Prime Models of Computably Enumerable Degree

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Abstract

We examine the computably enumerable (c.e.) degrees of prime models of complete atomic decidable (CAD) theories. A structure has degree $\mathbf{d}$ if $\mathbf{d}$ is the degree of its elementary diagram. We show that if a CAD theory T has a prime model of c.e. degree $\mathbf{c}$, then T has a prime model of strictly lower c.e. degree $\mathbf{b}$, where, in addition, $\mathbf{b}$ is low ($\mathbf{b}' = \mathbf{0}'$). This extends Csima's result that every CAD theory has a low prime model. We also prove a density result for c.e. degrees of prime models. In particular, if $\mathbf{c}$ and $\mathbf{d}$ are c.e. degrees with $\mathbf{d} < \mathbf{c}$ and $\mathbf{c}$ not low$_2$ ($\mathbf{c}'' > \mathbf{0}''$), then for any CAD theory T, there exists a c.e. degree $\mathbf{b}$ with $\mathbf{d} < \mathbf{b} < \mathbf{c}$ such that T has a prime model of degree $\mathbf{b}$, where $\mathbf{b}$ can be chosen so that $\mathbf{b}'$ is any degree c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{b}'$. As a corollary, we show that for any degree $\mathbf{c}$ with $\mathbf{0} < \mathbf{c} < \mathbf{0}'$, every CAD theory has a prime model of low c.e. degree incomparable with $\mathbf{c}$. We show also that every CAD theory has prime models of low c.e. degree that form a minimal pair, extending another result of Csima. We then discuss how these results apply to homogeneous models.

1 Introduction

One of the primary objectives of computable model theory is the study of the complexity of theorems in model theory from the perspective of computability theory. All languages, theories, and structures in this paper are countable. Vaught [1961] introduced the notion of a prime model. We say a model $\mathcal{A}$ of a theory T is a *prime model* if for all models $\mathcal{B}$ of T, $\mathcal{A}$ elementarily embeds into $\mathcal{B}$. All prime models of a given theory are isomorphic. As an example, the algebraic numbers form a prime model of the theory of algebraically closed fields of characteristic 0. It is known that every complete atomic theory has a prime model. However, Millar [1978] showed that there is a complete atomic decidable (CAD) theory with no decidable prime model. (Definitions of terms appear in Section 2.) This leads to the question of which classes of Turing degrees contain the degree of a prime model for any complete atomic decidable theory.

The computably enumerable (c.e.) degrees are a particularly important class of Turing degrees. They have been studied extensively since Post first asked in

1944 whether there is a c.e. degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. Such degrees were found independently by Friedberg and Muchnik in the 1950's using the priority method, which we will use here. The c.e. sets can be equivalently defined as the domains of partial computable functions and as Σ^0_1 sets. The definition we use in this paper is that a set C is *computably enumerable* (c.e.) if there is a uniformly computable sequence of computable sets $\{C_s\}_{s\in\omega}$ such that $C = \cup_s C_s$. A degree is said to be c.e. if it contains a c.e. set.

Another important class of degrees below $\mathbf{0}'$ are the low degrees. A degree $\mathbf{d} < \mathbf{0}'$ is called *low* if $\mathbf{d}' = \mathbf{0}'$. The concept of lowness has been generalized to reflect the behavior of the n^{th} jump of $\mathbf{d}$. In particular, a degree $\mathbf{d}$ is low_n if $\mathbf{d}^{(\mathbf{n})} = \mathbf{0}^{(\mathbf{n})}$.

Csima, Hirschfeldt, Knight, and Soare [2004] showed that a degree $\mathbf{d} \leq_\text{T} \mathbf{0}'$ computes a prime model for any CAD theory if and only if $\mathbf{d}$ is nonlow_2 ($\mathbf{d}'' > \mathbf{0}''$). Thus, any class of degrees that contains a nonlow_2 degree can compute a prime model of any CAD theory. It remains to determine which subsets of the low_2 degrees can always compute a prime model. We say a set of degrees $\mathbf{D}$ is a *basis for prime models* if every complete atomic decidable theory has a prime model of some degree $\mathbf{d} \in \mathbf{D}$. Bases for prime models have been studied previously, but never within the c.e. degrees. Some of the results in this paper strengthen previous non-c.e. results, and some are true only for the c.e. degrees. We state the main results below and prove them in Sections 3 and 4.

1.1 Continuity of Prime Models of C.E. Degree

There are questions we can ask about the degree spectra of prime models that do not involve bases for prime models. One such question is whether, given a prime model $\mathcal{M}$ of a particular theory T, we can always find prime models of T of degree above or below the degree of $\mathcal{M}$. Knight [1986] showed that if a CAD theory T that is nontrivial in the sense of Theorem 2.2 has a prime model of degree $\mathbf{d}$, then T has a prime model of any degree above $\mathbf{d}$. This answers one half of the question and leaves the question of whether the degree $\mathbf{d}$ of a prime model can always be pushed down to a strictly lower degree of a prime model of the same theory. If $\mathbf{d}$ is not c.e., it is not always possible to find a prime model of strictly lower degree. For instance, there are theories with prime models of minimal degree $\mathbf{d}$ and no decidable prime model. In contrast, there are no c.e. degrees of minimal degree. In Section 3 we show that given a prime model of c.e. degree, we can always find a prime model of strictly lower c.e. degree as well. In fact, the new prime model can be constructed to be low, which tells us that the low c.e. degrees form a basis for prime models.

We call this a "continuity" theorem, using the terminology of Harrington and Soare [1992], who showed that minimal pairs and degrees that cup to $\mathbf{0}'$ both have continuity properties. The name comes from the fact that for any c.e. degree $\mathbf{c}$ of a prime model of a theory T, every interval around $\mathbf{c}$ contains an open subinterval around $\mathbf{c}$ that consists entirely of degrees of prime models of T. This theorem also shows that there is no minimal c.e. degree of a prime model of a given theory.

Theorem 3.1 (Continuity of Prime Models of C.E. Degree). *Let T be a complete atomic decidable theory and let $\mathbf{c} > \mathbf{0}$ be the c.e. degree of a prime model of T. Then there is a prime model of T with low c.e. degree $\mathbf{b} < \mathbf{c}$.*

The proof of this theorem, which appears in Section 3, uses the finite injury priority method. We start with a $\mathbf{c}$-computable listing of the principal types of T and build a listing of the principal types with degree strictly below $\mathbf{c}$. The proof relies on the fact that a principal type is completely determined by a single formula, so that each type in our new listing is determined at a finite stage, allowing the injury to be finite.

It follows from Theorem 3.1 that the low c.e. degrees form a basis for prime models since every CAD theory has a prime model of some c.e. degree $\mathbf{c}$ (in fact, any nonlow$_2$ c.e. degree). This strengthens the result in [Csima, 2004] that the low degrees form a basis for prime models.

1.2 Density of Prime Models of C.E. Degree

One of the most significant results about the computably enumerable degrees is the Sacks Density Theorem [1964], which states that between any two c.e. degrees $\mathbf{d} < \mathbf{c}$ there is another c.e. degree. This raises the question of whether, for any two c.e. degrees $\mathbf{d} < \mathbf{c}$, every CAD theory has a prime model of c.e. degree between $\mathbf{d}$ and $\mathbf{c}$. However, we know that this could only possibly be true if the upper degree $\mathbf{c}$ were nonlow$_2$, because nonlow$_2$ degrees are the only degrees below $\mathbf{0}'$ that can compute prime models for every CAD theory, by [CHKS, 2004]. For $\mathbf{c}$ nonlow$_2$, we already know that the set of c.e. degrees strictly between $\mathbf{d}$ and $\mathbf{c}$ form a basis for prime models because it contains nonlow$_2$ degrees. Thus we strengthen our question to ask if we can also choose the jump degree of our prime model to be any degree $\mathbf{s}$ that is c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$. Such degrees exist between $\mathbf{d}$ and $\mathbf{c}$ by the Robinson Jump Interpolation Theorem [1971].

