Definability, Invariance and Automorphisms of the Computably Enumerable Sets

Rachel Epstein
Department of Mathematics, University of Chicago
e-mail: rachel@math.uchicago.edu
Completed under the guidance of Dr. Robert Soare

What is computability?

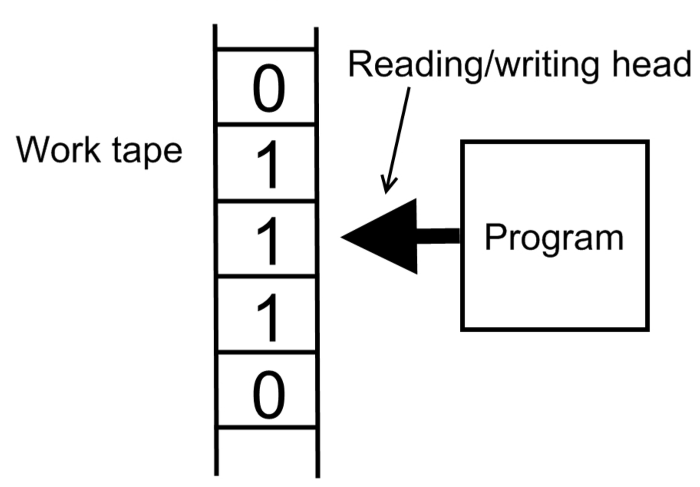
In the 1930's, mathematicians such as Kurt Gödel worked on finding a formal definition for when a set of natural numbers is computable. In 1936, Alan Turing came up with a notion of a theoretical computer that we now call a **Turing machine**. He argued that the computable sets are those that can be computed by some Turing machine. This was the first notion of computability that Gödel accepted. Any computer program, no matter how advanced, can be coded into a Turing machine.

Computability theory is mainly concerned with **noncomputable** sets

Examples of computable sets

- The even numbers
- The prime numbers
- Any finite set

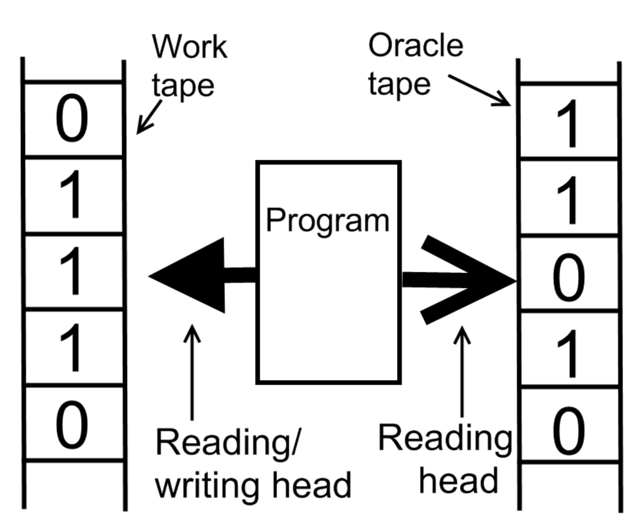
Turing Machine



A noncomputable set

Each Turing machine takes as inputs any natural number. However, some Turing machines do not give an output for every input. There are only countably many Turing machines, so we can label each with a natural number. Define the **halting set** K to be the set of all n such that the nth Turing machine gives some output on input n. Gödel showed that the halting set is a noncomputable set.

Oracle Turing Machine



Turing degrees

An **oracle Turing machine** is a Turing machine that can ask questions of an infinite set, called an oracle. If an oracle Turing machine can compute the set A given the oracle set B, then we say A is **Turing reducible** to B, and we write $A \leq_T B$. If A and B are Turing reducible to each other, we *say* A is **Turing equivalent** to B. The **degree** $\boldsymbol{a}$ of A is the set of all B such that A is Turing equivalent to B. The degree of the computable sets is **0**. The degree of the halting set K is **0'**

Jump classes of degrees

The primary way of classifying degrees is by jump classes. We say a degree is **low** if it has a low level of information content. The degree **0** is a low degree. We say a degree is **high** if it has a high level of information content. The degree **0'** is a high degree. We can generalize this idea to the classes **low$_n$** and **high$_n$**, where low$_{n+1}$ contains low$_n$ and high$_{n+1}$ contains high$_n$. These and their complements are called jump classes.

