1}{\varphi_H^{N'}}\right)\right) \tag{A.14}$$

by (3.6) and boundedness of $R_{N'+1}/R_1$. Let $\eta_{P+1}(\kappa, \lambda) := \eta^*(\kappa, \phi_H(k, \lambda))$, which is a sublog function with

$$\eta_{P+1}(k, L) = \eta^*(k, \varphi_H). \tag{A.15}$$

Let $\bar{H} := (h; \eta_1, \eta_2, \ldots, \eta_P, \eta_{P+1})$, which is a $(P+1)$-ply RISC with the same $N$-residual $\psi$ because

$$\sum_{p=0}^{P} \frac{|\eta_{p+1}(k, L) - \eta_p(k, L)|}{L^{P+1-p}} + |\eta^*(k, \varphi_{\bar{H},P}) - \eta_{P+1}(k, L)|$$
$$= \frac{1}{L} O\left(\frac{\psi}{L^P}\right) + \frac{|\eta^*(k, \varphi_{H,P}) - \eta_P(k, L)|}{L} - \frac{|\eta^*(k, \varphi_{H,P-1}) - \eta_P(k, L)|}{L} + 0$$
$$= O\left(\frac{\psi}{L^{P+1}}\right)$$

by (A.9) and (A.10). Recalling the notation $\varphi_{\bar{H}} := \varphi_{\bar{H},P+1}$ and $\varphi_H := \varphi_{H,P} = \varphi_{\bar{H},P}$, we have by Definitions 5.3 and 5.4 and by (A.15), (5.13) and (5.12) that

$$e^{\varphi_{\bar{H}}} C(k,V) = e^{h(k,\varphi_H)+\eta_{P+1}(k,L)} = e^{h^*_N(k,\varphi_H)}$$
$$= \begin{cases} \frac{R_1(k/\varphi_H)}{2\sqrt{\pi}\varphi_H^{1/2}} \exp\left(\sum_{n=1}^{N-1} \frac{B_n(k/\varphi_H)}{\varphi_H^n}\right) & \text{if } N>0, \\ \frac{1}{\varphi_H^{1/2}} & \text{if } N=0. \end{cases} \tag{A.16}$$

Combining (A.14) and (A.16), if $N>0$, then

$$C(k,V_H) = C(k,V)\exp\Bigg(\varphi_{\bar{H}} - \varphi_H + \log\Bigg(1+\sum_{n=1}^{N-1}\frac{A_n(k/\varphi_H)}{\varphi_H^n} + O\Big(\frac{1}{\varphi_H^N}\Big)\Bigg) - \sum_{n=1}^{N-1}\frac{B_n(k/\varphi_H)}{\varphi_H^n}\Bigg),$$

and if $N=0$, then

$$C(k,V_H) = C(k,V)R_1(k/\varphi_H)\exp\big(\varphi_{\bar{H}} - \varphi_H + O(1)\big).$$

Combining (2.1), and $\varphi_{\bar{H}} - \varphi_H = \varphi_{\bar{H},P+1} - \varphi_{\bar{H},P} = O(\psi/L^P)$ by (A.7), and $\psi/L^P = O(1)$ by (5.15), we get

$$C(k,V_H) - C(k,V) = C(k,V)\,O\Big(\frac{\psi}{L^P} + \frac{1}{L^N}\Big) \tag{A.17}$$

for $N \ge 0$, where the $N=0$ case uses boundedness of $R_1$.

By the mean value theorem,

$$|V_H(\theta) - V(\theta)| \le \frac{|C(k(\theta),V_H(\theta)) - C(k(\theta),V(\theta))|}{\min_{v\in I}|DC(k(\theta),v)|} \tag{A.18}$$

where $I$ denotes the interval $[V_H \wedge V, V_H \vee V]$. Indeed, the min may be taken over the set $\{V_H, V\}$ because minimizing $|DC(k(\theta),\cdot)|$ is equivalent to maximizing $F(k(\theta),\cdot)$, which by convexity attains its maximum at an endpoint. Abbreviating $F(k,V)$ as $F$ and taking $N=1$ in (3.8), which applies to $(k,V)$ by Lemma 4.1, we have