Theorem 4.1 (Density of Prime Models of C.E. Degree). *Let T be a complete atomic decidable theory, let $\mathbf{c}$ be a c.e. nonlow$_2$ degree, and let $\mathbf{d}$ be a c.e. degree with $\mathbf{d} < \mathbf{c}$. Then for any degree $\mathbf{s}$ c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$, there is a c.e. degree $\mathbf{b}$ with $\mathbf{d} < \mathbf{b} < \mathbf{c}$ and $\mathbf{b}' = \mathbf{s}$ such that $\mathbf{b}$ is the degree of a prime model of T.*

To prove this, we instead prove that we can make $\mathbf{b}' = \mathbf{d}'$ and use the Robinson Jump Interpolation Theorem [1971] and Knight's Upward Closure Theorem 2.2 to show that we can push the degree of a prime model up so that its jump degree is $\mathbf{s}$. The proof, which will be given in Section 4, uses the infinite injury priority method combined with a permitting argument that ensures $\mathbf{b}$ is computable in $\mathbf{c}$. In order to enumerate an element into our c.e. set B of degree $\mathbf{b}$, we must have a low enough restraint function and simultaneously have permission from $\mathbf{c}$. We balance these requirements by allowing $\mathbf{c}$ to permit in a given window of opportunity so that it does not need to permit at the exact moment that the restraint function is low enough for the desired element to enter B.

As a corollary, we show that for any degree $\mathbf{c}$ with $\mathbf{0} < \mathbf{c} < \mathbf{0}'$, the low c.e. degrees not comparable with $\mathbf{c}$ form a basis for prime models. This strengthens a result of Csima [2004] that says that for $\mathbf{c}$ low, the low degrees not comparable with $\mathbf{c}$ form a basis for prime models.

We show in a corollary of either Theorem 3.1 or Theorem 4.1 that for every CAD theory T, there exists a minimal pair of low c.e. degrees of prime models of T. This also strengthens a result of Csima [2004] that gives the existence of a minimal pair of low degrees of prime models, not necessarily c.e.

Theorem 4.1 and its corollaries all show that the c.e. degrees of prime models are scattered throughout the c.e. degrees.

1.3 Homogeneous Models

In the final section, we discuss homogeneous models. A model $\mathcal{M}$ is *homogeneous* if every finite partial automorphism of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. We say a homogeneous model has a $\mathbf{0}$-basis if there is a computable listing of computable indices of all types realized in $\mathcal{M}$. Using a theorem of Lange [ip] that relates homogeneous models with a $\mathbf{0}$-basis to CAD theories, we can transform Theorem 4.1 and its corollaries to theorems about degrees of isomorphic copies of homogeneous models.

2 Method of Approach

2.1 Atomic Models and Binary Trees

Let T be a theory in the language $\mathcal{L}$. T is *complete* if for all sentences φ in the language $\mathcal{L}$, $\varphi \in T$ or $\neg\varphi \in T$. An *n-type* p of T is a maximal consistent set of formulas in n free variables. Let $\mathcal{M}$ be a model of T and p an n-type of T. We say $\mathcal{M}$ *realizes* the type p if there exists an n-tuple $\overline{m}$ in $\mathcal{M}$ such that $\mathcal{M} \models \varphi(\overline{m})$ for all $\varphi \in p$.

A formula $\theta(\overline{x})$ consistent with T is an *atom* if for all $\psi(\overline{x})$, either $T \vdash (\theta \to \psi)$ or $T \vdash (\theta \to \neg\psi)$, but not both. A type p is *principal* if there exists an atom θ in p. We say a theory T is *atomic* if for each $\varphi(\overline{x})$ consistent with T, there exists an atom $\theta(\overline{x})$ such that $T \vdash (\theta \to \varphi)$. A model $\mathcal{M}$ of a theory is *atomic* if each n-tuple in $\mathcal{M}$ satisfies an *atom* θ of T. Thus, $\mathcal{M}$ is atomic if the only types realized in $\mathcal{M}$ are the principal types.

It is a well-known theorem of model theory that for a theory T, $\mathcal{M}$ is a prime model of T if and only if $\mathcal{M}$ is an atomic model of T. Therefore, to build a prime model of T, it suffices to build a model that realizes only the principal types of T.

Let $\mathcal{M}$ be a model of a theory T over the language $\mathcal{L}$. Let $\mathcal{L}(\mathcal{M})$ be the language given by the union of $\mathcal{L}$ with a set of new constant symbols for each element of $\mathcal{M}$. The *elementary diagram* of $\mathcal{M}$, $D(\mathcal{M})$, is the set of all sentences in $\mathcal{L}(\mathcal{M})$ that are true in $\mathcal{M}$.

A complete theory T is *decidable* if the set of sentences in T is computable. We say that a model $\mathcal{M}$ of T is *decidable* if its elementary diagram $D(\mathcal{M})$ is computable, and that $\mathcal{M}$ has degree $\mathbf{d}$ if the Turing degree of $D(\mathcal{M})$ is $\mathbf{d}$. This is a somewhat nonstandard definition, as the degree of a model is often defined to be the degree of its atomic diagram. The following theorem gives motivation for our definition, and will be used throughout this paper.

Theorem 2.1 (**Goncharov and Nurtazin (1973), Harrington (1974)**). *If T is a complete, atomic, decidable theory, then the following are equivalent:*

$$T \text{ has a decidable prime model.} \tag{1}$$

$$\text{There is a computable listing of the principal types of } T. \tag{2}$$

This theorem can be relativized to state that T has a $\mathbf{d}$-decidable prime model if an only if there is a $\mathbf{d}$-computable listing of the principal types of T.

By looking at types as paths through trees, we can prove theorems about prime models while working only with trees. Let $\mathcal{T} \subset 2^{<\omega}$ be a binary tree. We say $\mathcal{T}$ is *extendible* if there are no terminal nodes on $\mathcal{T}$. A node $\tau \in \mathcal{T}$ is an *atom* if for all σ_1, σ_2 extending τ on $\mathcal{T}$, either $\sigma_1 \subseteq \sigma_2$ or $\sigma_2 \subset \sigma_1$. That is, there is no splitting on the tree above τ. If $\mathcal{T}$ is extendible and τ is an atom, then there is exactly one path extending τ on the tree. We say that a path p through $\mathcal{T}$ is *isolated* if there exists an atom $\tau \subset p$. The tree $\mathcal{T}$ has *isolated paths dense* if for every $\sigma \in \mathcal{T}$, there is an isolated path p such that $\sigma \subset p$.

We may think of n-types of a complete theory T as paths through a binary tree $\mathcal{T}_n(T)$ whose nodes correspond to formulas in n variables consistent with T. The isolated paths of $\mathcal{T}_n(T)$ correspond to the principal n-types of T. As shown in [CHKS, 2004], if T is a complete, atomic, decidable theory, then $\mathcal{T}_n(T)$ is a computable, extendible tree with isolated paths dense. By combining the trees $\mathcal{T}_n(T)$ for all $n \in \omega$ onto one tree in an appropriate manner, as in [CHKS, 2004], the resulting tree $\mathcal{T}(T)$ is a computable, extendible tree with isolated paths dense. Furthermore, given a listing of the isolated paths of $\mathcal{T}(T)$, we can compute a listing of the principal types of T. Thus, to show that a certain set of degrees $\mathbf{D}$ can compute a prime model of any CAD theory, it suffices to show that for every computable, extendible tree $\mathcal{T}$ with isolated paths dense, there is a $\mathbf{d}$-computable listing of the isolated paths of $\mathcal{T}$ for some $\mathbf{d} \in \mathbf{D}$. The converse is also true, as shown in [CHKS, 2004]. That is, if $\mathbf{D}$ is a basis for prime models, then $\mathbf{D}$ is a basis for listings of the isolated paths of computable, extendible trees with isolated paths dense.

