Research Statement - Rachel Epstein

1. Computability Theory

The study of computability theory began with the search for a formal definition of the intuitively computable functions and sets. For example, the set of even numbers is a computable set because we can determine whether or not a number is even. A set is not computable if no algorithm or computer program could tell you whether or not a number is in the set. Kleene came up with six schemes defining the class of computable functions. Turing defined these same functions using machines with a single infinite tape that can be read and changed one cell at a time, which we call *Turing machines*.

Turing also showed how his machines could be used to compute functions and sets relative to some non-computable set. The concept of relative computability allows us to study and classify the noncomputable functions and sets. We say a set A is *Turing reducible to* B and write $A \leq_{\mathrm{T}} B$ if A can be computed from an algorithm that can ask whether or not any element is in B. The sets A and B are *Turing equivalent*, $A \equiv_{\mathrm{T}} B$, if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. The degree $\mathbf{a}$ of A is $\mathbf{a} = \{B \mid A \equiv_{\mathrm{T}} B\}$. $\mathbf{0}$ is the degree of the computable sets.

Computability theory is concerned mainly with sets and functions that are not computable. One particularly important class of sets is the class of computably enumerable sets. A *computably enumerable (c.e.) set* A is a set where there is an effective algorithm that can list the elements of A in some order. The c.e. sets can be equivalently defined as the domains of partial computable functions, ranges of computable functions, or Σ^0_1 sets. Note that all computable sets are c.e. We say a degree is c.e. if it contains a c.e. set. An important c.e. degree is the degree $\mathbf{0}'$ of the halting set, which is the set that tells you which algorithms halt on which input. Every c.e. set is Turing reducible to $\mathbf{0}'$. When a set has degree $\mathbf{0}'$, we say it is *complete*. The c.e. degrees have been studied extensively since Post first asked in 1944 whether there is a c.e. degree strictly between $\mathbf{0}$ and $\mathbf{0}'$, but there are still many interesting open questions involving the c.e. sets and degrees.

There are also many applications of c.e. sets to other areas of mathematics. The word problem in algebra is one example. The solution to Hilbert's 10th problem also uses c.e. sets. Perhaps the most well-known use of c.e. sets was Gödel's Incompleteness Theorem. While Gödel did not use the words "computably enumerable," his proof heavily relies on the concept of c.e. sets. More recently, Nabutovsky and Weinberger have found applications of c.e. sets in differential geometry.

The primary way of classifying degrees is by their jumps. The *jump* A' of a set A is the degree of the halting set relative to A. We call a c.e. set A *high* if $\mathbf{0}'' \leq_{\mathrm{T}} \mathbf{A}'$ and *low* if $A' \leq_{\mathrm{T}} \mathbf{0}'$. The high sets have information content close to that of $\mathbf{0}'$ and the low sets have low information content. They can be thought of as nearly computable. Another class of sets that is especially important in computable model theory is the class of low$_2$ sets, where a set A is low$_2$ if $A'' \leq_{\mathrm{T}} \mathbf{0}''$. The low$_2$ sets have slightly more information content than the low sets. We can generalize these definitions so that the low$_n$ sets have more information content than the low$_{n-1}$ sets, and the high$_n$ sets have less information content than the high$_{n-1}$ sets. We often abbreviate these classes by L_n and H_n. We call these *jump classes*.

My areas of research. Definability is one of the most fundamental themes in computability theory, and indeed in all of logic. My research involves studying which jump classes of degrees are definable in the structure of the c.e. sets. In addition, I have worked in computable model theory, examining which classes of c.e. degrees can compute prime models, which can be described as the smallest models of a theory. This had not previously been studied for c.e. degrees, and the unique properties of the c.e. degrees led to many interesting results. I plan to continue research in both of these areas. I have also included some ideas at the end of this statement for undergraduate research projects.

2. Definability and invariance

I will first discuss my work on the structure of the c.e. sets, which led to the solution to a 40-year-old problem about the definability of degree classes. We can examine the c.e. sets and degrees by looking at the structure $\mathcal{R}$ of the c.e. degrees under Turing reducibility or at the structure $\mathcal{E}$ of the c.e. sets under inclusion. We say a class of sets $\mathcal{S}$ is definable in $\mathcal{E}$ if we can describe $\mathcal{S}$ in the language of set inclusion, and a class of sets $\mathcal{S}$ is definable in $\mathcal{R}$ if we can describe $\mathcal{S}$ in the language of Turing reducibility. A class of degrees $\mathbf{D}$ is definable in $\mathcal{E}$ or $\mathcal{R}$ if there is a class of sets $\mathcal{S}$ definable in $\mathcal{E}$ or $\mathcal{R}$ such that $\mathbf{D} = \{\deg(W) \mid W \in \mathcal{S}\}$. The question of which classes of degrees are definable in each structure has been the topic of much research.