$$\begin{aligned} Q_\pm(k,V) &:= C_\pm(k,V)\Big/\Big(\frac{e^{-F}}{\sqrt{2\pi}}\frac{1}{(2F)^{1/2}}\Big) \\ &= R_1^\pm(k/F) \pm (2F)^{1/2}e^F\int_F^\infty e^{-u}D^2G_\pm(k,u)\,\mathrm{d}u \\ &\le R_1^\pm(k/F), \end{aligned} \tag{A.19}$$

because $\pm D^2 G_\pm < 0$ by (3.6). So for all $\theta$ such that $F(k, V_H) \le F(k, V)$, we have by (3.3) and (A.19) that

$$|DC(k,V)|^{-1} = \sqrt{2\pi} e^{F(k,V)} = \frac{Q_\pm(k,V)}{C(k,V)(2F(k,V))^{1/2}}$$
$$\le \frac{R_1^\pm(k/F(k,V))}{C(k,V)(F(k,V_H))^{1/2}} = O\Big(\frac{R_1^\pm(k/L)}{C(k,V)L^{1/2}}\Big), \qquad \text{(A.20)}$$

where the last expression's denominator is by (A.13), and the numerator in the $(+)$ case is because $R_1^+ \in [1,2]$, and in the $(-)$ case is because $R_1^-$ is increasing, hence

$$R_1^-\big(k/F(k,V)\big) \le R_1^-\big(k/F(k,V_H)\big) \sim R_1^-(k/L)$$

by (A.3) and (A.13). On the other hand, for all $\theta$ such that $F(k, V_H) \ge F(k, V)$,

$$|DC(k,V_H)|^{-1} = \sqrt{2\pi} e^{F(k,V_H)} = \sqrt{2\pi} e^{\varphi_{H,P}} = O\Big(\frac{R_{1\wedge N}^\pm(k/L)}{C(k,V)L^{1/2}}\Big) \qquad \text{(A.21)}$$

by (A.11). Both (A.20), (A.21) are $O((1 \wedge (k/L_-)^{N\wedge 1})C_-^{-1}L_-^{-1/2})$ in Case $(-)$, and both are $O(C_+^{-1}L_+^{-1/2})$ in Case $(+)$. Combining this with (A.17) and (A.18) produces the conclusions (5.17) and (5.18).

The final statement is by Example 5.5. □

Lemma A.5 and Corollary 5.7 convert errors of $V$ approximations into errors of $V^2$ approximations.

**Lemma A.5** *For any $\Lambda \sim L_\pm$,*

$$G_-(k,\Lambda) = O\Big(\frac{k}{L_-^{1/2}}\Big), \qquad G_+(k,\Lambda) = O\big(L_+^{1/2} + k^{1/2}\big), \qquad \text{(A.22)}$$

$$DG_-(k,\Lambda) = O\Big(\frac{1\wedge(k/L_-)}{L_-^{1/2}}\Big), \qquad DG_+(k,\Lambda) = O\Big(\frac{1}{L_+^{1/2}}\Big), \qquad \text{(A.23)}$$

$$DG_-^2(k,\Lambda) = O\Big(\frac{k}{L_-} \wedge \frac{k^2}{L_-^2}\Big), \qquad DG_+^2(k,\Lambda) = O\Big(1 + \frac{k^{1/2}}{L_+^{1/2}}\Big). \qquad \text{(A.24)}$$

*Proof* Obtain (A.22) from

$$G_-(k,\Lambda) = \frac{\sqrt{2}k}{\sqrt{\Lambda+k}+\sqrt{\Lambda}}, \qquad G_+(k,\Lambda) = O\big(\sqrt{\Lambda+k}\big),$$

and (A.23) from (A.2), (A.3) and

$$|DG_\pm(k,\Lambda)| = \frac{R_1^\pm(k/\Lambda)}{(2\Lambda)^{1/2}} \sim \frac{R_1^\pm(k/L)}{(2L)^{1/2}}.$$

