Research Statement - Rachel Epstein

Introduction. Definability is one of the most fundamental themes in computability theory, and indeed in all of logic. My research involves studying which classes of Turing degrees are definable in the structure of the computably enumerable (c.e.) sets. A set of natural numbers is *c.e.* if there is an algorithm that can list the elements of the set in some order. The c.e. sets and degrees have been central to computability theory since they were introduced in the 1930's by Gödel and Kleene, and were extensively developed by Sacks and others in the 1960's. We consider only c.e. sets below.

In addition to studying definability of classes of c.e. sets, I have worked in computable model theory, examining which classes of c.e. degrees can compute prime models, which are models that can be embedded into every other model of the theory. This had not previously been studied for c.e. degrees, and the unique properties of the c.e. degrees led to many interesting results. I plan to continue research in both of these areas.

A set of natural numbers is *computable* if there is some algorithm that can determine whether or not any element is in the set. Computability theory is mainly concerned with noncomputable sets. The concept of relative computability allows us to study and classify the noncomputable sets. We say a set A is *Turing reducible to* B and write $A \leq_{\mathrm{T}} B$ if A can be computed from an algorithm that can ask whether or not any element is in B. The sets A and B are *Turing equivalent*, $A \equiv_{\mathrm{T}} B$, if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. The degree $\mathbf{a}$ of A is $\mathbf{a} = \{B \mid A \equiv_{\mathrm{T}} B\}$. $\mathbf{0}$ is the degree of the computable sets.

1. Definability and invariance

I will first discuss my work on the structure of the c.e. sets, which led to the solution to a 40-year-old problem about the definability of degree classes. We can examine the c.e. sets and degrees by looking at the structure $\mathcal{R}$ of the c.e. degrees under Turing reducibility or at the structure $\mathcal{E}$ of the c.e. sets under inclusion. We say a class of sets $\mathcal{S}$ is definable in $\mathcal{E}$ if we can describe $\mathcal{S}$ in the language of set inclusion, and a class of sets $\mathcal{S}$ is definable in $\mathcal{R}$ if we can describe $\mathcal{S}$ in the language of Turing reducibility. A class of degrees $\mathbf{D}$ is definable in $\mathcal{E}$ or $\mathcal{R}$ if there is a class of sets $\mathcal{S}$ definable in $\mathcal{E}$ or $\mathcal{R}$ such that $\mathbf{D} = \{\deg(W) \mid W \in \mathcal{S}\}$. The question of which classes of degrees are definable in each structure has been the topic of much research.

Jump classes of degrees. We consider the jump of A, A', which is the halting problem relative to A. If A' has degree $\mathbf{0}'$, we say that A is *low*. The highest possible jump of a c.e. set is $\mathbf{0}''$, so if A' has the degree $\mathbf{0}''$, we call A a *high* set. We can generalize this idea to multiple jumps. We call these classes L_n and H_n, where L_1 and H_1 are the low and high degrees, respectively. These and their complements make up the *jump classes.* The jump classes are the primary way of classifying c.e. degrees.

Definable jump classes. It is natural to ask which jump classes of degrees are definable in the structures $\mathcal{E}$ and $\mathcal{R}$. Nies, Shore, and Slaman have worked on this problem for the structure of degrees $\mathcal{R}$, solving the problem for all but the low degrees. Cholak, Harrington, Lachlan, Millar, Shoenfield, and Soare have all contributed to the results for the structure of the c.e. sets $\mathcal{E}$ over the past 40 years. The only jump class in $\mathcal{E}$ for which the definability question remained open was the nonlow degrees. I have shown that the nonlow degrees are not definable in $\mathcal{E}$, using a two-phase method that should be applicable to other open problems in the area. My work completed the problem of determining which jump classes are definable in $\mathcal{E}$.

The primary method of showing whether a class is definable is to find a definition for the class. However, to show that a class is not definable in $\mathcal{E}$, we show that it is noninvariant. We say a class of sets $\mathcal{S}$ is *invariant* if every automorphism of $\mathcal{E}$ takes $\mathcal{S}$ to itself. A class of c.e. degrees $\mathbf{D}$ is *invariant* if there is an invariant class of sets $\mathcal{S}$ such that $\mathbf{D} = \{\deg(W) \mid W \in \mathcal{S}\}$. Every definable class is invariant because automorphisms must preserve the structure of $\mathcal{E}$. To show that a class $\mathbf{D}$ is noninvariant, we must show there is some degree $\mathbf{d} \in \mathbf{D}$ such that for all sets $D \in \mathbf{d}$, there is an automorphism of $\mathcal{E}$ taking D outside of $\mathbf{D}$.