2.2 Upward Closure

Theorem 2.2 (**Upward Closure Theorem, Knight (1986)**). *Let $\mathcal{M}$ be a countable structure with degree $\mathbf{d}$. Then either there exists for any $\mathbf{b} > \mathbf{d}$ a* *structure $\mathcal{B} \cong \mathcal{M}$ such that $\mathcal{B}$ has degree $\mathbf{b}$, or all isomorphic copies of $\mathcal{M}$ have the same degree. The latter case occurs only in the trivial case where there is a finite subset S of the universe of $\mathcal{M}$ such that all permutations that fix S are automorphisms of $\mathcal{M}$.*

Since all prime models of a complete theory are isomorphic, this theorem says that either the set of degrees of prime models of T are closed upward or that all prime models of T have the same degree. In the latter case, all prime models of T are computable, so we will thus focus only on the former nontrivial case. All theories in this paper will be assumed to have prime models in some noncomputable degree.

Now, in order to show that $\mathbf{D}$ is a basis for prime models, it suffices to show that for every CAD theory T, T has a prime model computable in some $\mathbf{d} \in \mathbf{D}$. Combining this with the method of converting models to trees, we need only show that for every computable, extendible tree $\mathcal{T}$ with isolated paths dense, there is a listing of the isolated paths of $\mathcal{T}$ computable in some $\mathbf{d} \in \mathbf{D}$.

2.3 C.E. Sets and Permitting Arguments

Throughout this paper, we show that CAD theories have prime models of certain c.e. degrees. It is important to note that we are not claiming that all CAD theories have prime models whose elementary diagrams are themselves c.e. *sets*. Let $\mathcal{M}$ be a structure in a language $\mathcal{L}$. If $D(\mathcal{M})$ is a c.e. set, then for any sentence φ in the language $\mathcal{L}(\mathcal{M})$, we can compute whether $\varphi \in D(\mathcal{M})$ by enumerating the elements of $D(\mathcal{M})$ and waiting until either φ or $\neg\varphi$ enters $D(\mathcal{M})$. Thus $D(\mathcal{M})$ is a c.e. set if and only if it is computable.

Therefore, we instead find models $\mathcal{M}$ of c.e. *degree*. That is, $D(\mathcal{M}) \equiv_{\mathrm{T}} C$ for some c.e. set C. In this paper, we find Δ^0_2 listings of isolated paths that are computable in some c.e. set B with certain desired properties. Then, by Upward Closure, we know there is a prime model $\mathcal{M}$ with $D(\mathcal{M}) \equiv_{\mathrm{T}} B$, so $\mathcal{M}$ has c.e. degree.

In order to ensure that a listing of isolated paths A is computable in a c.e. set B, we will use a permitting argument as in [Soare, 1987, Section V.3]. Whenever $A_{s+1}(n) \neq A_s(n)$, we enumerate a certain element into B_{s+1} so that when B has settled through a certain point, $A(n)$ has also settled.

2.4 Conventions and Notation

Let $C \subset \omega$. Then C can be viewed as an element of 2^ω by means of its characteristic function. Let $C \upharpoonright n \in 2^n$ be the first n bits of C. That is, $C \upharpoonright n = C(0)C(1)\ldots C(n-1)$. Let $C \upharpoonright\upharpoonright n = C \upharpoonright (n+1)$.

Let Φ^C_e be the eth Turing functional with oracle C. We define the *use* $\varphi^C_e(n)$ of the computation $\Phi^C_e(n)$ by

$$\varphi^C_e(n) = \begin{cases} m+1, & \text{if } \Phi^C_e(n)\downarrow, \text{ where } m \text{ is the largest bit of } C \\ & \text{queried in the computation } \Phi^C_e(n) \\ 0, & \text{otherwise.} \end{cases}$$

Let C be a c.e. set with computable enumeration $C = \cup_{s\in\omega} C_s$. We write $\Phi_e^C(n)[s]$ to mean $\Phi_{e,s}^{C_s}(n)$, the result of the Turing functional Φ_e with oracle C_s on input n, after s steps of computation. Similarly, we write $\varphi_e^C(n)[s]$ for the use of that computation.

For a given enumeration $\{C_s\}_{s\in\omega}$ of a c.e. set C, we define, for $s > 0$,

$$\widehat{\Phi}_e^C(n)[s] = \begin{cases} \Phi_e^C(n)[s], & \text{if } \Phi_e^C(n)[s]\downarrow \text{ and } C_s\restriction\varphi_e^C(n)[s] = C_{s-1}\restriction\varphi_e^C(n)[s] \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

For notational simplicity, given an enumeration of C, we will assume we have converted $\Phi_e^C[s]$ to $\widehat{\Phi}_e^C[s]$, and write it without the hat. By this convention, if $\Phi_e^C(n)$ diverges, then $\Phi_e^C(n)[s]$ diverges for infinitely many s. This is called the "hat trick".

Let $\langle a,b,c\rangle = \langle\langle a,b\rangle,c\rangle$, where $\langle x,y\rangle$ is the standard pairing function from ω^2 to ω. This allows us to view sets as three-dimensional matrices, where each "row" of the matrix is itself a matrix.

3 Continuity of Prime Models of C.E. Degree

Knight's Upward Closure Theorem 2.2 tells us that given a prime model of degree $\mathbf{c}$, we can find a prime model with higher degree. In this section, we will show that if $\mathbf{c}$ is c.e., we can always find a prime model of strictly lower degree $\mathbf{b}$ such that $\mathbf{b}$ is low. Using the language of [Harrington and Soare, 1992], we call this a continuity result for c.e. prime models.

For non-c.e. degrees, this result does not hold. Hirschfeldt [2006] showed that for any CAD theory T with all types computable, T has a prime model of every noncomputable degree. Thus, T has a prime model of minimal degree $\mathbf{d}$. That is, there is no degree between $\mathbf{0}$ and $\mathbf{d}$. By [Millar, 1978], there exists a theory T with all types computable and no decidable prime model, so T has a prime model of degree $\mathbf{d}$ and no prime model of strictly lower degree.

Theorem 3.1 (Continuity of Prime Models of C.E. Degree). *Let T be a complete atomic decidable theory and let $\mathbf{c} > \mathbf{0}$ be the c.e. degree of a prime model of T. Then there is a prime model of T with low c.e. degree $\mathbf{b} < \mathbf{c}$.*

Using the techniques of Section 2, it suffices to show the following equivalent theorem.

Theorem 3.2. *Let $\mathcal{T}$ be a computable tree with isolated paths dense which is extendible (i.e. has no terminal nodes). Let C be a noncomputable c.e. set such that C computes a listing of the isolated paths of $\mathcal{T}$. Then there exist sets A and B such that $A \leq_{\mathrm{T}} B <_{\mathrm{T}} C$ with B c.e. and low, and A a listing of the isolated paths of $\mathcal{T}$.*

Proof. Given a noncomputable c.e. set C, we will build A and B to satisfy the theorem. Let $\{\sigma_e\}_{e\in\omega}$ be a computable listing of the nodes of $\mathcal{T}$. Let $A^{[e]}$ denote the e^{th} row of A, and let $A_s^{[e]}$ be the approximation of $A^{[e]}$ at stage s.

We will meet the requirements:

P_e: $A^{[e]} = \lim_s A_s^{[e]}$ is an isolated path of $\mathcal{T}$ extending σ_e.

N_e^1 ($C \not\leq_{\mathrm{T}} B$): $\Phi_e^B = C \implies C$ is computable.

N_e^2 (B is low): $(\Phi_e^B(e)[s]\!\downarrow$ for infinitely many $s) \implies \Phi_e^B(e)\!\downarrow$.

R: $A \leq_{\mathrm{T}} B \leq_{\mathrm{T}} C$.

Let $M \leq_{\mathrm{T}} C$ be a matrix listing the isolated paths of $\mathcal{T}$ such that $M^{[e]}$ extends the node σ_e of $\mathcal{T}$. We will build A so that its e^{th} row $A^{[e]}$ is a row in M that extends σ_e, satisfying requirement P_e. Since A will include rows extending each atom of $\mathcal{T}$, A will be a listing of the isolated paths in $\mathcal{T}$.

To do this, we need to use a computable approximation of M. Let $C = \cup_s C_s$ be a computable enumeration of C. Using the enumeration of C and the reduction from M to C, define a computable approximation $\{M_s\}_{s\in\omega}$ of M with $\lim_s M_s(x) = M(x)$ such that $\sigma_e \subset M_s^{[e]}$ and $M_s^{[e]}\!\restriction\! s \in \mathcal{T}$ for all $e \in \omega$.