Multiply (A.22) and (A.23) to produce (A.24). □

*Proof* Proof of Corollary 5.7 We have

$$
\begin{aligned}
|V_{H,\pm}^2 - V^2| &= |V_{H,\pm} + V|\,|V_{H,\pm} - V| \\
&\le (2|V_{H,\pm}| + |V_{H,\pm} - V|)\,|V_{H,\pm} - V|. \qquad \text{(A.25)}
\end{aligned}
$$

The last factor in (A.25) is already estimated by the FAT in Proposition 5.6. The other factor is, using (A.22) in Case $(-)$,

$$
\begin{aligned}
2|V_{H,-}| + |V_{H,-} - V| &= O(kL_-^{-1/2}) + O\Big(\frac{1 \wedge (k/L_-)^{N\wedge 1}}{L_-^{1/2}}\Big) O\Big(\frac{1}{L_-^N} + \frac{\psi}{L_-^P}\Big) \\
&= \begin{cases} O(kL_-^{-1/2}) & \text{if } N > 0, \\ O(kL_-^{-1/2} + L_-^{-1/2}) & \text{if } N = 0 \end{cases}
\end{aligned}
$$

as $\theta \to \infty$, where the last equality is by definition of $\psi$. In Case $(+)$,

$$
\begin{aligned}
2|V_{H,+}| + |V_{H,+} - V| &= O\big(L_+^{1/2} + k^{1/2}\big) + O\Big(\frac{1}{L_+^{1/2}}\Big) O\Big(\frac{1}{L_+^N} + \frac{\psi}{L_+^P}\Big) \\
&= O\big(L_+^{1/2} + k^{1/2}\big)
\end{aligned}
$$

as $\theta \to \infty$, again using (A.22). □

Lemmas A.6 and 5.8 use the mean value theorem to estimate the impact of replacing $L$ by a proxy $\hat{L}$. This completes Step 3 of the argument outlined in the Sect. 5.1 overview.

**Lemma A.6** *For any functions $\Lambda_0(\theta)$ and $\Lambda_1(\theta)$ such that $\Lambda_0 \sim \Lambda_1 \sim L_\pm$, let $\varepsilon := |\Lambda_0 - \Lambda_1|$. Then in Case $(-)$,*

$$
\begin{aligned}
|G_-(k, \Lambda_0) - G_-(k, \Lambda_1)| &= O\Big(\frac{1 \wedge (k/L_-)}{L_-^{1/2}}\varepsilon\Big), \\
|G_-^2(k, \Lambda_0) - G_-^2(k, \Lambda_1)| &= O\Big(\Big(\frac{k}{L_-} \wedge \frac{k^2}{L_-^2}\Big)\varepsilon\Big),
\end{aligned}
$$

*and in Case $(+)$,*

$$
\begin{aligned}
|G_+(k, \Lambda_0) - G_+(k, \Lambda_1)| &= O\Big(\frac{1}{L_+^{1/2}}\varepsilon\Big), \\
|G_+^2(k, \Lambda_0) - G_+^2(k, \Lambda_1)| &= O\Big(\Big(1 + \frac{k^{1/2}}{L_+^{1/2}}\Big)\varepsilon\Big).
\end{aligned}
$$

*Proof of Lemma A.6* For each $\theta$, the mean value theorem gives some $\Lambda(\theta)$ between $\Lambda_0$ and $\Lambda_1$, hence $\Lambda \sim \Lambda_0 \sim \Lambda_1$, with $|G_\pm^n(k, \Lambda_0) - G_\pm^n(k, \Lambda_1)| = \varepsilon|DG_\pm^n(k, \Lambda)|$, where $n \in \{1, 2\}$. Now apply Lemma A.5. □

*Proof of Lemma* 5.8 For each $\theta$, by the mean value theorem, there exists $\Lambda(\theta)$ between $L$ and $\hat{L}$, hence $\Lambda \sim L \sim \hat{L}$, such that

$$|\phi_H(k, L) - \phi_H(k, \hat{L})| = |D\phi_H(k, \Lambda)|\,|L - \hat{L}| \sim |L - \hat{L}|$$

by (A.6). By (A.5), $\phi_H(k, L) \sim \phi_H(k, \hat{L}) \sim L$, so Lemma A.6 implies the conclusion. □

Lemma 5.11 completes Step 4 of the argument outlined in the Sect. 5.1 overview.