For the downward closed jump classes, the problem of definability in $\mathcal{E}$ has been solved. The class of computable sets, L_0, is definable. Harrington and Soare [1996b] showed that all other downward closed jump classes are noninvariant, and thus not definable.

For the upward closed jump classes, Martin [1966] showed that the high degrees are definable, and Lachlan [1968] and Shoenfield [1976] showed that $\overline{L_2}$ is definable. Cholak and Harrington [2002] solved the problem for all but $\overline{L_1}$ by proving that for $n \geq 2$, H_n and $\overline{L_n}$ are definable. The only remaining jump class was $\overline{L_1}$. This is analogous to the case of the structure of c.e. degrees $\mathcal{R}$. As shown by Nies, Shore, and Slaman [1996], all classes except L_1 are definable in $\mathcal{R}$. It is still unknown whether L_1 is definable in $\mathcal{R}$.

$\overline{L_1}$ is not definable in $\mathcal{E}$.

Conjecture 1 (Harrington-Soare (1996a))**.** $\overline{L_1}$ *is noninvariant.*

We prove this conjecture by the following theorem.

Theorem 1.1 (Epstein (ip2))**.** *There exists a nonlow set D such that for all c.e. $A \leq_{\mathrm{T}} D$, there exists a low set B such that A is taken to B by an automorphism of $\mathcal{E}$.*

Corollary 1.2 (Epstein (ip2))**.** $\overline{L_1}$ *is the only upward closed jump class that is not definable.*

Harrington and Soare believed that their automorphism method in [1996b] would be enough to prove their conjecture. Indeed, other researchers have used this method for other important results. However, they soon realized that their method would not work here, due to the need to incorporate restraint into the automorphism machinery. Therefore, the Harrington-Soare conjecture remained open until [Epstein, ip2].

The new automorphism method. My $\overline{L_1}$ theorem is the only automorphism theorem that involves sending a set A *down* to a low set B. Cholak [1995] and Harrington-Soare [1996b] showed that for every noncomputable set A, there is an automorphism of $\mathcal{E}$ taking A to a high set B. In addition, Harrington-Soare [1996b] showed that all prompt sets are automorphic to complete sets. Both of these theorems took the given sets *up* to high sets and complete sets. They did this using coding methods. No restraint was involved in these theorems. In contrast, my theorem involves a complicated restraint mechanism to ensure that every $A \leq_{\mathrm{T}} D$ is automorphic to a low set B. I developed new machinery that works within $\overline{A}$ and $\overline{B}$ in order to handle the problem of restraint while building a partial automorphism of $\mathcal{E}$. In addition, I produced a new way to extend partial automorphisms to automorphisms of $\mathcal{E}$. These two techniques together prove the theorem.

The Tree Extension Theorem. Within this theorem, I have proven a Tree Extension Theorem, which can be used for other automorphism problems. If we construct a partial automorphism of $\mathcal{E}$ on an appropriate tree, then we can apply this theorem to extend to an automorphism of $\mathcal{E}$ that takes A to B. There have been several extension theorems in the past, but this is the first that works explicitly on a tree. It is useful to have an extension theorem that works on a tree because we often build partial automorphisms on a tree, and this allows us to refer to the Tree Extension Theorem instead of constructing a full automorphism. This means that this technique is potentially very useful for other open problems, because in order to show that A can be taken to B, we only need to show that we can build a partial automorphism of $\mathcal{E}$, within $\overline{A}$ and $\overline{B}$.

Planned work on degree structures. I plan to work on several questions on the structure of c.e. sets, as well as related problems on the structures of the n-c.e. sets. I showed that there exists a nonlow set D such that for all $A \leq_{\mathrm{T}} D$, A is automorphic to a low set. I would like to explore this problem more by asking if every low$_2$ set D is automorphic to a low set. If not, is there a set in every L_2 degree that is automorphic to a low set? Soare [1982] showed that the structure of the c.e. sets restricted to the complement of a low set is isomorphic to $\mathcal{E}$. I would like to show if this holds for every low$_2$ set. My proof that $\overline{L_1}$ is noninvariant shows that this holds for all $A \leq_{\mathrm{T}} D$, for some nonlow D.