In order to ensure $A \leq_{\mathrm{T}} B$, whenever $A_{s+1}^{[e]}(n) \neq A_s^{[e]}(n)$ we will enumerate an element of the form $\langle y, m, e\rangle$ into B, for some $m \leq n$, thus permitting the change. We can imagine a movable marker that rests on the least element of the form $\langle y, m, e\rangle$ not yet enumerated into B, for a given m and e. If t is large enough such that all markers have stopped moving for a fixed e and for all $m \leq n$ by stage t, then $A_t^{[e]}(n) = A^{[e]}(n)$.

We will construct A by having $A_s^{[e]}$ copy $M_s^{[e]}$ as long as it can while preserving initial segments of A needed for the requirements N_i^1 and N_i^2, $i \leq e$, which have higher priority than P_e. If copying $M_s^{[e]}$ would lead to injuring a higher-priority restraint, then $A_s^{[e]}$ will begin copying a higher row of M_s. We will continue this process of choosing higher rows for $A^{[e]}$ to copy until eventually, as we will see, $A^{[e]}$ will settle on a row of M, so that $A^{[e]} = M^{[j]}$ for some $j \geq e$.

To meet requirement N_e^1, we will build restraints on B. Let

$$l(e,s) = \max\{x \mid (\forall y < x)(\Phi_e^B(y)[s]\!\downarrow = C_s(y))\}.$$

This is the least place of disagreement between $\Phi_e^B[s]$ and C_s. Let

$$r^1(e,s) = \max\{\varphi_e^B(y)[s] \mid y \leq l(e,s)\},$$

the maximum amount of B_s used in the computations up through the least disagreement, and let

$$R^1(e,s) = \max\{r^1(i,v) \mid i \leq e \text{ and } v \leq s\}.$$

To meet requirement N_e^2, let

$$r^2(e,s) = \varphi_e^B(e)[s].$$

We will need our restraints to be non-decreasing, so let

$$R^2(e,s) = \max\{r^2(i,s) \mid i \leq e \text{ and } v \leq s\}$$

Combine the negative restraints to get:

$$R(e,s) = \max\{R^1(e,s), R^2(e,s)\}.$$

In fact, the restraint $R^2(e,s)$ is unnecessary. As shown in [Soare, 1987, p.125] it is possible to prove that B is low using only the restraint for the requirement that B does not compute C.

We will define $k(e,s)$ to be the number of the row in M that $A^{[e]}$ is trying to copy at stage s.

Construction.

Stage s=0. Let $A^{[e]}_0 = \emptyset$ and $k(e,0) = e$ for all e, and let $B_0 = \emptyset$.

Stage s+1. For each $e \leq s$:

Let $k(e,s+1)$ be the least $k \geq k(e,s)$ such that $M^{[k]}_{s+1} \restriction R(e,s) = A^{[e]}_s \restriction R(e,s)$ and $\sigma_k \supseteq \sigma_e$. Let $A^{[e]}_{s+1} = M^{[k(e,s+1)]}_{s+1} \restriction s+1$. Note that $A^{[e]}$ will continue to copy $M^{[k(e,s)]}$ unless there is a change at stage $s+1$ in $M^{[k(e,s)]}$ in the first $R(e,s)$ bits. When there is such a change, $A^{[e]}_{s+1}$ will begin copying the next highest row of M that agrees with $A^{[e]}_s$ in the first $R(e,s)$ bits. If $A^{[e]}_{s+1}(n) \neq A^{[e]}_s(n)$ for some $n < s$, then for the least such n, enumerate $\langle y_s, n, e\rangle$ into B_{s+1}, where y_s is the least value of y such that $\langle y, n, e\rangle \notin B_s$.

For each $e > s$, let $A^{[e]}_{s+1} = A^{[e]}_s$ and $k(e,s+1) = k(e,s)$.

Verification.

Let $B^{[e]}$ be the set of all elements in B of the form $\langle y, n, e\rangle$ for any $y, n \in \omega$, and let $B^{[e]}_s$ be the set of all such elements in B_s. Let $B^{[<e]} = \cup_{i<e} B^{[i]}$ and $B^{[<e]}_s = \cup_{i<e} B^{[i]}_s$. Note that the requirement P_e enumerates elements only into $B^{[e]}$. Note also that restraining the first $R(e,s)$ bits of $A^{[e]}_s$ simultaneously restrains the first $R(e,s)$ bits of $B^{[e]}_s$ because $\langle y, n, e\rangle \geq n$.

Lemma 3.3. *For all $e \in \omega$, $\Phi^B_e \neq C$, $\lim_s R(e,s)$ exists, only finitely many elements are enumerated into B for each P_e, and each P_e is satisfied.*

Proof. The proof is by induction on e. Suppose true for all $i < e$. By the inductive hypothesis, there is a stage s_e by which $B^{[i]}_{s_e} = B^{[i]}$ for all $i < e$.

Suppose $\Phi^B_e = C$. Then we can compute C given $B^{[<e]}$ as follows. To compute $C(x)$, find a stage s such that $l(e,s) > x$ and $B^{[<e]}_s = B^{[<e]}$. Then the R^1 restraint ensures that nothing can injure $\Phi^B_e(x)[s]$, so $\Phi^B_e(x)[s] = \Phi^B_e(x) = C(x)$. Now since $B^{[i]}$ is finite for all $i < e$, C is computable in a finite set, so C is computable, contradicting the assumption that $\emptyset <_{\mathrm{T}} C$. Thus $\Phi^B_e \neq C$.

Let d be the least place of disagreement between C and Φ_e^B. That is, for all $y < d$, $C(y) = \Phi_e^B(y)$, and either $\Phi_e^B(d)$ diverges or $C(d) \neq \Phi_e^B(d)$. Then at some stage $v > s_e$, d is the least place of disagreement between $\Phi_e^B[v]$ and C_v, and if $\Phi_e^B(d)[v]$ diverges, then it never later converges. Such a stage exists because if $\Phi_e^B(d)[v]$ converges, then the computation is preserved by the R^1 restraint. Note that $R^1(e,s)$ will preserve $\Phi_e^B[v] \Uparrow d$ for all $s \geq v$ so $l(e,s) = d$ for all $s \geq v$ and $\lim_s r^1(e,s)$ is finite. Thus, $\lim_s R^1(e,s)$ is also finite, by induction.

If $\Phi_e^B(e)[s]\downarrow$ for any $s > s_e$, then $r^2(e,s)$ will preserve this computation and thus $R^2(e,s)$ will stay the same thereafter. If $\Phi_e^B(e)[s]$ diverges for each $s > s_e$, then $\lim_s r^2(e,s) = 0$. Thus, $R^2(e,s)$ and $R(e,s)$ come to a limit.