*Proof of Lemma* 5.11 Let $g(z) := (\sqrt{2+z} \pm \sqrt{2-z})^2$. By applying Taylor's theorem to $g$ at $z = 0$, we have for all $|z| < 2$ that

$$g(z) = \sum_{m=-1}^{M} a_m^{\pm} z^{m+1} + \frac{D^{M+3} g(z_0(z))}{(M+3)!} z^{M+3}$$

for some $z_0(z) \in [0, z]$. Let $\bar{z} \in (\zeta(\limsup(k/L)), 2)$, where $\zeta(x) := 2x/(2+x)$. Let $\bar{g} := \sup\{|D^{M+3} g(z)|/(M+3)! : \ |z| \le \bar{z}\} < \infty$. Then for all $\kappa$ and all $u > 0$ with $|\kappa/u| < \bar{z}$, we have

$$\begin{aligned}
\left| G_{\pm}^2(\kappa, u - \kappa/2) - \sum_{m=-1}^{M} \frac{a_m^{\pm} \kappa^{m+1}}{u^m} \right| &= \left| u g(\kappa/u) - \sum_{m=-1}^{M} \frac{a_m^{\pm} \kappa^{m+1}}{u^m} \right| \\
&= u \frac{|D^{M+3} g(z_0(\kappa/u))|}{(M+3)!} \left(\frac{\kappa}{u}\right)^{M+3} \\
&\le \bar{g} \frac{\kappa^{M+3}}{u^{M+2}}.
\end{aligned}$$

Therefore

$$G_{\pm}^2(k, \Lambda) - \sum_{m=-1}^{M} \frac{a_m^{\pm} k^{m+1}}{(\Lambda + k/2)^m} = O\left(\frac{k^{M+3}}{(\Lambda + k/2)^{M+2}}\right) = O\left(\frac{k^{M+3}}{L^{M+2}}\right),$$

because $k/(\Lambda + k/2) = \zeta(k/\Lambda) < \bar{z}$ eventually. □

*Proof of Lemma* 5.12 This is clear from $(\hat{a} - a)(\hat{a} + a) = \hat{a}^2 - a^2$. □

*Proof of Corollary* 6.1 Define the 1-ply RISC $(h_{N=0}^*; \eta_1)$ by $\eta_1 := 0$. Then $\eta^* = 0$ and

$$|\eta_1(k, L) - \eta_0(k, L)| = |h_{N=0}^*(k, L)| = O(\log L),$$

so the RISC has 0-residual $\psi = \log L$. With $N = 0$ and $P = 1$, the FAT in Proposition 5.6 implies (6.1).

For (6.2), let $H := (h^*_{N=1}; \eta_1)$ where $\eta_1 := 0$. Then $\eta^* = 0$, so $H$ has 1-residual

$$\psi = |\eta_1(k, L) - \eta_0(k, L)| = |h^*_{N=1}(k, L)| = \left| \log \frac{1}{\sqrt{4\pi L}} + \log R_1^-\Big(\frac{k}{L}\Big) \right|$$
$$= O\Big( \log L + \max\big(0, \log(L/k)\big) \Big)$$

by (A.4). By Proposition 5.6 and the boundedness of $\max(0, \log x)/x$, we have

$$|V_H - V| = O\Big( \frac{1 \wedge (k/L)}{L^{1/2}} \Big( \frac{\log L + \max(0, \log(L/k))}{L} + \frac{1}{L} \Big) \Big) = O\Big( \frac{\log L}{L^{3/2}} \Big)$$

as claimed. □

*Proof of Corollary* 6.3 For any $\Lambda \sim L$, we have

$$h^*_{N=1}(k, \Lambda) = -\frac{1}{2} \log \Lambda + \log \frac{1}{\sqrt{\pi}} + \log \frac{R_1^+(k/\Lambda)}{2} = h(k, \Lambda) + O\Big(1 \wedge \frac{k}{L}\Big),$$