A longstanding open question about $\mathcal{E}$ is whether we can always avoid an upper cone. Specifically, for all $A <_{\mathrm{T}} \mathbf{0}'$ and all noncomputable sets C, does there exist a set B such that $C \not\leq_{\mathrm{T}} B$ and A is automorphic to B? Miller [2002] showed that this holds for all low noncomputable sets A. I plan to examine whether it holds for all low$_2$ sets A as well. It is likely that the Tree Extension Theorem could simplify this problem.

A set A is n-c.e. if there is a computable approximation $\{A_s\}_{s\in\omega}$ of A such that for each $x \in \omega$, $A_s(x) \neq A_{s-1}(x)$ at most n times. A degree is called n-c.e. if it contains an n-c.e. set. Arslanov, Kalimullin, and Lempp [ta] have shown that the structures of the 2-c.e. degrees and the 3-c.e. degrees are not elementarily equivalent. However, it is still unknown whether the structures of the n-c.e. degrees for $n \geq 3$ are elementarily equivalent. My experience with complicated tree arguments may be useful in attempting to find elementary differences between these theories.

2. Computable Model Theory

The combination of computability theory and model theory is a natural one, allowing us to explore the computability of mathematical structures. This is another area of my research that I plan to develop further.

In 1961, Vaught began the study of countable models of complete theories, and introduced the notions of prime, homogeneous, and saturated models. A *prime model* is a model $\mathcal{M}$ that embeds into every other model of a theory T by a map that preserves the sentences true in $\mathcal{M}$. We say a theory T is *atomic* if the isolated points are dense in the Stone space of its types. All atomic theories have prime models. I have been studying the c.e. degree spectra of prime models, extending some non-c.e. theorems and proving other theorems that only hold for the c.e. degrees.

A theory T is *decidable* if the set of sentences in T is computable. Let $\mathcal{M}$ be a model of T. Define the *elementary diagram* of $\mathcal{M}$, $D^e(\mathcal{M})$, to be the set of all formulas true in $\mathcal{M}$.

We say the model $\mathcal{M}$ is *decidable* if its elementary diagram is computable. Similarly, we say a model has degree $\mathbf{d}$ if its elementary diagram has degree $\mathbf{d}$. (An alternate definition is the degree of the atomic diagram. Since the atomic diagram is Turing reducible to the elementary diagram, many results for the elementary diagram carry over to the atomic diagram.)

As mentioned above, every atomic theory has a prime model. This leads to the question of whether every complete atomic decidable (CAD) theory has a decidable prime model. Goncharov-Nurtazin [1973] and Millar [1978] independently showed that there is a CAD theory with no decidable prime model. This raises question of which degrees can compute prime models.

This problem has been investigated for the degrees below $\mathbf{0}'$, but never before for the c.e. degrees. It is a more difficult problem for the c.e. degrees because it requires a computably enumerable construction, and most of the work in the area uses oracle constructions.

Prime models of computably enumerable degree. Csima [2004] showed that every complete atomic decidable theory has a prime model of low degree. Her proof uses an oracle argument that does not work for the c.e. degrees. The following theorem extends her result, and uses a priority argument.

Theorem 2.1 (Epstein (2008))**.** *Every complete atomic decidable theory has a prime model of low c.e. degree.*

There are many other questions we can ask about the degree spectra of prime models. One such question is whether, given a prime model $\mathcal{M}$ of a particular theory T, we can always find prime models of T of degree above or below the degree of $\mathcal{M}$. Knight [1986] showed that if a nontrivial CAD theory T has a prime model of degree $\mathbf{d}$, then T has a prime model of any degree above $\mathbf{d}$. This answers one half of the question and leaves the question of whether the degree $\mathbf{d}$ of a prime model can always be pushed down to a strictly lower degree of a prime model of the same theory. If $\mathbf{d}$ is not a c.e. degree, then it is not always possible to find a prime model of strictly lower degree, so this is an example of a question that can be asked only of the c.e. degrees.