The computably enumerable sets

A set of natural numbers is called **computably enumerable** (**c.e.**) if its elements can be computably listed in some order. The c.e. sets play an important role in computability theory. All computable sets are c.e. The halting set K is also c.e. In fact, every c.e. set is Turing reducible to K. We say a degree is c.e. if it contains a c.e. set.

Examples of c.e. sets

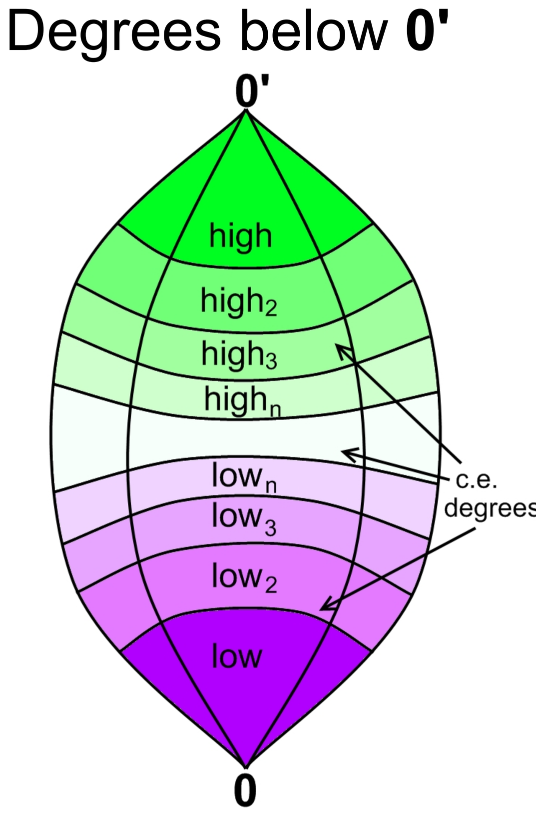
Computably enumerable sets appear in many areas of mathematics. In algebra, the word problem for groups is a c.e. set. There are even real world examples of c.e. sets. The set of all natural numbers that will ever be used on a calculus test is a c.e. set. An immortal observer could make a list of these by looking at every calculus test. However, there is no way for the observer to know if a particular number will ever be on the list. He or she can wait until the number appears, but if it never appears, the observer will wait forever.

Maximal sets and high degrees

A c.e. set M is called **maximal** if the only c.e. sets containing M are only finitely different from either M or the set ω of all natural numbers.

Theorem (Martin, 1966): The degrees of the maximal sets are exactly the high degrees.

Degrees below **0'**



Definability

We say a class of sets is **definable** if it can be defined in terms of set inclusion ⊂. A class of degrees is **definable** if it is exactly the degrees of a definable class of sets.

For example, the maximal sets are a definable class, so the high degrees are definable by Martin's theorem.

Which jump classes are definable?

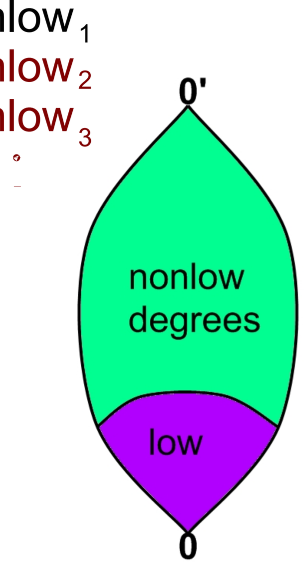
Theorem (Harrington-Soare, 1996): The downward closed jump classes, low$_n$ and nonhigh$_n$ are not definable.

Theorem (Cholak-Harrington, 2002): The upward closed jump classes, high$_n$ and nonlow$_n$, for n ≥ 2, are definable.