where

$$h(\kappa, \lambda) := -\frac{1}{2} \log \lambda + \log \frac{1}{\sqrt{\pi}}.$$

Define the 2-ply RISC $H := (h; \eta_1, \eta_2)$, where

$$\eta_1(\kappa, \lambda) := \log \sqrt{\pi} \qquad \Rightarrow \qquad \varphi_{H,1} = L - \frac{\log L}{2},$$
$$\eta_2(\kappa, \lambda) := \frac{1}{2} \log \Big(1 - \frac{\log \lambda}{2\lambda}\Big) + \frac{\log \lambda}{4\lambda} \qquad \Rightarrow \qquad \varphi_{H,2} = L - \frac{\log L}{2} + \frac{\log L}{4L} - \frac{\log \pi}{2}.$$

Then

$$\begin{aligned}
|\eta^*(k, \Lambda) - \eta_2(k, L)| &= |h^*_{N=1}(k, \Lambda) - h(k, \Lambda) - \eta_2(k, L)| \\
&= O\big((k \wedge L)/L + (\log L)^2/L^2\big), \\
|\eta_2(k, L) - \eta_1(k, L)| &= O(1), \\
|\eta_1(k, L) - \eta_0(k, L)| &= |\eta_1(k, L) + h(k, L)| = O(\log L),
\end{aligned}$$

so $H$ has 1-residual $\psi$ such that

$$\frac{\psi}{L^2} = O\Big( \frac{\log L}{L^2} + \frac{1}{L} + (k \wedge L)/L + (\log L)^2/L^2 \Big).$$

By the FAT in Proposition 5.6, we have (6.4) and (6.6). Then (6.3) and (6.5) follow from Lemma A.6. □

*Proof of Corollary* 7.1 For any $\Lambda \sim L$, we have $k/\Lambda \to 0$, hence

$$\log R_1^-(k/\Lambda) = \log\big(k/(2\Lambda)\big) + O(k/L)$$

as $\theta \to \infty$. So

$$h^*_{N=1}(k, \Lambda) = -\frac{1}{2}\log\Lambda + \log\frac{1}{2\sqrt{\pi}} + \log R_1^-(k/\Lambda) = h(k, \Lambda) + O(k/L),$$

where

$$h(\kappa, \lambda) := -\frac{3}{2}\log\lambda + \log\frac{\kappa}{4\sqrt{\pi}}.$$

To show that $h$ is a sublog function, note that $Dh(\kappa, \lambda) = -3/(2\lambda)$ verifies (5.11), and that $\log(1 \wedge k) = o(L)$ by (4.3), and $\log(1 \vee k) = O(k) = O(L \times k/L) = o(L)$; therefore $|\log k| = o(L)$, and for all $\Lambda \sim L$,

$$h(k, \Lambda) = O(\log L + |\log k|) = o(L), \tag{A.26}$$

which verifies (5.10). Define the 2-ply RISC $H := (h; \eta_1, \eta_2)$, where

$$\begin{aligned}
\eta_1(\kappa, \lambda) &:= -\log\frac{\kappa}{4\sqrt{\pi}} \quad \Rightarrow \quad \varphi_{H,1} = L - \frac{3}{2}\log L,\\
\eta_2(\kappa, \lambda) &:= \frac{3}{2}\log\left(1 - \frac{3\log\lambda}{2\lambda}\right) + \frac{9\log\lambda}{4\lambda}\\
&\Rightarrow \quad \varphi_{H,2} = L - \frac{3}{2}\log L + \log\frac{k}{4\sqrt{\pi}} + \frac{9\log L}{4L}.
\end{aligned}$$