Definable jump classes. It is natural to ask which jump classes of degrees are definable in the structures $\mathcal{E}$ and $\mathcal{R}$. Nies, Shore, and Slaman have worked on this problem for the structure of degrees $\mathcal{R}$, solving the problem for all but the low degrees. Cholak, Harrington, Lachlan, Millar, Shoenfield, and Soare have all contributed to the results for the structure of the c.e. sets $\mathcal{E}$ over the past 40 years. The only jump class in $\mathcal{E}$ for which the definability question remained open was the nonlow degrees. I have shown that the nonlow degrees are not definable in $\mathcal{E}$, proving a conjecture of Harrington and Soare [1996]. My work completed the problem of determining which jump classes are definable in $\mathcal{E}$.

Theorem 2.1 (Epstein (ip))**.** *$\overline{L_1}$ is not definable.*

Planned work on degree structures. My work showing that $\overline{L_1}$ was not definable used automorphisms of the structure $\mathcal{E}$. I would like to explore further questions on automorphisms of $\mathcal{E}$. The machinery I used in my theorem may be applicable to other open questions in the area. For example, I am interested in the question of whether every low$_2$ set D can be taken to a low set by an automorphism of $\mathcal{E}$. A longstanding open question about $\mathcal{E}$ is whether we can always avoid an upper cone. Specifically, for all $A <_{\mathrm{T}} \mathbf{0}'$ and all noncomputable sets C, does there exist a set B such that $C \not\leq_{\mathrm{T}} B$ and A can be taken to B by an automorphism of $\mathcal{E}$? Miller [2002] showed that this holds for all low noncomputable sets A. I plan to examine whether it holds for all low$_2$ sets A as well.

3. Computable Model Theory

The combination of computability theory and model theory is a natural one, allowing us to explore the computability of mathematical structures. This is another area of my research that I plan to develop further.

Before we can define a model, we must define language and theory. A *language* $\mathcal{L}$ is a set of symbols representing constants, functions and relations. An *$\mathcal{L}$-formula* $\theta(\bar{x})$ is a formula in the language $\mathcal{L}$ with free variables $\bar{x} = (x_0, \ldots, x_{n-1})$. We call a formula with no free variables a *sentence*. An *$\mathcal{L}$-theory* T is a collection of consistent sentences in $\mathcal{L}$. We say T is a *complete theory* if it is a maximal consistent collection of sentences in $\mathcal{L}$. Since we are studying computable model theory, we will limit our discussion to countable theories and languages.

A *model* $\mathcal{M}$ of an $\mathcal{L}$-theory T consists of a set M called the *universe* of $\mathcal{M}$ along with interpretations of each symbol in $\mathcal{L}$ such that all the sentences in T are true in $\mathcal{M}$. For example, $\mathbb{Q}$ and $\mathbb{R}$ are both models of the theory of dense linear orderings.

In 1961, Vaught began the study of countable models of complete theories, and introduced the notions of prime, homogeneous, and saturated models. A *prime model* is a model $\mathcal{M}$ that embeds into every other model of a theory T by a map that preserves the sentences true in $\mathcal{M}$. For example, $\mathbb{Q}$ is a prime model of the theory of dense linear orders, and the algebraic numbers form a prime model of the theory of algebraically closed fields of characteristic 0. I have been studying the c.e. degree spectra of prime models, extending some non-c.e. theorems and proving other theorems that only hold for the c.e. degrees.

A theory T is *decidable* if the set of sentences in T is computable. Let $\mathcal{M}$ be a model of T. Define the *elementary diagram* of $\mathcal{M}$, $D^e(\mathcal{M})$, to be the set of all formulas true in $\mathcal{M}$. We say the model $\mathcal{M}$ is *decidable* if its elementary diagram is computable. Similarly, we say a model has degree $\mathbf{d}$ if its elementary diagram has degree $\mathbf{d}$. (An alternate definition is the degree of the atomic diagram. Since the atomic diagram is Turing reducible to the elementary diagram, many results for the elementary diagram carry over to the atomic diagram.)

It is well known that every atomic theory has a prime model. This leads to the question of whether every complete atomic decidable (CAD) theory has a decidable prime model. Goncharov-Nurtazin [1973] and Millar [1978] independently showed that there is a CAD theory with no decidable prime model. This leads to the question of which degrees can compute prime models.

This problem has been investigated for the degrees below $\mathbf{0}'$, but never before for the c.e. degrees. It is a more difficult problem for the c.e. degrees because it requires a computably enumerable construction.

Prime models of computably enumerable degree. Csima [2004] showed that every complete atomic decidable theory has a prime model of low degree. Her proof uses an oracle argument that does not work for the c.e. degrees. The following theorem extends her result, and uses a priority argument.