Theorem 2.2 (Epstein (2008), **Continuity of Prime Models of C.E. Degree**)**.** *Let T be a complete atomic decidable theory and let $\mathbf{c} > \mathbf{0}$ be the c.e. degree of a prime model of T. Then there is a prime model of T with low c.e. degree $\mathbf{b} < \mathbf{c}$.*

One of the most significant results about the computably enumerable degrees is the Sacks Density Theorem [1964], which states that between any two c.e. degrees $\mathbf{d} < \mathbf{c}$ there is another c.e. degree. In fact, Robinson [1971] showed that all degrees c.e. in $\mathbf{c}$ and above $\mathbf{d}'$ are the jump of some degree between $\mathbf{d}$ and $\mathbf{c}$. A degree $\mathbf{c}$ is *prime bounding* if every CAD theory T has a $\mathbf{c}$-decidable prime model. Csima, Hirschfeldt, Knight, and Soare [2004] showed that for $\mathbf{c} < \mathbf{0}'$, $\mathbf{c}$ is prime bounding if and only if $\mathbf{c} \in \overline{\mathbf{L_2}}$. We show that between c.e. degrees $\mathbf{d}$ and $\mathbf{c}$, we can always find a prime model of c.e. degree whose jump is given, as long as $\mathbf{c}$ is prime bounding.

Theorem 2.3 (Epstein (2008), **Density of Prime Models of C.E. Degree**)**.** *Let T be a complete atomic decidable theory, let $\mathbf{c}$ be a c.e. nonlow$_2$ degree, and let $\mathbf{d}$ be a c.e. degree with $\mathbf{d} < \mathbf{c}$. Then for any degree $\mathbf{s}$ c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$, there is a c.e. degree $\mathbf{b}$ with $\mathbf{d} < \mathbf{b} < \mathbf{c}$ and $\mathbf{b}' = \mathbf{s}$ such that $\mathbf{b}$ is the degree of a prime model of T.*

Note that this theorem is unique for the c.e. degrees because if either degree **d** or **c** were not c.e., then there may not always be a degree **b** between **d** and **c**.

As a corollary, we show that for any degree **c** with $\mathbf{0} < \mathbf{c} < \mathbf{0}'$, every CAD theory has a prime model of low c.e. degree not comparable with **c**. It is also a corollary that for every CAD theory T, there exists a minimal pair of low c.e. degrees of prime models of T, where **c** and **d** form a minimal pair if the only sets reducible to both of them are computable. These both strengthen results of Csima [2004].

Planned work in computable model theory. The *atomic diagram* of a model $\mathcal{M}$ is the set of all quantifier-free formulas true in the model $\mathcal{M}$. Computable model theorists often study the degrees of atomic diagrams of models because it gives an idea of how much information is needed to build a model. Since the atomic diagram is Turing reducible to the elementary diagram of $\mathcal{M}$, some of our results translate to atomic diagrams. However, it is unknown whether, given a prime model with atomic diagram of c.e. degree **c**, we can always find a prime model with atomic diagram strictly below **c**. I plan to work on this and other problems regarding the degrees of atomic diagrams of models of decidable theories in the coming years.

Along with prime models, Vaught also defined homogeneous and saturated models. A model $\mathcal{M}$ is *homogeneous* if every finite partial automorphism of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. A model $\mathcal{M}$ of a theory T is *countable saturated* if every countable model of T embeds into $\mathcal{M}$. There is still little known about the c.e. degrees of homogeneous and saturated models.

We say a homogeneous model has a **0**-basis if there is a computable listing of computable indices of all types realized in $\mathcal{M}$. Using a theorem of Lange [ip] that relates homogeneous models with a **0**-basis to prime models, we can transform Theorem 2.3 and its corollaries to theorems about the degrees of isomorphic copies of homogeneous models. However, it is still unknown whether there is a homogeneous analog of Theorem 2.2. Little is known about homogeneous models of c.e. degree, and I would like to investigate this subject. I already have some preliminary results in [Epstein, ip1].

For saturated models, theorems of Jockusch [1972] show that the high degrees and the degrees of models of Peano Arithmetic are saturated bounding. Harris [ip] showed that no c.e. degree in L_n is saturated bounding, which Montalban improved in [Harris, ip] by showing that no nonhigh c.e. degree is saturated bounding. However, it is still unknown exactly which non-c.e. degrees are saturated bounding. This is another problem that I hope to make progress toward using my experience with prime models.

Another problem that I have begun to examine is whether we can strengthen any of the results on prime, homogeneous, and saturated models to stronger reducibilities. For example, we know that every complete atomic decidable theory T has a prime model of low degree, so we ask if it has a prime model of superlow degree, or even a prime model bounded Turing reducible to $\mathbf{0}'$. I have some results in this area already, and I plan to expand on my work.