Then

$$\begin{aligned}
|\eta^*(k, \Lambda) - \eta_2(k, L)| &= |h^*_{N=1}(k, \Lambda) - h(k, \Lambda) - \eta_2(k, L)|\\
&= O\big(k/L + (\log L)^2/L^2\big),\\
|\eta_2(k, L) - \eta_1(k, L)| &= O\big((\log L)^2/L^2 + |\log k|\big),\\
|\eta_1(k, L) - \eta_0(k, L)| &= |\eta_1(k, L) + h(k, L)| = O(\log L),
\end{aligned}$$

so $H$ has 1-residual $\psi$ such that

$$\frac{\psi}{L^2} = O\Big(\frac{\log L}{L^2} + \frac{(\log L)^2/L^2 + |\log k|}{L} + k/L + (\log L)^2/L^2\Big).$$

By the FAT in Proposition 5.6, with $N = 1$, we have (7.3) and (7.5). Then (7.4) and (7.3) follow from Lemma A.6. □

*Proof of Corollary* 7.2 By (7.3) and Lemma 5.8 and (5.19) with $\Lambda := \phi_H(k, \hat{L})$,

$$\begin{aligned}
&\left|\frac{k^2}{2\hat{L} + k - 3\log\hat{L} + \log(k^2/(16\pi)) + (9\log\hat{L})/(2\hat{L})} - V^2\right|\\
&\quad = O\left(\frac{k^2}{L^2}\Big(|L - \hat{L}| + \frac{|\log k| + k^2}{L}\Big)\right).
\end{aligned}$$

Then factor $k^2/(2\hat{L})$ out of the fraction, apply Taylor's theorem to $x \mapsto 1/(1 + x)$, and recalling (4.3), drop terms of order $O((k^2 + (\log k)^2)/L^2)$ to obtain (7.7).

Similar reasoning produces (7.8) from (7.5) and Lemma 5.8 and (5.20). □

*Proof of Corollary* 7.4 Insert $\hat{L} := \alpha_{-1}/T - (3\log T)/2 + \alpha_0$ into Corollary 7.2, divide by $T$, apply Taylor's theorem, drop terms of order $O(T^2 + \varepsilon T)$, and note that the coefficients of $T\log T$, $T^2 \log T$ and $T^2(\log T)^2$ all vanish. □

*Proof of Proposition* 7.5 The function

$$v \mapsto C_-(0, v) = 2\mathcal{N}(v/2) - 1 = \operatorname{erf}\big(v/\sqrt{8}\big)$$

extends to an entire function on $\mathbb{C}$, with nonzero derivative at 0. By the inverse function theorem, it has an analytic inverse $\sqrt{8}\,\mathrm{inverf}(\cdot)$ on some open neighborhood of $C_-(0,0) = 0$. So for all $\theta$ sufficiently large, we have $V = \sqrt{8}\,\mathrm{inverf}(C_-)$, which has the convergent power series representation (7.9). □

*Proposition of Corollary* 7.6 By Proposition 7.5 and Taylor's theorem,

$$\begin{aligned} V &= \sqrt{2\pi}\Big(C_- + \frac{\pi}{12}C_-^3\Big) + O\big(T^{5/2}\big) \\ &= \sqrt{2\pi}\bigg(\alpha_{1/2}T^{1/2} + \Big(\frac{\pi}{12}\alpha_{1/2}^3 + \alpha_{3/2}\Big)T^{3/2}\bigg) + O\big(T^{5/2}\big). \end{aligned}$$

Squaring this produces the result for $V^2$. □

*Proof of Corollary* 7.7 To simplify notation, we suppress the $\pm$ on each $\varrho, \varphi, R, L$. By definition,

$$h^*_{N=2}(\kappa, \lambda) = -\frac{1}{2}\log\lambda + \log\frac{1}{\sqrt{\pi}} + \log\frac{R_1(\kappa/\lambda)}{2} - \frac{1}{2}\frac{R_2(\kappa/\lambda)}{\lambda R_1(\kappa/\lambda)},$$

and let

$$h(\kappa, \lambda) := -\frac{1}{2}\log\lambda + \log\frac{1}{\sqrt{4\pi}} + \log R_1(\kappa/\lambda).$$