Blue = Not Definable   Red = Definable

Downward Closed Jump Classes		Upward Closed Jump Classes	
low$_1$	nonhigh$_1$	high$_1$	nonlow$_1$
low$_2$	nonhigh$_2$	high$_2$	nonlow$_2$
low$_3$	nonhigh$_3$	high$_3$	nonlow$_3$
⋮	⋮	⋮	⋮

The only jump class whose definability was unknown after 2002 was the class of **nonlow** degrees.



The following theorem was conjectured by Harrington and Soare in 1996.

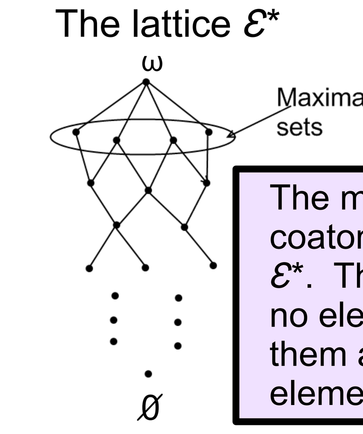
Theorem (Epstein): The nonlow degrees are not definable.

To show this, we instead show that the nonlow degrees are not invariant under automorphisms of the lattice $\mathcal{E}$ of the computably enumerable sets.

The lattices $\mathcal{E}$ and $\mathcal{E}^*$

We consider the structure of the c.e. sets under set inclusion. This forms a lattice $\mathcal{E}$. The finite sets form an ideal in $\mathcal{E}$, so we can consider the quotient lattice $\mathcal{E}^*$ of the c.e. sets modulo finite difference. For our purposes, $\mathcal{E}$ and $\mathcal{E}^*$ are interchangeable.

The lattice $\mathcal{E}^*$



The maximal sets are coatoms of the lattice $\mathcal{E}^*$. That is, there are no elements between them and the top element, ω.

Invariance and definability

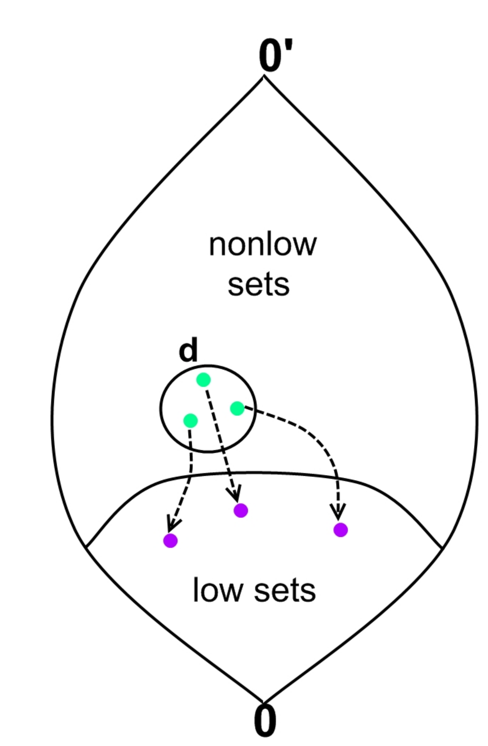
A class of sets S is **invariant** if every automorphism of $\mathcal{E}$ (or equivalently $\mathcal{E}^*$) maps S to itself. A class of degrees is **invariant** if it is exactly the degrees of an invariant class of sets.

Every definable class is invariant. For example, the maximal sets are invariant, so the high degrees are invariant because they are the degrees of the maximal sets.

To show that the nonlow degrees are not definable, it suffices to show that they are not invariant.

To show the nonlow degrees are not invariant, we can show that there is a nonlow degree **d** such that every set A in **d** can be taken by an automorphism of $\mathcal{E}$ to a low set. This guarantees that there is no invariant class of sets S such that the nonlow degrees are the degrees of S.

We construct a nonlow degree **d** such that every set in **d** can be taken by an automorphism of $\mathcal{E}$ to a low set.



Conclusion

The nonlow degrees are the **only** upward closed jump class that is not definable. This completes the 40-year-old problem of determining which jump classes are definable.