If $k(e,s)$ rises high enough, it will land on a row k such that $|\sigma_k| > \lim_s R(e,s)$, and then it will never increase again. Thus $k(e,s)$ comes to a limit $k(e)$. Let s be a stage such that $k(e,s) = k(e)$ and $M_s^{[k(e)]}$ has settled through an atom. That is, there exists an atom $\tau \subset M_s^{[k(e)]}$ and for all $t \geq s$, $M_t^{[k(e)]} \restriction |\tau| = M_s^{[k(e)]} \restriction |\tau|$. Then no new elements will enter $B^{[e]}$ after stage s. Therefore, only finitely many elements are enumerated into $B^{[e]}$. Requirement P_e is satisfied because $A^{[e]} = M^{[k(e)]}$, which is an isolated path extending σ_e. □

Lemma 3.4. *B is low. That is, $B' \equiv_{\mathrm{T}} 0'$.*

Proof. Define

$$g(e,s) = \begin{cases} 1, & \text{if } \Phi_e^B(e)[s]\downarrow \\ 0, & \text{otherwise.} \end{cases}$$

We have shown in Lemma 3.3 that $\lim_s \varphi_e^B(e)[s] = \lim_s r^2(e,s)$ exists. If $\Phi_e^B(e)$ converges, then there exists t such that for all $s \geq t$, $\varphi_e^B(e)[s] = \varphi_e^B(e)$, so for all $s \geq t$, $g(e,s) = 1$. Similarly, if $\Phi_e^B(e)$ diverges, then there exists t such that for all $s \geq t$, $\Phi_e^B(e)[s]\uparrow$. Thus, for all $s \geq t$, $g(e,s) = 0$. Therefore $\lim_s g(e,s) = B'$. By the Limit Lemma (see [Soare, 1987]), $\lim_s g(e,s) \leq_{\mathrm{T}} 0'$, so $B' \leq_{\mathrm{T}} 0'$. Thus, B is low. □

Lemma 3.5. *$A \leq_{\mathrm{T}} B$.*

Proof. We can B-computably determine $A^{[e]} \restriction n$ uniformly in n as follows. For each $m \leq n$, find

$$s(m) = (\mu s)[B_s \Uparrow \langle y_s, m, e\rangle = B \Uparrow \langle y_s, m, e\rangle],$$

where y_s is the least y such that $\langle y, m, e\rangle \notin B_s$. This exists for each m since only finitely much is enumerated into each $B^{[e]}$. Now after stage $s(m)$, no new elements of the form $\langle y, m, e\rangle$ are enumerated into $B^{[e]}$. Let $s_n = \max_{m \leq n}\{s(m), n\}$. Then $A_{s_n}^{[e]} \restriction n = A^{[e]} \restriction n$. Thus $A \leq_{\mathrm{T}} B$. □

Lemma 3.6. *$B \leq_{\mathrm{T}} C$.*

Proof. We can C-computably find a stage, as follows, by which $\langle y, n, e\rangle$ will have entered B if it ever does. Note that such a stage depends only on n and e, not on y.

Define $k_0 = e$ and $k_{i+1} = k(e, s_i)$ where s_i is the smallest $s \geq s_{i-1}$ such that $M_s^{[k_i]} \upharpoonright n = M_t^{[k_i]} \upharpoonright n$ for all $t \geq s$. Note that $\{k_i\}_{i\in\omega}$ is computable in C. Let j be the least value of i such that $k_{i+1} = k_i$. Such a j exists since $k(e, s)$ comes to a limit by the proof of Lemma 3.3.

If $R(e, s) \leq n$ for all $s \geq s_j$, then since $M_s^{[k_j]} \upharpoonright n$ has settled, $\lim_s k(e, s) = k_j$. Now if $R(e, s) > n$ for any $s \geq s_j$ then $A_s^{[e]} \upharpoonright n$ will be preserved forever by the restraint, even if $k(e, s)$ changes. Thus, in either case, $A_{s_j}^{[e]} \upharpoonright n$ will be preserved. Hence $\langle y, n, e\rangle \in B \iff \langle y, n, e\rangle \in B_{s_j}$.

$\square$

$\square$

4 Density of Prime Models of C.E. Degree

The Sacks Density Theorem [1964] states that between any two c.e. degrees, there is another c.e. degree. In order for the set of degrees between any two c.e. degrees $\mathbf{d} < \mathbf{c}$ to be a basis for prime models, it is necessary that $\mathbf{c}$ be nonlow$_2$, as otherwise, there is a CAD theory that has no prime model below $\mathbf{c}$ by [CHKS, 2004]. If we choose $\mathbf{c}$ to be a nonlow$_2$ c.e. degree, and $\mathbf{d}$ to be any c.e. degree below $\mathbf{c}$, then the set of c.e. degrees strictly between $\mathbf{d}$ and $\mathbf{c}$ contains nonlow$_2$ degrees by the Jump Interpolation Theorem [Robinson, 1971]. Therefore, the set of c.e. degrees between $\mathbf{d}$ and $\mathbf{c}$ is trivially a basis for prime models. However, if $\mathbf{d}$ is low$_2$, then we can ask if the set of c.e. degrees between $\mathbf{d}$ and $\mathbf{c}$ that have the same jump as $\mathbf{d}$ form a basis. We show that it is a basis, and that in fact we can choose the jump of a prime model between $\mathbf{d}$ and $\mathbf{c}$ to be any degree c.e. in $\mathbf{c}$ and above $\mathbf{d}'$.

Theorem 4.1 (Density of Prime Models of C.E. Degree). *Let T be a complete atomic decidable theory, let $\mathbf{c}$ be a c.e. nonlow$_2$ degree, and let $\mathbf{d}$ be a c.e. degree, $\mathbf{d} < \mathbf{c}$. Then for any degree $\mathbf{s}$ c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$, there is a c.e. degree $\mathbf{b}$ with $\mathbf{d} < \mathbf{b} < \mathbf{c}$ and $\mathbf{b}' = \mathbf{s}$ such that $\mathbf{b}$ is the degree of a prime model of T.*

We will prove the theorem in terms of trees:

Theorem 4.2. *Let $\mathcal{T}$ be a computable, extendible tree with isolated paths dense. Let C be c.e. and nonlow$_2$, and let D be c.e. with $D <_{\mathrm{T}} C$. Then there is a c.e. set B with $D <_{\mathrm{T}} B <_{\mathrm{T}} C$ and $D' \equiv_{\mathrm{T}} B'$ such that B computes a listing of the isolated paths of $\mathcal{T}$.*

Note that for the special case where D is computable, this is Theorem 3.1 with C nonlow$_2$.

Theorem 4.1 follows by the methods of Section 2 and by the Robinson Jump Interpolation Theorem [1971], which states that given c.e. sets $B <_{\mathrm{T}} C$ and S

c.e. in C with $B' \leq_{\mathrm{T}} S$, then there is a c.e. set $\widehat{B}$ between B and C such that $\widehat{B}' \equiv_{\mathrm{T}} S$. By Knight's Upward Closure Theorem 2.2, if B is the same degree as a prime model of T, then so is $\widehat{B}$.

Proof. Let $C = \cup_s C_s$ and $D = \cup_s D_s$ be computable enumerations of C and D. Let $\{\sigma_e\}_{e \in \omega}$ be a listing of the nodes of $\mathcal{T}$. Let $A^{[e]}$ be the e^{th} row of A and let $A^{[e]}_s$ denote the approximation of $A^{[e]}$ at stage s.

We will meet the requirements:

P$_0$: $D \leq_{\mathrm{T}} B^{[0]}$.

P$_e$ ($e > 0$): $A^{[e]} = \lim_s A^{[e]}_s$ is an isolated path of $\mathcal{T}$ extending σ_e.

N: $B' \leq_{\mathrm{T}} D'$.

R: $A \leq_{\mathrm{T}} B \leq_{\mathrm{T}} C$.

Note that these requirements only guarantee that $D \leq_{\mathrm{T}} B \leq_{\mathrm{T}} C$, and so it is possible that $D \equiv_{\mathrm{T}} B$ or $B \equiv_{\mathrm{T}} C$. If $D \equiv_{\mathrm{T}} B$, we can use the Jump Interpolation Theorem to find a new $\widehat{B}$ strictly between B and C such that $\widehat{B}' \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} D'$, and by Knight's Upward Closure Theorem, $\widehat{B}$ will satisfy the theorem. If $B \equiv_{\mathrm{T}} C$, then D is nonlow$_2$ by requirement N, so the theorem is trivially satisfied by any c.e. set strictly between D and C.

To meet the negative requirement, we build a restraint preserving the use of the computation $\Phi^B_e(e)[s]$. For $e > 0$, let

$$r(e,s) = \varphi^B_e(e)[s], \quad R(e,s) = \max\{r(i,s) \mid i \leq e\}.$$

These walls of restraint may be injured infinitely often, but we will show that they will still ensure $B' \leq_{\mathrm{T}} D'$.

Let $\tau_{e,s}$ be our guess of an atom extending σ_e at stage s. We will build our approximation $A^{[e]}_s$ to an isolated path extending σ_e so that $A^{[e]}_s$ lies on the lexicographically least path extending $\tau_{e,s}$ and $A^{[e]}_s$ has length s. By ensuring that $\tau_{e,s}$ comes to a limit on an actual atom extending σ_e, $\lim_s A^{[e]}_s = A^{[e]}$ will be an isolated path extending σ_e, as desired.