Define the 2-ply RISC $H := (h; \eta_1, \eta_2)$, where

$$\begin{aligned} \eta_1(\kappa, \lambda) &:= \log\frac{\varrho_1}{R_1(\kappa/\lambda)}, \\ \eta_2(\kappa, \lambda) &:= \frac{1}{2}\log\Big(1 + \frac{h(\kappa,\lambda) + \eta_1(\kappa,\lambda)}{\lambda}\Big) - \frac{h(\kappa,\lambda) + \eta_1(\kappa,\lambda)}{2\lambda} - \frac{1}{2}\frac{\varrho_2}{\varrho_1\lambda}. \end{aligned}$$

Then

$$\begin{aligned} \varphi_{H,1} &= L - \frac{\log L}{2} + \log\frac{\varrho_1}{\sqrt{4\pi}}, \\ \varphi_{H,2} &= L - \frac{\log L}{2} + \frac{\log L}{4L} - \frac{\log(\varrho_1/\sqrt{4\pi})}{2L} + \log\frac{R_1(k/\varphi_{H,1})}{\sqrt{4\pi}} - \frac{1}{2}\frac{\varrho_2}{\varrho_1 L}. \end{aligned}$$

Note that

$$\left|\frac{k}{\varphi_{H,1}} - \lim\frac{k}{L}\right| \le \delta + \frac{k}{L}\left|\frac{1}{\varphi_{H,1}/L} - 1\right| = \delta + O\Big(\frac{\log L}{L}\Big),$$

so

$$\begin{aligned}|\eta^*(k,\varphi_{H,1})-\eta_2(k,L)| &= |h^*_{N=2}(k,\varphi_{H,1})-h(k,\varphi_{H,1})-\eta_2(k,L)|\\ &= O(\delta/L)+O\big((\log L)^2/L^2\big),\\ |\eta_2(k,L)-\eta_1(k,L)| &= O\big((\log L)^2/L^2+1/L\big)+O(\delta),\\ |\eta_1(k,L)-\eta_0(k,L)| &= |\eta_1(k,L)+h(k,L)| = O(\log L).\end{aligned}$$

Therefore $H$ has 2-residual $\psi$ such that

$$\frac{\psi}{L^2}=O\Big(\frac{\log L}{L^2}+\frac{\delta}{L}+\frac{(\log L)^2/L^2}{1}\Big)=O\Big(\frac{\delta}{L}+\frac{(\log L)^2}{L^2}\Big).$$

The FAT in Proposition 5.6 then implies (7.11) and the corresponding result for $V$; and Lemma A.6 implies (7.10) and the corresponding result for $V$. □

*Proof of Corollary* 7.8 Under the $k/L\to 0$ assumption, we have (7.10) and (7.11) with $\varrho_1=\varrho_2=2$ and $\delta=k/L$. Then (7.12) follows from (7.10) plus the $O(k^2/L)$ error from (5.21). Next, (7.13) follows by combining (7.11) with the $O(k^4/L^3)$ expansion in (5.21), and then applying Taylor's theorem, with an additional error of $O((k^3+k^2\log L)/L^2)$. Taking the square root of the $V^2$ expansion then approximates $V$ according to Lemma 5.12. □

*Proof of Corollary* 7.10 Corollary 7.7 applies with $\delta=O(k^{-1/2})$, resulting in the error $O(k^{-1})$ for the implied volatility using exact $L$. Retaining the terms up to $\beta_{-1/2}k^{-1/2}$ results in an additional error $O(k^{-r-1/2})$ by Taylor's theorem. □

*Proof of Corollary* 7.11 Substitute (7.16) into Corollaries 7.7 and 7.8, apply Taylor's theorem, and drop terms of order $O((\log\theta)^2/\theta^2)$ and $O((\log\theta)^2/\theta^{5/2})$, respectively. □

*Proof of Corollary* 8.1 By Example 4.2, the Sect. 4.1 conditions are satisfied. By Friz at al. ([8], Eq. (4.2)), equation (7.14) here holds with $r=1/4$. The conclusion of Corollary 7.10 therefore holds with error $O(k^{-3/4})$.

