A gamma swap on an underlying $Y$ is a weighted variance swap on $\log Y$, with weight function

$$w(y) := y/Y_0.$$  

In practice, the gamma swap monitors $Y$ discretely, typically daily, for some number of periods $N$, annualizes by a factor such as $252/N$, and multiplies by notional, for a total payoff

$$\text{Notional} \times \text{Annualization} \times \sum_{n=1}^{N} \frac{Y_n}{Y_0} \left( \log \frac{Y_n}{Y_{n-1}} \right)^2.$$  

If the contract makes dividend adjustments (as typical for single-stock gamma swaps but not index gamma swaps), then the term inside the parentheses becomes $\log((Y_n + D_n)/Y_n)$, where $D_n$ denotes the dividend payment, if any, of the $n$th period.

Gamma swaps allow investors to acquire variance exposures proportional to the underlying level. One application is dispersion trading of a basket’s volatility against its components’ single-name volatilities; as a component’s value increases, so does its proportion of the total basket value, and hence so does the desired volatility exposure of the single-name contract; this variable exposure to volatility is provided by gamma swaps, according to point 1 below. A second application is to trade the volatility skew; for example, to express a view that the skew slopes too steeply downward, the investor can go long a gamma swap and short a variance swap, to create a weighting $y/Y_0 - 1$, which is short downside variance and long upside variance. A third application is to trade single-stock variance without the caps often embedded in variance swaps to protect the seller from crash risk; in a gamma swap, the weighting inherently dampens the downside variance, so caps are typically regarded as unnecessary.

**Model-free replication and valuation**

The continuously-monitored gamma swap admits model-free replication by a static position in options and dynamic trading of shares, under conditions specified in the weighted variance swap article, which include all positive continuous semimartingale share prices $Y$ under deterministic interest rates and proportional dividends.
Explicitly, one replicates by using that article’s (7), with payoff function
\[
\lambda(y) = \frac{2}{Y_0} \left[ y \log(y/\kappa) - y + \kappa \right] = \int_0^\infty \frac{2}{Y_0K} \text{Van}(y, K) dK,
\]
where \(\text{Van}(y, K) := (K - y)^+ I_{K<\kappa} + (y - K)^+ I_{K>\kappa}\) for an arbitrary put/call separator \(\kappa\). Forms of this payoff were derived in, for instance, [2] and [3].

Therefore, in the case that the interest rate equals the dividend yield (otherwise, see the weighted variance swap article), a replicating portfolio statically holds \(2/(Y_0K) dK\) out-of-the-money vanilla calls or puts at each strike \(K\). The gamma swap model-independently has the same initial value as this portfolio of Europeans. Additionally, the replication strategy trades shares dynamically according to a “zero-vol” delta-hedge, meaning that its share holding equals the negative of what would be the European portfolio’s delta under zero volatility.

**Further properties**

Points 2–5 follow from (3). Point 1 uses only the definition (1).

1. For an index \(Y_t := \sum_{j=1}^J \theta_j Y_{j,t}\), let \(\alpha_{j,t} := \theta_j Y_{j,t}/Y_t\) be the fraction of total index value due to the quantity \(\theta_j\) of the \(j\)th component \(Y_{j,t}\). Define the cumulative dispersion \(D_t\) by
\[
\int D_t = \sum_{j=1}^J \alpha_{j,t} d[\log Y_j]_t - d[\log Y]_t.
\]

Then going long \(\alpha_{j,0}\) gamma swaps (non-dividend-adjusted) on each \(Y_j\) and short a gamma swap on \(Y\) creates the payoff
\[
\sum_{j=1}^J \alpha_{j,0} \int_0^T \frac{Y_{j,t}}{Y_{j,0}} d[\log Y_j]_t - \int_0^T \frac{Y_t}{Y_0} d[\log Y]_t = \int_0^T \frac{Y_t}{Y_0} dD_t,
\]
as noted in [2]. Hence a static combination of gamma swaps produces cumulative index-weighted dispersion.

2. By Corollary 2.7 in [1], if the implied volatility smile is symmetric in log-moneyness, and the dividend yield equals the interest rate \((q_t = r_t)\), and there are no discrete dividends, then a gamma swap has the same value as a variance swap.

3. Assuming that \(Y_T = Y_t R_{t,T}\) for all \(t\), where the time-\(t\) conditional distribution of each \(R_{t,T}\) does not depend on \(Y_t\), the gamma swap has time-\(t\) gamma equal to a discounting/dividend-dependent factor times the risk-neutral expectation
\[
\frac{2}{Y_0} \mathbb{E}_t \left( \frac{\partial^2}{\partial y^2} \bigg|_{y=Y_t} y R_{t,T} \log(y R_{t,T}) \right) = \frac{2 \mathbb{E}_t R_{t,T}}{Y_0 Y_t}.
\]

Therefore its *share gamma*, defined to be \(Y_t\) times the gamma, does not depend on \(Y_t\). This property motivates the term *gamma swap*.
4. Within the family of weight functions proportional to \( w(y) = y^n \), the gamma swap takes \( n = 1 \). In that sense, the gamma swap is intermediate between the usual logarithmic variance swap (which takes \( n = 0 \)) and an arithmetic variance swap (which, in effect, takes \( n = 2 \)).

Expressed in terms of put and call holdings, the replicating portfolios in these three cases hold, at each strike \( K \), a quantity proportional to \( K^{n-2} \). The gamma swap \( O(1/K) \) is intermediate between logarithmic variance \( O(1/K^2) \) and arithmetic variance \( O(1) \).

5. Let \( F_{X_t} \) be the characteristic function of \( X_t := \log Y_t \). Then

\[
\mathbb{E}_t \log Y_t = -iF'_{X_t}(-i).
\]

Gamma swap valuations are therefore directly computable in continuous models for which \( F_{X_t} \) is known, such as the Heston model.

References

