1 Trigonometric Sums

The main goal of this note is to establish certain bounds of Gauss and Kloosterman sums using étale cohomology.

1.1 Gauss Sums

Definition 1.1 (Gauss Sum). Let $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be an additive character and let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a multiplicative character, then the Gauss sum is defined to be

$$\tau(\chi, \psi) = - \sum_{x \in \mathbb{F}_q^\times} \psi(x)\chi^{-1}(x).$$

We will prove the following two theorems.
Theorem 1.2. Let $\psi$ and $\chi$ as above, then

$$|\tau(\chi, \psi)| = q^{1/2}.$$ 

Theorem 1.3 (Hasse-Davenport). Let $\mathbb{F}_{q^N}$ be a degree $N$ field extension of $\mathbb{F}_q$ and denote $\text{Tr} = \text{Tr}_{\mathbb{F}_{q^N}/\mathbb{F}_q}$ and $N = N_{\mathbb{F}_{q^N}/\mathbb{F}_q}$ the trace and the norm resp. of this extension. Then,

$$\tau(\chi \circ N, \psi \circ \text{Tr}) = - \sum_{x \in \mathbb{F}_{q^N}} \psi(\text{Tr}(x)) \chi^{-1}(N(x)) = (\tau(\chi, \psi))^N.$$ 

1.2 Kloosterman Sums

Definition 1.4 (Kloosterman Sum). Let $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ be a non-trivial additive character and let $a \in \mathbb{F}_q$, then the Kloosterman sum is defined as

$$K_{n,a} = \sum_{x_1 x_2 \cdots x_n = a} \psi(x_1 + x_2 + \cdots + x_n).$$

We will prove the following theorem.

Theorem 1.5. We have the following estimates

(i) When $a = 0$, then $K_{n,0} = (-1)^{n-1}$.

(ii) When $a \neq 0$, then $|K_{n,a}| \leq nq^{(n-1)/2}$.

Remark 1.6. These trigonometric sums are classically defined to take values in $\mathbb{C}$. However, since we hope to use étale cohomology to analyze them, we think of them as a number inside $\mathbb{Q}_l$ by fixing an isomorphism $\mathbb{Q}_l \cong \mathbb{C}$.

2 Geometrization

2.1 A Lemma on Torsors

We will move between different torsors of different groups. The following lemma is basic, but helpful when one thinks about such situations.

Lemma 2.1. Let $X$ be a scheme and $G, G'$ be smooth group schemes over $X$. Let $T, T'$ be $G$ and $G'$ torsors resp. over $X$. Let $\varphi_G : G \to G'$ be a morphism of group schemes over $X$ and $\varphi_T : T \to T'$ a morphism of schemes over $X$ compatible with $\varphi_G$ in the obvious way. Let $Y$ be an $X$-scheme, on which $G'$ (and hence $G$) acts, then

$$Y \times^G T \cong Y \times^{G'} T'.$$

Proof. Obvious: just write down the descent datum. \qed
Corollary 2.2. Suppose we have the following sequence

\[
\begin{array}{ccccccc}
1 & \rightarrow & H & \rightarrow & G & \rightarrow & G/H & \rightarrow & 1 \\
\downarrow & & \downarrow & & \varphi & & \downarrow & \\
1 & \rightarrow & H' & \rightarrow & G' & \rightarrow & G'/H' & \rightarrow & 1 \\
\end{array}
\]

where \(G, H, G', H'\) are algebraic groups over a field \(k\). Let \(X\) be a scheme on which \(H'\), and hence \(H\), acts, then

\[G \times^H X \cong (\varphi^* G') \times^{H'} X\]
as \(X\)-bundles over \(G/H\).

Proof. Note that \(G \rightarrow G/H\) and \(G' \rightarrow G'/H'\) are \(H\) and \(H'\)-torsors over \(G/H\) and \(G'/H'\) respectively. This is a direct consequence of the above. \(\Box\)

2.2 Artin-Shreier Sheaves

Goal: produce a sheaf whose local Frobenius traces are precisely the summands in the Gauss/Kloosterman sums, so we hope to use the other side of the trace formula to analyze these sums.