The expression for $A$ comes from Friz at al. ([8], Eqs. (3.3) and (3.11)); indeed,

$$\begin{aligned}A=\frac{1}{2\sqrt{\pi}}\Big(\frac{2v_0}{c^2\sigma(s_+)}\Big)^{1/4-a/c^2}\exp\bigg(&-v_0\Big(\frac{\chi(s_+)}{c^2}-\frac{\sigma'(s_+)}{c^2\sigma^2(s_+)}\Big)+\frac{2a}{c^2}\log\frac{T}{\sigma(s_+)}\\ &+a\int_0^T\Big(\Psi(s_+,t)-\frac{2}{c^2(T-t)}\Big)\,\mathrm{d}t\bigg),\end{aligned}$$

together with

$$\Psi(s,t)=\frac{(s^2-s)\sin(\sqrt{-\Delta(s)}t/2)}{\sqrt{-\Delta(s)}\cos(\sqrt{-\Delta(s)}t/2)-\chi(s)\sin(\sqrt{-\Delta(s)}t/2)}$$

from del Baño Rollin et al. ([4], Eq. (11)). □

*Proof of Corollary* 8.1 First, by Figueroa-Lopez and Forde ([5], Proposition 2.2), we have $C_- = aT + O(T^2)$ so that $L = \log \frac{1}{aT} + O(T)$. By Example 4.3, the Sect. 4.1 conditions are satisfied. Let $\hat{L} := \log \frac{1}{aT}$. By Corollary 7.2,

$$\left|W(k, \hat{L}) - V^2\right| = O\left(\frac{1}{L^3} + \frac{T}{L^2}\right) = O\left(\frac{1}{L^3}\right)$$

because $T = O(L^{-m})$ for all $m > 0$. □

*Proof of Lemma* 8.5 To prove that existence of a density suffices for (8.3)–(8.5), we can assume $b = 0$ because the case of general $b$ follows immediately. Next observe that $y \mapsto \exp(\mathcal{L}_0(u_* + iy) - \mathcal{L}_0(u_*))$ is the characteristic function of $X_1$ under the measure $\mathbb{P}^*$ defined by $\mathrm{d}\mathbb{P}^*/\mathrm{d}\mathbb{P} = e^{u_* X_1}/\mathbb{E}e^{u_* X_1}$. Because $X_1$ still admits a density under $\mathbb{P}^*$, the Riemann–Lebesgue theorem implies (8.5). In particular, $X_1$ does not have a lattice distribution, so (8.4) follows. Finally, $X_1$ is not a point mass and $\mathcal{L}_0$ is a cumulant generating function, which implies (8.3).

Proceeding with the proof of (8.6), let each $\pm$ be read as $(+)$ in the case $u_* < 1$, or $(-)$ in the case $u_* > 1$. By standard option pricing results of Fourier type (in, for instance, [14] or [20]),

$$C_\pm = \pm \int_{u_*-i\infty}^{u_*+i\infty} -iQ(z)e^{T\mathcal{L}_b(z)}\,\mathrm{d}z. \tag{A.27}$$

Then by the saddlepoint result in, for instance, Olver ([16], Chap. 4, Theorem 7.1), which is valid under the hypotheses (8.3)–(8.5),

$$C_\pm = \pm T^{-1/2}e^{T\mathcal{L}_b(u_*)}\big(\gamma_0 + \gamma_{-1}T^{-1} + O(T^{-2})\big). \tag{A.28}$$

(We use only the first few terms of the expansion in [16], which suffice here. Including all terms in Olver would give a full expansion for option prices.) The result follows by taking logs and noting that $\pm\gamma_0 = |\gamma_0|$. □

*Proof of Corollaries* 8.6 *and* 8.7 Combine Lemma 8.5 with Corollary 7.11 to obtain (8.7) and (8.9). In (8.7), $\log \gamma_0$ and $Q'/Q$ are affine in $k$, and $Q''/Q$ is quadratic in $k$; so $\alpha_1$ and $\alpha_\ell$ in (8.7) do not depend on $k$, while $\alpha_0$ and $\alpha_{-1}$ in (8.7) are, respectively, affine and quadratic in $k$, which implies (8.8). Finally, (8.10) follows by direct substitution. □

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