Corridor Variance Swap

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A corridor variance swap, with corridor $C$, on an underlying $Y$ is a *weighted variance swap* on $X := \log Y$ (unless otherwise specified), with weight given by the corridor’s indicator function:

$$w(y) := \mathbb{I}_{y \in C}. \quad (1)$$

For example, one may define an up-variance swap by taking $C = (H, \infty)$, and a down-variance swap by taking $C = (0, H)$, for some agreed $H$.

In practice, the corridor variance swap monitors $Y$ discretely, typically daily, for some number of periods $N$, annualizes by a factor such as $252/N$, and multiplies by notional, for a total payoff

$$\text{Notional} \times \text{Annualization} \times \sum_{n=1}^{N} \mathbb{I}_{Y_n \in C} \left( \log \frac{Y_n}{Y_{n-1}} \right)^2. \quad (2)$$

If the contract makes dividend adjustments (as typical for contracts on single stocks but not on indices), then the term inside the parentheses becomes $\log((Y_n + D_n)/Y_n)$, where $D_n$ denotes the dividend payment, if any, of the $n$th period.

Corridor variance swaps accumulate only the variance that occurs while price is in the corridor. The buyer therefore pays less than the cost of a full variance swap. Among the possible motivations for a volatility investor to accept this trade-off, and to buy up (or down) variance are the following. First, the investor may be bullish (bearish) on $Y$. Second, the investor may have the view that the market’s downward volatility skew is too steep (flat), making down-variance expensive (cheap) relative to up-variance. Third, the investor may be seeking to hedge a short volatility position that worsens as $Y$ increases (decreases).

Model-free replication and valuation

The continuously-monitored corridor variance swap admits model-free replication by a static position in options and dynamic trading of shares, under conditions specified in the *weighted variance swap* article, which include all positive continuous semimartingale share prices $Y$ under deterministic interest rates and proportional dividends.
Explicitly, one replicates using that article’s (7), with payoff derived in [3]:

\[
\lambda(y) = \int_{K \in C} \frac{2}{K^2} \text{Van}(y, K) dK,
\]

(3)

where \( \text{Van}(y, K) := (K - y)^+ I_{K < \kappa} + (y - K)^+ I_{K > \kappa} \) for an arbitrary put/call separator \( \kappa \).

Therefore, in the case that the interest rate equals the dividend yield (otherwise, see the weighted variance swap article), a replicating portfolio statically holds \( 2/K^2 dK \) out-of-the-money vanilla calls or puts at each strike \( K \) in the corridor \( C \). The corridor variance swap model-independently has the same initial value as this portfolio of Europeans. Additionally, the replication strategy trades shares dynamically according to a “zero-vol” delta-hedge, meaning that its share holding equals the negative of what would be the European portfolio’s delta under zero volatility.

For corridors of the type \( C = (0, H) \) or \( C = (H, \infty) \) where \( H > 0 \), taking \( \kappa := H \) in (3) yields

\[
\lambda(y) = (-2 \log(y/H) + 2y/H - 2) I_{y \in C}.
\]

(4)

This \( \lambda \), with \( H \) chosen arbitrarily, is also valid for the variance swap \( C = (0, \infty) \).

**Further properties**

1. For a small interval \( C = (a, b) \), the corridor variance swap approximates a contract on local time, in the following sense. Corridor variance satisfies

\[
V^{(a,b)}_T := \int_0^T \mathbb{1}_{X_t \in (\log a, \log b)} d(X)_t = \int_{\log a}^{\log b} L^T_x dx,
\]

by the occupation time formula, where \( L^T_x \) denotes (an \( x \)-cadlag modification of) the local time of \( X \). Therefore, at any point \( a \),

\[
\frac{1}{\log b - \log a} V^{(a,b)}_T \longrightarrow L^a_T, \quad \text{as } b \downarrow a.
\]

2. Corridor variance can arise from imperfect replication of variance. The replicating portfolio for a standard variance swap holds options at all strikes \( K \in (0, \infty) \). In practice, not all of those strikes actually trade. If we truncate the portfolio to hold only the strikes in some interval \( C \), then the resulting value does not price a full variance swap but rather a \( C \)-corridor variance swap. (Moreover, in practice not even an interval of strikes actually trade, but rather a finite set, which can replicate instead a strike-to-strike notion of corridor variance, as shown in [1].)

3. In the case \( C = (H, \infty) \) where \( H > 0 \), we rewrite (4) as

\[
\lambda(y) = \frac{2}{H} (y - H)^+ - 2(y - \log H)^+.
\]

Thus the replicating portfolio is long calls on \( Y_T \) and short calls on \( \log Y_T \).
Let $F_{X_T}$ be the characteristic function of $X_T = \log Y_T$. Then techniques in [4] and [5] price the calls on $Y_T$ and $\log Y_T$ respectively. Specifically, assuming zero interest rates and dividends, we have the following semi-explicit formula for the corridor variance swap’s fair strike:

$$
\mathbb{E}\lambda(Y_T) - \lambda(Y_0) = \frac{2}{H\pi} \int_{0-\alpha i}^{\infty-\alpha i} \text{Re} \left( \frac{F_{X_T}(z-i)}{iz-z^2} e^{-iz\log H} \right) dz \\
+ \frac{2}{\pi} \int_{0-\beta i}^{\infty-\beta i} \text{Re} \left( \frac{F_{X_T}(z)}{z^2} e^{-iz\log H} \right) dz - \lambda(Y_0),
$$

(5)

for arbitrary positive $\alpha, \beta$ such that $\alpha + 1, \beta < \sup\{p : \mathbb{E}Y_T^p < \infty\}$.

In the case $C = (0, \infty)$, equation (4) implies the fair strike formula

$$
\mathbb{E}\lambda(Y_T) - \lambda(Y_0) = -2\mathbb{E} \log(Y_T/Y_0) = 2iF'_{X_T}(0) + 2 \log Y_0.
$$

(6)

In the case $C = (H_1, H_2)$ where $0 \leq H_1 < H_2$, subtract the formula for $C = (H_2, \infty)$ from the formula for $C = (H_1, \infty)$.

In the case of nonzero interest rates or dividends, add to (5) a correction involving payoffs at all expiries in $(0, T)$, as specified in weighted variance swap article’s (7a); and in (6) replace the $Y_0$ by the forward price.

4. With discrete monitoring, the question arises, how to define up-variance and down-variance, and in particular how much variance to recognize, given a discrete move that takes $Y$ across $H$. Definition (2) recognizes the full square of each move that ends in the corridor. Alternatively, the contract specifications in [2] treat the movements of $Y$ across $H$ by recognizing a fraction of the squared move. The fraction is defined in a way that admits approximate discrete hedging, in the sense that the time-discretized implementation of the continuous replication strategy has in each period a hedging error of only third-order in that period’s return.

References