Definition. Let \(G_0\) be a commutative, connected algebraic group over \(\mathbb{F}_q\), where the group operation is written multiplicatively. Then Lang isogeny is defined to be

\[
\mathcal{L} : G_0 \rightarrow G_0 \\
g \mapsto Fg \cdot g^{-1}
\]

We see easily that this map is an étale map. Thus, its image is an open subgroup of \(G_0\). But since \(G_0\) is connected, this is actually an étale covering. We have the following exact sequence

\[
0 \rightarrow G_0(\mathbb{F}_q) \rightarrow G_0 \rightarrow G_0 \rightarrow 0.
\]

Examples. Applied to the case \(\mathbb{G}_a\) and \(\mathbb{G}_m\), we get the following

\[
0 \rightarrow \mathbb{F}_q \rightarrow \mathbb{G}_a \rightarrow \mathbb{G}_a \rightarrow 0
\]

\[
0 \rightarrow \mu_{q-1} \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0.
\]

Frobenius Trace. Note that \(\mathcal{L} : G_0 \rightarrow G_0\) is a smooth sheaf, and hence, we can talk about trace of the Frobenius at closed points of \(G_0\).

Let \(\gamma \in G_0(\mathbb{F}_q)\), then for any \(g \in \mathcal{L}^{-1}(\gamma)\), we have \(Fg = Fgg^{-1}g = \mathcal{L}(g)g = \gamma g\). Thus, the action of the geometric Frobenius on the stalk at \(\gamma\) is multiplication by \(\gamma^{-1} : g \mapsto g\gamma^{-1}\).
Twisting. Let $f_0 : X_0 \to G_0$ be a morphism and $\chi : G_0(\mathbb{F}_q) \to \mathbb{Q}_l^\times$ be a character. Then one can form a $\mathbb{Q}_l$-sheaf on $X_0$ by twisting $\mathbb{Q}_l$ with the Lang torsor using the inverse action $\chi^{-1}$ on $\mathbb{Q}_l$. This inverse is to cancel out the inverse in the Frobenius action. And thus, if we denote this sheaf $\mathcal{F}(\chi, f_0)$, then the action of the Frobenius at a point $\gamma \in G_0(\mathbb{F}_q)$ is $\chi(\gamma)$.

Since $\chi$ is a character of a finite group, we see easily that $|\chi(\gamma)| = 1$ and hence, we see that all Artin-Shreier sheaves are pure of weight 0.

Functorialities. All of these are natural consequences of the lemma on torsors above.

(i) $\mathcal{F}(\chi, f_0') \cdot f_0'' = \mathcal{F}(\chi, f_0') \otimes \mathcal{F}(\chi, f_0'')$.

(ii) $\mathcal{F}(\chi' \cdot \chi'', f_0) = \mathcal{F}(\chi', f_0) \otimes \mathcal{F}(\chi'', f_0)$.

(iii) Let $u_0 : G_0 \to H_0$ be a morphism of groups, and $\chi : H(\mathbb{F}_q) \to \mathbb{Q}_l^\times$, then

$\mathcal{F}(\chi, u_0 f_0) \cong \mathcal{F}(\chi u_0, f_0)$.

(iv) Let $G_0 = \prod_{i \in I} G_i^i, \chi = (\chi_i)_{i \in I}$, and $f_0 = (f_i^0)_{i \in I}$, then

$\mathcal{F}(\chi, f_0) = \bigotimes_{i \in I} \mathcal{F}(\chi_i, f_i^0)$.

Base Field Extension. Let $G_1 = G_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, then we have the following morphism of exact sequences:

$$
\begin{array}{cccccc}
0 & \longrightarrow & G_0(\mathbb{F}_q) & \longrightarrow & G_1 & \longrightarrow & G_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_0(\mathbb{F}_{q^n}) & \longrightarrow & G_1 & \longrightarrow & G_1 \\
\end{array}
$$

and hence, $\mathcal{L}(\mathbb{Q}) = N\mathcal{L}(\mathbb{Q^n})$.