Theorem 3.1 (Epstein (2008)). *Every complete atomic decidable theory has a prime model of low c.e. degree.*

There are many other questions we can ask about the degree spectra of prime models. One such question is whether, given a prime model $\mathcal{M}$ of a particular theory T, we can always find a prime model of T of degree below the degree of $\mathcal{M}$. I showed that this is true for $\mathcal{M}$ of c.e. degree.

One of the most significant results about the computably enumerable degrees is the Sacks Density Theorem [1964], which states that between any two c.e. degrees $\mathbf{d} < \mathbf{c}$ there is another c.e. degree. I proved a sort of density theorem for the degrees of prime models, showing that there is always a prime model between any two appropriate c.e. degrees. This theorem is unique for the c.e. degrees because if either degree $\mathbf{d}$ or $\mathbf{c}$ were not c.e., then there may not always be a degree strictly between $\mathbf{d}$ and $\mathbf{c}$.

As a corollary, I showed that for any degree $\mathbf{c}$ with $\mathbf{0} < \mathbf{c} < \mathbf{0}'$, every CAD theory has a prime model of low c.e. degree not comparable with $\mathbf{c}$. This strengthens a result of Csima [2004].

Planned work in computable model theory. The *atomic diagram* of a model $\mathcal{M}$ is the set of all quantifier-free formulas true in the model $\mathcal{M}$. Computable model theorists often study the degrees of atomic diagrams of models because it gives an idea of how much information is needed to build a model. I plan to work on problems regarding the degrees of atomic diagrams of models of decidable theories.

Along with prime models, Vaught also defined homogeneous and saturated models. A model $\mathcal{M}$ is *homogeneous* if every finite partial automorphism of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. A model $\mathcal{M}$ of a theory T is *countable saturated* if every countable model of T embeds into $\mathcal{M}$. There is still little known about the c.e. degrees of homogeneous and saturated models, and I would like to explore this topic.

Another problem that I have begun to examine is whether we can strengthen any of the results on prime, homogeneous, and saturated models to stronger reducibilities. For example, we know that every complete atomic decidable theory T has a prime model of low degree, so we ask if it has a prime model of superlow degree, or even a prime model bounded Turing reducible to $\mathbf{0}'$. I have some results in this area already, and I plan to expand on my work.

In addition to studying the degrees of models, I would like to explore model theory from the proof-theoretic perspective of reverse mathematics. Reverse mathematics is the proof-theoretic study of the relative strengths of theorems, by comparing them over a basic set of axioms.

4. Ideas for undergraduate research projects

Apart from Turing reducibility, there are other reducibilities such as truth-table reducibility and bounded Turing reducibility. An undergraduate could examine some of the basic proofs in computability theory regarding Turing reducibility, and check if they hold for other reducibilities. In addition, the student could develop his or her own reducibility and compare it to the existing reducibility. He or she could discuss the benefits and drawbacks of using the new reducibility, and explore which results hold with it in place of Turing reducibility.

One of the principal proof techniques of computability theory is the priority argument. These include finite injury arguments. There are several different ways mathematicians have developed to implement injury arguments, including the tree method and the pinball-method. A student could take one of the basic theorems using finite injury and prove it using each of the methods. They could then discuss the advantages of each method. If the student is sufficiently advanced, he or she could try this for infinite injury arguments.

Students could also study Martin-Löf randomness, and how it relates to martingales and probability theory. There are many interesting topics related to randomness, such as effective dimension, and some results in the area would be accessible to an advanced and motivated undergraduate.

In addition to these ideas, students could also combine computability with other interests and study the computability of infinite countable structures, such as linear orderings, graphs, and boolean algebras.

References

[1] [Csima, 2004]
B. F. Csima, Degree spectra of prime models, *J. Symbolic Logic*, **69** (2004), 430–442.

[2] [Epstein, 2008]
R. Epstein, Prime models of computably enumerable degree, *J. Symbolic Logic*, **73** (2008), pp. 1373–1388.

[3] [Epstein, ip]
R. Epstein, Invariance and automorphisms of the computably enumerable sets, in preparation.

[4] [Goncharov and Nurtazin, 1973]
S. S. Goncharov and A. T. Nurtazin, Constructive models of complete decidable theories, *Algebra and Logic*, **12** (1973), pp. 67–77.

[5] [Harrington-Soare, 1996]
L. Harrington and R. I. Soare, Definability, automorphisms, and dynamic properties of computably enumerable sets, *Bull. Symbolic Logic*, **2** (1996), 199–213.

[6] [Millar, 1978]
T. S. Millar, Foundations of recursive model theory, *Ann. Math. Logic*, **13** (1978), 45–72.

[7] [Miller, 2002]
R. Miller, Orbits of computably enumerable sets: low sets can avoid an upper cone, *Ann. Pure and Applied Logic*, **118** (2002), 61–85.

[8] [Sacks, 1964]
G. E. Sacks, The recursively enumerable degrees are dense, *Ann. of Math. (2)*, **80** (1964), 300–312.