Let $\gamma^s_{e,m}$ be a marker on an element of the form $\langle y, m, e\rangle$ not yet in B. We will only change $A^{[e]}_s(n)$ if a new element is enumerated into B below $\gamma^s_{e,m}$ for some $m \leq n$ (that is, B permits the change in A).

Let τ_σ be the smallest atom extending σ. Let τ^s_σ be the length lexicographically least string on $\mathcal{T}$ extending the node σ such that there is no pair of nodes $\tau_1 \neq \tau_2$ on $\mathcal{T}$, $|\tau_1| = |\tau_2| = s$, such that both τ_1 and τ_2 extend τ^s_σ. That is, τ^s_σ is the least extension of σ such that no splitting of length s extends it.

With strings on $\mathcal{T}$ ordered length lexicographically, let

$$h(n) = (\mu s)(\forall \sigma)_{|\sigma| \leq n}(\forall \tau \leq \tau^s_\sigma)$$
$$[\tau \text{ has no splittings of length } s \iff \tau \text{ is an atom}].$$

Thus $\tau_\sigma^{h(n)} = \tau_\sigma$ for all σ with $|\sigma| \leq n$.

C is nonlow$_2$, so by [Martin, 1966], C escapes domination by h since $h \leq_{\mathrm{T}} \emptyset'$. That is,

$$(\exists f \leq_{\mathrm{T}} C)(\exists \text{ an infinite set } I)(\forall n \in I)[h(n) \leq f(n)].$$

Hence,

$$(\forall n \in I)(\forall \sigma)_{|\sigma| \leq n}[\tau_\sigma^{f(n)} = \tau_\sigma].$$

We define a computable sequence of computable functions $\{f_s\}_{s \in \omega}$ such that $\lim_s f_s(n) = f^*(n) \geq f(n)$, so that if $n \in I$ and $|\sigma| \leq n$, then for all $m \geq n$, $\tau_\sigma^{f^*(m)} = \tau_\sigma$ since f can be assumed to be increasing. Suppose $f = \Psi^C$ for some Turing reduction Ψ. Define $q_s(n) = (\mu s' \geq s)\Psi^C(n)[s']\downarrow$. Let

$$f_s(n) = \begin{cases} 0 & \text{if } s < n \\ \max\{\Psi^C(m)[s'] \mid m \leq n \ \& \ s' \leq q_s(m)\} & \text{if } n \leq s. \end{cases}$$

Note that $f_s(n)$ increases as s increases, and as n increases as long as $n \leq s$.

Construction.

Stage $s = 0$. Let $A_0^{[e]} = \sigma_e = \tau_{e,0}$ for all $e \in \omega$. Let $B_0 = \emptyset$. Let $\gamma_{e,m}^0$ be the least x such that $x = \langle y, m, e \rangle$ for some $y \in \omega$ and $x > \gamma_{e,n}^0$ for all $n < m$.

Stage s+1.

Step 1: Enumerate $\langle x, 0 \rangle$ in B if $x \in D_s$.

Step 2: For each e with $|\sigma_e| \leq s$, define $m(e,s)$ to be the least element m, with $|\sigma_e| \leq m < s$, that satisfies the following conditions:

(i) $\gamma_{e,m}^s \geq R(e,s)$

(ii) for $s' < s$ the last stage such that $R(e,s') \leq R(e,s)$, there exists $n \leq m$ such that $f_{s+1}(n) \neq f_{s'}(n)$

If no such m satisfies both conditions, let $m(e,s) = s$.

Condition (i) assures that $A_s^{[e]}(m)$ is not restrained from changing by $R(e,s)$, and condition (ii) assures that $A_s^{[e]}(m)$ has been permitted to change by a change in f occurring after the last stage that the restraint was at least as low as it is at the current stage.

Choose as our new guess of an atom $\tau_{e,s+1} = \tau_{A_s^{[e]} \upharpoonright m(e,s)}^{f_{s+1}(s+1)}$. That is, $\tau_{e,s+1}$ is the least node extending $A_s^{[e]} \upharpoonright m(e,s)$ that has no splittings of length $f_{s+1}(s+1)$. Let $A_{s+1}^{[e]}$ be $\tau_{e,s+1} \upharpoonright s+1$, or the lexicographically least extension of $\tau_{e,s+1}$ such that $|A_{s+1}^{[e]}| = s+1$. If $A_{s+1}^{[e]} \upharpoonright s \neq A_s^{[e]} \upharpoonright s$, enumerate $\gamma_{e,m(e,s)}^s$ into B_{s+1}, and for each $n \geq m(e,s)$, let $\gamma_{e,n}^{s+1} = x$ for the smallest x such that $x = \langle y, n, e \rangle$ for some $y \in \omega$, $x > \gamma_{e,n}^s$, and $x > \gamma_{e,m}^{s+1}$ for all $m < n$.

For all $\gamma_{e,n}^{s+1}$ not yet defined, let $\gamma_{e,n}^{s+1} = \gamma_{e,n}^s$. For each e with $|\sigma_e| > s$, let $\tau_{e,s+1} = \tau_{e,s}$ and $A_{s+1}^{[e]} = A_s^{[e]}$.

Verification.

Let $B^{[e]}$ be all elements in B of the form $\langle y, m, e\rangle$ for any $y, m \in \omega$. Let $B^{[<e]} = \cup_{i<e} B^{[i]}$. Let

$$b_t = \begin{cases} \min\{B_t^{[<e]} - B_{t-1}^{[<e]}\} & \text{if } B_t^{[<e]} \neq B_{t-1}^{[<e]},\ t \neq 0 \\ t & \text{otherwise.} \end{cases}$$

Define $T^e = \{t \mid B_t^{[<e]} \restriction b_t = B^{[<e]} \restriction b_t\}$. This is the set of "true stages" of $B^{[e]}$. Note that T^e is infinite.

Lemma 4.3. *For all* $e \in \omega$, $\liminf_s R(e, s) = \lim_{t \in T^e} R(e, t) < \infty$.

Proof. The proof is by induction on e. Assume true for all i, $i < e$. If $\Phi_e^B(e)$ converges, then cofinitely often, $r(e, s) = \varphi_e^B(e)$. Conversely, if $\Phi_e^B(e)[t]$ converges for any t in T^e, then by the hat trick of Section 2.4, $\varphi_e^B(e)[t] < b_t$, and thus $\Phi_e^B(e)$ converges. Thus if $\Phi_e^B(e)$ diverges, then for all $t \in T^e$, $r(e, t) = 0$. Therefore $\liminf_s r(e, s) = \lim_{t \in T^e} r(e, s)$ exists and is finite. Since $T^e \subseteq T^i$ for $i < e$, $\liminf_s R(e, s) = \lim_{t \in T^e} R(e, t)$ is finite. □

Lemma 4.4. *For all* $e > 0$, *the positive requirement* P_e *enumerates finitely many elements into* B *and is satisfied.*

Proof. Let $e \in \omega, e > 0$. Define $R(e) = \liminf_s R(e, s)$. Let s'_e be the smallest stage such that for all $s \geq s'_e$, $R(e, s) \geq R(e)$. There exist infinitely many $n \in I$ such that $f(n) > f_{s'_e}(n)$ and $R(e) \leq \gamma_{e,n}^{s'_e}$. Let n_e be the least such $n \in I$, with $n \geq |\sigma_e|$, and let $s_e \geq s'_e$ be the least stage such that $R(e, s_e) = R(e)$ and $f_{s_e+1}(n_e) = \lim_s f_s(n_e) = f^*(n_e)$.

Claim 1: $m(e, s_e) \leq n_e$.