Abuse of Notation. Instead of writing $\mathcal{F}(\chi, f_0)$, we write $\mathcal{F}(\chi f_0)$. The properties above allows no ambiguity. Moreover, we also write

$$
\mathcal{F}\left(\prod_i \chi_i f_i^0\right) = \mathcal{F}(\chi, f_0) = \bigotimes_{i \in I} \mathcal{F}(\chi_i, f_i^0).
$$

The case of $\mathbb{A}^1_{\mathbb{Q}}$. Recall that when $\psi$ is non-trivial, then $\mathcal{L}(\psi)$ is a smooth sheaf on $\mathbb{A}^1_{\mathbb{Q}}$ of rank 1, with Swan conductor 1 at $\infty$. Moreover, $H^*(\mathbb{A}^1_{\mathbb{Q}}, \mathcal{F}(\psi)) = 0$. In particular, the monodromy at $\infty$ is totally wildly ramified.
The case of $G_m$. In this case, $Fg \cdot g = g^{q-1}$. Thus, we get a covering of $G_m$ that is tamely ramified at both 0 and $\infty$.

In general, for $n$ relatively prime to $p$, we have the following exact sequence

$$0 \longrightarrow \mu_n \longrightarrow G_m \longrightarrow G_m \longrightarrow 0.$$ 

For any character $\chi : \mu_n \to \overline{\mathbb{Q}}_l$, we can form the Kummer sheaf $\mathcal{K}_n(\chi)$ that twists $\overline{\mathbb{Q}}_l$ with the $\mu_n$-torsor $G_m \to G_m$ via $\chi^{-1}$. $\mathcal{K}_n(\chi)$ is a smooth sheaf on $G_m$. When $\chi$ is non-trivial, it is tamely ramified at both 0 and $\infty$ (since essentially, it’s just from the extension $k(t)[x]/(x^n-t)$.

**Geometrization of Gauss Sums.** Let $\chi$ and $\psi$ as in the definition of Gauss sum. From what we have said above,

$$\tau(\chi, \psi) = -\sum_{x \in \mathbb{F}_q^*} \chi^{-1}(x)\psi(x) = -\sum_{x \in G_m(\mathbb{F}_q)} \text{Tr}(F, \mathcal{F}(\chi^{-1}) \otimes \mathcal{F}(\psi)) = -\sum \text{Tr}(F, \mathcal{F}(\chi^{-1}\psi)),$$

where $\mathcal{F}(\chi^{-1})$ is the Artin-Shreier sheaf on $G_m$ associated to $\chi$, and $\mathcal{F}(\psi)$ the restriction of the Artin-Shreier sheaf on $\mathbb{A}_0^1$ associated to $\psi$. Note that the last equality is due to our convention (of notation abuse).

**Geometrization of Kloosterman Sums.** For $a \in \mathbb{F}_q$, let $V_a \subset \mathbb{A}_0^n$ defined by $x_1x_2 \cdots x_n = a$. Let $\sigma : \mathbb{A}_0^n \to \mathbb{A}_0^1$ defined by the sum of the coordinates. Then

$$K_{n,a} = \sum_{x \in V_a(\mathbb{F}_q)} \text{Tr}(F, \mathcal{F}(\psi\sigma)).$$

**2.3 Some Cohomological Results**

**Theorem 2.3.** Let $X_0$ be a smooth, connected curve over a finite field $k$ of characteristic $p$, $U_0$ an open subscheme of $X$ and $\mathcal{F}_0$ an $\ell$-adic sheaf on $U_0$ such that the natural map $j_!\mathcal{F}_0 \to j_*\mathcal{F}_0$ is an isomorphism. Then, $j_!\mathcal{F}_0 \to Rj_*\mathcal{F}_0$ is also an isomorphism.