In addition to studying the degrees of models, I would like to explore model theory from the proof-theoretic perspective of reverse mathematics. Reverse mathematics is the proof-theoretic study of the relative strengths of theorems, by comparing them over a basic set of axioms.

References

[1] [Arslanov, et al, ta]
M. M. Arslanov, I. S. Kalimullin, and S. Lempp, On Downey's Conjecture, *J. Symbolic Logic*, ta.

[2] [Cholak, 1995]
P. Cholak. Automorphisms of the lattice of recursively enumerable sets, *Mem. Amer. Math. Soc.*, **113** (1995), (541) viii+151.

[3] [Cholak-Harrington, 2002]
P. Cholak and L. A. Harrington. On the definability of the double jump in the computably enumerable sets, *J. Math. Logic*, **2**(2) (2002), pp. 261–296.

[4] [Csima, 2004]
B. F. Csima, Degree spectra of prime models, *J. Symbolic Logic*, **69** (2004), 430–442.

[5] [Csima, et al, 2004]
B. F. Csima, D. R. Hirschfeldt, J. F. Knight, and R. I. Soare, Bounding prime models, *J. Symbolic Logic*, **69** (2004), 1117–1142.

[6] [Epstein, 2008]
R. Epstein, Prime models of computably enumerable degree, *J. Symbolic Logic*, **73** (2008), pp. 1373–1388.

[7] [Epstein, ip1]
R. Epstein, Degrees of prime and homogeneous models, in preparation.

[8] [Epstein, ip2]
R. Epstein, Invariance and automorphisms of the computably enumerable sets, in preparation.

[9] [Goncharov and Nurtazin, 1973]
S. S. Goncharov and A. T. Nurtazin, Constructive models of complete decidable theories, *Algebra and Logic*, **12** (1973), pp. 67–77.

[10] [Harrington-Soare, 1996a]
L. Harrington and R. I. Soare, Definability, automorphisms, and dynamic properties of computably enumerable sets, *Bull. Symbolic Logic*, **2** (1996), 199–213.

[11] [Harrington-Soare, 1996b]
L. Harrington and R. I. Soare, The Δ^0_3 automorphism method and noninvariant classes of degrees, *Jour. Amer. Math. Soc.*, **9** (1996), 617–666.

[12] [Harris, ip]
K. Harris, On bounding saturated models, in preparation.

[13] [Jockusch, 1972]
C. Jockusch, Degrees in which the recursive sets are uniformly recursive, *Canadian Journal of Mathematics*, **24** (1972), 1092-1099.

[14] [Knight, 1986]
J. F. Knight, Degrees coded in jumps of orderings, *J. Symbolic Logic*, **51** (1986), 1034–1042.

[15] [Lachlan, 1968]
A. H. Lachlan, Degrees of recursively enumerable sets which have no maximal superset, *J. Symbolic Logic* **33** (1968), 431–443.

[16] [Lange, ip]
K. Lange, The reverse mathematics of homogeneous models, in preparation.

[17] [Martin, 1966]
D. A. Martin, Classes of recursively enumerable sets and degrees of unsolvability, *Z. Math. Logik Grundlag. Math.* **12** (1966), 295–310.

[18] [Millar, 1978]
T. S. Millar, Foundations of recursive model theory, *Ann. Math. Logic*, **13** (1978), 45–72.

[19] [Miller, 2002]
R. Miller, Orbits of computably enumerable sets: low sets can avoid an upper cone, *Ann. Pure and Applied Logic*, **118** (2002), 61–85.

[20] [Nies, Shore, and Slaman, 1996]
A. Nies, R. Shore, and T. Slaman, Definability in the recursively enumerable degrees, *Bulletin of Symbolic Logic* **2** (1996), 392–404.

[21] [Robinson, 1971]
R. W. Robinson, Jump restricted interpolation in the recursively enumerable degrees, *Ann. of Math. (2)*, **93** (1971), 586–596.

[22] [Sacks, 1964]
G. E. Sacks, The recursively enumerable degrees are dense, *Ann. of Math. (2)*, **80** (1964), 300–312.

[23] [Shoenfield, 1976]
J. R. Shoenfield, Degrees of classes of r.e. sets, *J. Symbolic Logic* **41** (1976), 695–696.

[24] [Soare, 1982]
R. I. Soare, Automorphisms of the lattice of recursively enumerable sets, Part II: Low sets, *Ann. Math. Logic* **22** (1982), 69–107.