Proof of Claim 1: $R(e, s_e) = R(e) \leq \gamma_{e,n_e}^{s_e}$, so n_e satisfies condition (i) in the definition of $m(e, s_e)$. Now if the last stage $s' < s_e$ such that $R(e, s') \leq R(e, s_e)$ occurred before stage s'_e, then $f_{s_e+1}(n_e) > f_{s'_e}(n_e) \geq f_{s'}(n_e)$, so condition (ii) is satisfied. On the other hand, if $s' \geq s'_e$, then $R(e, s') = R(e)$, so we must have that $f_{s_e+1}(n_e) \neq f_{s'}(n_e)$, as otherwise we would contradict the minimality of s_e. Thus condition (ii) is also satisfied for n_e. This proves the claim.

Claim 2: For all $s > s_e$, $\tau_{e,s}$ extends the atom $\tau_{A_s^{[e]} \restriction m}$ for some $m \leq n_e$.

Proof of Claim 2: Induct on s.

For the base case, τ_{e,s_e+1} is the smallest node extending $A_{e,s_e} \restriction m(e, s_e)$ that looks like an atom up to $f_{s_e+1}(s_e+1)$ rows on the tree, and so $\tau_{e,s_e+1} = \tau_{A_s^{[e]} \restriction m(e,s)}$ since $f_{s_e+1}(s_e + 1) \geq f_{s_e+1}(n_e) = f^*(n_e)$ and $n_e \in I$.

Assume the induction hypothesis that $\tau_{e,s}$ extends the atom $\tau_{A_s^{[e]} \restriction m}$ for some $m \leq n_e$. For $s + 1$, if $|\tau_{e,s}| \leq m(e, s)$, then $\tau_{e,s+1}$ will extend $\tau_{e,s}$, and the claim will hold.

Suppose $m(e, s) < |\tau_{e,s}|$. If $m(e, s) \leq n_e$, then $\tau_{e,s+1}$ will be a real atom, possibly on a different isolated path than $\tau_{e,s}$.

Now suppose $n_e < m(e,s) < |\tau_{e,s}|$. We assume by induction that $\tau_{e,s}$ extends $\tau_{A_s^{[e]} \upharpoonright m}$ for some $m \leq n_e$. If $|\tau_{A_s^{[e]} \upharpoonright m}| \leq m(e,s)$ then $A_s^{[e]} \upharpoonright m(e,s)$ is an atom, so $\tau_{e,s+1}$ is an atom extending $\tau_{A_s^{[e]} \upharpoonright m}$, as desired. If $|\tau_{A_s^{[e]} \upharpoonright m}| > m(e,s)$, then since $m(e,s) > n_e \geq m$, $\tau_{A_s^{[e]} \upharpoonright m}$ is the smallest node that looks like an atom extending $A_s^{[e]} \upharpoonright m(e,s)$. Thus $\tau_{e,s+1} = \tau_{A_s^{[e]} \upharpoonright m}$. This proves the claim.

Note that the isolated path determined by $\tau_{e,s}$ can change after stage s_e only when $m(e,s) \leq n_e$, which can happen only finitely often since condition (ii) holds for any $m \leq n_e$ only finitely often. Therefore P_e enumerates only finitely many elements into B and is satisfied. ☐

Lemma 4.5. $A \leq_{\mathrm{T}} B$.

Proof. To B-computably determine $A^{[e]} \upharpoonright n$ uniformly in n:

Find $s(n) = (\mu s)(B_s \upharpoonright\upharpoonright \gamma_{e,n}^s = B \upharpoonright\upharpoonright \gamma_{e,n}^s)$. Such an s exists because $\lim_s \gamma_{e,n}^s < \infty$. Now, at $s' \geq s(n)$, $\gamma_{e,m}^{s'}$ is not enumerated into B by P_e for any $m \leq n$ since $\gamma_{e,m}^s \leq \gamma_{e,n}^s$ for all s. Let $s_n = \max\{s(n), n\}$. Then $A_{e,s_n} \upharpoonright n = A_e \upharpoonright n$. Thus $A \leq_{\mathrm{T}} B$. ☐

Lemma 4.6. $B' \leq_{\mathrm{T}} D'$.

Proof. Claim: D' can uniformly compute a D-index for T^e. That is, there exists $g \leq_{\mathrm{T}} D'$ such that $\Phi_{g(e)}^D = T^e$.

Given the claim, D' can determine if there exists $t \in T^e$ such that $\Phi_e^B(e)[t]\downarrow$. If so, $e \in B'$ since computations at true stages are preserved, and if not, $e \notin B'$.

Proof of Claim:

It suffices to show that D' can uniformly compute a D-index for $B^{[e]}$, since T^e is uniformly computable in $B^{[<e]}$. Clearly D' knows a D-index for $B^{[0]}$. Suppose D' has computed a D-index for $B^{[<e]}$ and thus for T^e. Then D' can compute $s(e) = (\mu s \in T^e)(\forall t \geq s)(t \in T^e \implies R(e,t) = R(e,s))$, and so D' can compute $R(e) = R(e, s(e))$.

Now for all $t \in T^e$ and all $s > t$, $R(e,s) \geq R(e,t)$, so D' can find the least stage s'_e such that for all $s \geq s'_e$, $R(e,s) \geq R(e)$, as defined in the proof of Lemma 4.4. D' can compute $0'$, which can compute the set I, so D' can find n_e, the least $n \in I$, $n \geq |\sigma_e|$, such that $f(n) > f_{s'_e}(n)$ and $R(e) \leq \gamma_{e,n}^{s'_e}$. D' also knows f^*, so D' can find a stage t such that $(\forall s \geq t)(\forall n \leq n_e)(f_{s'}(n) = f^*(n))$, where $s' < s$ is the last stage such that $R(e,s') \leq R(e,s)$. This guarantees that after stage t, condition (ii) of the construction will not hold for any $m \leq n_e$. Thus, by the proof of Lemma 4.4, for all $s \geq t$, $\tau_{e,s}$ and $\tau_{e,t}$ will lie on the same isolated path. Therefore, for any $y, m \in \omega$, $\langle y, m, e\rangle \in B^{[e]} \iff \langle y, m, e\rangle \in B_{t+1}^{[e]}$.

Thus D' can uniformly determine computable indices for $B^{[e]}$, $e > 0$, and thus D-indices for $B^{[<e]}$ and T^e. ☐

Lemma 4.7. $B \leq_{\mathrm{T}} C$.

Proof. C computes D, so $B_{[0]} <_{\mathrm{T}} C$. For $e > 0$, assume C can uniformly compute $B^{[<e]}$, and thus T^e. To determine whether $\langle y, m, e\rangle$ is in $B^{[e]}$, C

finds some stage s_m by which $f_{s_m}(n)$ has reached its limit for all $n \leq m$, and then finds some $t > s_m$, $t \in T^e$. Then $R(e,t) \leq R(e,s)$ for all $s \geq t$, so P_e is not permitted to enumerate $\langle y, m, e\rangle$ into B after stage $t+1$. Thus $\langle y, m, e\rangle \in B \iff \langle y, m, e\rangle \in B_{t+1}$. □

□

Csima [2004] proved that for any low degree $\mathbf{c}$, every CAD theory has a prime model Turing incomparable with $\mathbf{c}$. We generalize this theorem in two ways. We allow $\mathbf{c}$ to be any degree strictly between $\mathbf{0}$ and $\mathbf{0}'$, and we show that we can always get a prime model of *computably enumerable* degree incomparable with $\mathbf{c}$.

Corollary 4.8. *For any degree* $\mathbf{c}$, *with* $\mathbf{0} < \mathbf{c} < \mathbf{0}'$, *every CAD theory* T *has a prime model of low c.e. degree* $\mathbf{b}$ *such that* $\mathbf{c}|\mathbf{b}$ *(i.e.,* $\mathbf{c} \not\leq \mathbf{b}$ *and* $\mathbf{b} \not\leq \mathbf{c}$*).*

Proof. Let T be a CAD theory. We split the proof into two cases.