**Proof.** It suffices to show that for all $x \in X - U$, $0 = (R^lj_*\mathcal{F})_x \cong H^i(\text{Spec} \mathcal{O}_x^\text{sh}, \mathcal{F})$. But since $\mathcal{F}_x$ is supported at the generic point of $\text{Spec} \mathcal{O}_x^\text{sh}$, this cohomology is just $H^i(I, \mathcal{F}_x)$.

where $I$ is the inertia group at $x$. Since $H^j(P, \mathcal{F}_{\overline{\eta}}) = 0$ for all $j > 0$ (since it’s the cohomology of a pro $p$ group in a pro $\ell$ thing—look at finite subquotients of $P$, since that’s how cohomology of a profinite group is computed, and see that the cohomology must be both $p$ and $\ell$ torsion), the spectral sequence $H^j(I/P, H^i(P, \mathcal{F}_{\overline{\eta}})) \Rightarrow H^{i+j}(I, \mathcal{F}_{\overline{\eta}})$ implies that

$$H^i(I, \mathcal{F}_{\overline{\eta}}) \cong H^i(I/P, \mathcal{F}_{\overline{\eta}}^P).$$

Now, $I/P$ is a pro-infinite cyclic group, its cohomology concentrates at degree 0 and 1: there is a resolution of 2 terms $A \to A$, like in the infinite cyclic case. Thus, $\dim H^0 = \dim H^1$. But $\dim H^0 = 0$ since $j_! \cong j_*$. Thus, so is $H^1$, and we are done. \qed

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Corollary 2.4. Let \( X_0, U_0, F_0 \) be as above, such that \( j_! F_0 \cong j_* F_0 \). Then the natural map \( H^i_c(U, F) \to H^i(U, F) \) is an isomorphism for all \( i \).

Proof. From the previous theorem 2.3, we know that \( j_! F \cong Rj^* F \). Thus,

\[
R\Gamma_c(U, F) = R\Gamma(X, j_! F) \cong R\Gamma(X, Rj^* F) \cong R\Gamma(U, F).
\]

The following theorem is a cohomological reflection of the following fact: let \( \chi : G \to \mathbb{Q}_l^\times \) be a non-trivial character of a finite group, then

\[
\sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(gh) = \chi(h) \sum_{g \in G} \chi(g).
\]

Since \( \chi \) is non-trivial, we can choose \( h \) such that \( \chi(h) \neq 1 \). Thus,

\[
(\chi(h) - 1) \sum_{g \in G} \chi(g) = 0
\]

and hence

\[
\sum_{g \in G} \chi(g) = 0.
\]

Theorem 2.5. Let \( G_0 \) be a connected commutative group over \( \mathbb{F}_q \) and \( \chi : G_0(\mathbb{F}_q) \to \mathbb{Q}_l^\times \) a non-trivial character. Then \( H^*_c(G_0, F(\chi)) = 0 \). Hence, \( H^*(G, F(\chi)) = 0 \), by Poincaré duality.

Proof. Let \( x \in G_0(\mathbb{F}_q) \), and denote \( t_x \) the translation by \( x \), then we have

\[
\mathcal{L} \circ t_x = \mathcal{L},
\]

since \( \mathcal{L}(x) = 1 \). Thus, \( t_x \) is a morphism of the Lang torsor \( \mathcal{L} \) of \( G_0 \). This induces a morphism on \( \mathcal{L}(\chi) \) by multiplication by \( \chi(g)^{-1} \), which also induces a morphism on \( H^*(G, \mathcal{L}(\chi)) \) by multiplication by \( \chi(g)^{-1} \). Since \( \chi \) is non-trivial, we can choose \( g \) such that \( \chi(g) \neq 1 \). Thus, if we can show that the action of \( \chi(g) \) is the same as the action of \( \chi(e) = 1 \), then we must have \( H^*(G, \mathcal{L}(\chi)) = 0 \). This is the cohomological reflection of the identity above.

This is done by a homotopy argument. First, we put all these morphisms into a family:

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\text{id}, t_x} & G \times G \\
\downarrow \text{id, } \mathcal{L} & & \downarrow \text{id, } \mathcal{L} \\
G \times G & \xrightarrow{\text{id, } t_x \chi(e)} & G \times G
\end{array}
\]

where \( x \) belongs to the first coordinate. The following lemma will finish the job.