Case 1: $\mathbf{c}$ is low$_2$. That is, $\mathbf{c}'' = \mathbf{0}''$. Let $\mathbf{d}$ be a nonlow$_2$ c.e. degree such that $\mathbf{c}|\mathbf{d}$. Such a degree $\mathbf{d}$ exists because otherwise all nonlow$_2$ c.e. degrees would be above $\mathbf{c}$, and by [Lachlan, 1966], there exists a minimal pair of nonlow$_2$ c.e. degrees, forcing $\mathbf{c}$ to be computable. Now $\mathbf{d} = \mathbf{d_1} \vee \mathbf{d_2}$ for $\mathbf{d_1}, \mathbf{d_2}$ low c.e. degrees because any c.e. degree is the join of two low c.e. degrees by [Sacks, 1963]. Clearly $\mathbf{c} \not\leq \mathbf{d_1}$ and $\mathbf{c} \not\leq \mathbf{d_2}$. Suppose $\mathbf{c} > \mathbf{d_2}$. Then $\mathbf{c} \not> \mathbf{d_1}$ since otherwise $\mathbf{c} > \mathbf{d}$. Thus $\mathbf{c}$ is incomparable with $\mathbf{d_1}$ or $\mathbf{d_2}$. Say $\mathbf{c}|\mathbf{d_1}$. By the theorem, there exists a prime model of T with c.e. degree $\mathbf{b}$ such that $\mathbf{d_1} < \mathbf{b} < \mathbf{d}$ and $\mathbf{b}' = \mathbf{d_1}' = \mathbf{0}'$. The degree $\mathbf{b}$ is incomparable with $\mathbf{c}$.

Case 2: $\mathbf{c}$ is nonlow$_2$. That is, $\mathbf{c}'' > \mathbf{0}''$. Let $\mathbf{d}$ be a low c.e. degree such that $\mathbf{c}|\mathbf{d}$. Such a degree $\mathbf{d}$ exists because otherwise all low c.e. degrees would be below $\mathbf{c}$, and by [Sacks, 1963], $\mathbf{0}'$ is the join of two low c.e. degrees, so $\mathbf{c}$ would equal $\mathbf{0}'$. By the theorem, there exists a prime model of T with c.e. degree $\mathbf{b}$ such that $\mathbf{d} < \mathbf{b} < \mathbf{0}'$ and $\mathbf{b}' = \mathbf{d}' = \mathbf{0}'$. Since $\mathbf{c}$ is not above $\mathbf{d}$ and since $\mathbf{b}$ is low and therefore not above $\mathbf{c}$, the degrees $\mathbf{b}$ and $\mathbf{c}$ are incomparable. □

We strengthen another result of Csima [2004] by showing that every CAD theory has a minimal pair of low prime models. This corollary follows from either Theorem 3.1 or Theorem 4.1.

Corollary 4.9 (Minimal Pair of Prime Models of Low C.E. Degree). *Let* T *be a complete atomic decidable theory. Then there are low c.e. degrees* $\mathbf{b}$ *and* $\widehat{\mathbf{b}}$ *such that* $\mathbf{b}$ *and* $\widehat{\mathbf{b}}$ *are degrees of prime models of* T *that form a minimal pair in the Turing degrees (i.e.,* $\mathbf{a} \leq \mathbf{b}$ *and* $\mathbf{a} \leq \widehat{\mathbf{b}}$ *implies* $\mathbf{a} = \mathbf{0}$*).*

Proof. By [Lachlan, 1966], there is a minimal pair of nonlow$_2$ (in fact, high) c.e. degrees $\mathbf{c}$ and $\widehat{\mathbf{c}}$. By either Theorem 3.1 or Theorem 4.1, there exist low c.e. degrees $\mathbf{b}$ and $\widehat{\mathbf{b}}$ with $\mathbf{b} < \mathbf{c}$ and $\widehat{\mathbf{b}} < \widehat{\mathbf{c}}$ such that $\mathbf{b}$ and $\widehat{\mathbf{b}}$ are the degrees of prime models of T. Thus, $\mathbf{b}$ and $\widehat{\mathbf{b}}$ form a minimal pair of low c.e. degrees of prime models of T. □

We can combine the statements of Theorem 3.1 and Theorem 4.1 to get the following open question.

Question 4.10. *Let T be a CAD theory and let $\mathbf{c} > \mathbf{0}$ be the c.e. degree of a prime model of T. Let $\mathbf{d}$ be a c.e. degree, $\mathbf{d} < \mathbf{c}$ and let $\mathbf{s}$ be c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$. Is there always a c.e. degree $\mathbf{b}$ with $\mathbf{d} < \mathbf{b} < \mathbf{c}$ and $\mathbf{b}' = \mathbf{s}$ such that $\mathbf{b}$ is the degree of a prime model of T?*

Note that if $\mathbf{c}$ is nonlow$_2$, this becomes 4.1, so we may assume $\mathbf{c}$ is low$_2$.

5 Homogeneous Models

Homogeneous models are another important class of models introduced by Vaught [1961]. A model $\mathcal{M}$ is *homogeneous* if every finite partial automorphism of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. Equivalently, if $\overline{a}$ and $\overline{b}$ realize the same type in a homogeneous model $\mathcal{M}$, and $c \in \mathcal{M}$, then there is an element $d \in \mathcal{M}$ such that $(\overline{a}, c)$ and $(\overline{b}, d)$ realize the same type in $\mathcal{M}$. Prime models are examples of homogeneous models, as are saturated models, which are models realizing every type in the theory over finitely many parameters.

Homogeneous models are determined up to isomorphism by the types realized in the model. In computable model theory, we say that a homogeneous model $\mathcal{M}$ with all types computable has a $\mathbf{0}$-*basis* if there is a computable listing of computable indices of the types realized in $\mathcal{M}$. Goncharov [1978], Millar [1980], and Peretyat'kin [1978] showed that there exists a homogeneous model $\mathcal{M}$ of a complete decidable (CD) theory such that $\mathcal{M}$ has a 0-basis but no decidable isomorphic copy. Lange [ta] showed that for every CD theory T, if $\mathcal{M}$ is a homogeneous model of T with a $\mathbf{0}$-basis, then $\mathcal{M}$ has a low isomorphic copy. Combining a theorem of Lange with the results in this paper, we show that if $\mathcal{M}$ is a homogeneous model with a $\mathbf{0}$-basis, then there are low c.e. copies of $\mathcal{M}$.

Theorem 5.1 (Lange (ip)). *Let $\mathcal{M}$ be a homogeneous model of a CD theory T with a $\mathbf{0}$-basis. Then there is a CAD theory T' such that if $\mathbf{d}$ computes a prime model of T', then $\mathbf{d}$ computes an isomorphic copy of $\mathcal{M}$.*

Combining Lange's theorem with Theorem 4.1, Corollary 4.8, and Corollary 4.9, we get the following results.

Corollary 5.2. *Let T be a CD theory and $\mathcal{M}$ a homogeneous model of T with a $\mathbf{0}$-basis. Let $\mathbf{c}$ be a c.e.* nonlow$_2$ *degree and let $\mathbf{d} < \mathbf{c}$ be a c.e. degree. Then for any $\mathbf{s}$ c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$, there is a c.e. degree $\mathbf{b}$ such that $\mathbf{d} < \mathbf{b} < \mathbf{c}$, $\mathbf{b}' = \mathbf{s}$, and $\mathbf{b}$ is the degree of an isomorphic copy of $\mathcal{M}$.*

Corollary 5.3. *Let T be a CD theory and $\mathcal{M}$ a homogeneous model of T with a $\mathbf{0}$-basis. Let $\mathbf{c}$ be a degree, $\mathbf{0} < \mathbf{c} < \mathbf{0}'$. Then there is a low c.e. degree $\mathbf{b}$ such that $\mathbf{c}|\mathbf{b}$ and $\mathbf{b}$ is the degree of an isomorphic copy of $\mathcal{M}$.*

Corollary 5.4. *Let T be a CD theory and $\mathcal{M}$ a homogeneous model of T with a $\mathbf{0}$-basis. Then there are low c.e. degrees $\mathbf{b}$ and $\widehat{\mathbf{b}}$ such that $\mathbf{b}$ and $\widehat{\mathbf{b}}$ are degrees of copies of $\mathcal{M}$, and $\mathbf{b}$ and $\widehat{\mathbf{b}}$ form a minimal pair in the Turing degrees.*

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