Lemma 2.6. Let \( X, Y \) be two schemes over an algebraically closed field \( k \), with \( X \) separated, of finite type, and \( Y \) connected. Let \( F \) be a sheaf over \( X \) and \( (\rho, \epsilon) \) a family of endomorphisms of \( (X, F) \) parametrized by \( Y \):

(i) \( \rho : Y \times_k X \to Y \times_k X \) is a \( Y \)-morphism.
(ii) \( \varepsilon : \rho^*pr_2^*\mathcal{F} \to pr_2^*\mathcal{F} \) a morphism of sheaves.

Suppose that \( \rho \) is proper. Then if we denote \( \rho_H(y)^* \) the endomorphism of \( H_c^i(X, \mathcal{F}) \) induced by \((\rho_y, \varepsilon_y)\), where \( y \in Y(k) \), then \( \rho_H(y)^* \) is independent of \( y \).

Proof. By proper base change, we know that \( R^p\text{pr}!pr_2^*\mathcal{F} \) is a constant sheaf on \( Y \), whose fibers are \( H^p(X, \mathcal{F}) \). Now, \( \rho_H(y)^* \) is the fiber of the following endomorphism

\[
R^p\text{pr}!pr_2^*\mathcal{F} \xrightarrow{\rho^*} R^p\text{pr}!\rho^*pr_2^*\mathcal{F} \xrightarrow{\varepsilon^*} R^p\text{pr}!pr_2^*\mathcal{F}.
\]

Now, note that a morphism of any locally constant sheaf is determined at a point (when the scheme is connected).

Remark 2.7. There is an alternative proof of theorem 2.5 using the same strategy as for the Artin-Scheier sheaf over \( \mathbb{A}^1 \). First, note that via the equivalence of categories

\[
\{\text{Smooth sheaves}\} \leftrightarrow \{\text{representation of } \pi_1\},
\]

if we have a finite connected étale Galois cover \( \pi : Y \to X \), then \( \pi_* \) is the same as the induction \( \text{Ind}^{\pi_1(X)}_{\pi_1(Y)} \). Thus, applied to \( \overline{\mathbb{Q}}_l \), the push forward is just the one corresponds to the regular representation of \( \text{Gal}(Y/X) \). Applied to \( \mathcal{L} : G_0 \to G_0 \) (connected, commutative group), and note that the Galois group of this is \( G_0(\mathbb{F}_q) \)

\[
\mathcal{L} \overline{\mathbb{Q}}_l \cong \bigoplus_{\chi \in G_0(\mathbb{F}_q)} \mathcal{F}(\chi).
\]

Then, the same proof as in the case of \( \mathbb{A}^1_0 \) carries over.

3 Estimates

3.1 A General Estimates from Weil Conjectures

As noted above, all the Artin-Shreier sheaves (and hence, also tensors, direct sums thereof) have weight 1. Thus, if \( \mathcal{F} \) is just a sheaf, \( H^i_c(X, \mathcal{F}) \) has weight \( \leq i \), by the Weil conjectures, and hence,

\[
\left| \sum_{x \in X_0(\mathbb{F}_q)} \text{Tr}(F, \mathcal{F}_x) \right| = \left| \sum_{i} (-1)^i \text{Tr}(F, H^i_c(X, \mathcal{F})) \right| \leq \sum_{i} q^{i/2} \dim H^i_c(X, \mathcal{F}).
\]

But for Gauss sums and Kloosterman sums, we can get a more precise information about the dimension and the weights of the cohomology groups.
3.2 Gauss Sums

From the above, we have
\[ \tau(\chi, \psi) = -\sum_{i=0}^{2} (-1)^i \text{Tr}(F, H^i_c(G_m, \mathcal{F}(\chi^{-1}\psi))). \]

We start with the following cohomological result.

**Lemma 3.1.** Let \( U_0 \subset X_0 \) be an open subscheme of a projective smooth curve, and \( \mathcal{F} \) an étale sheaf on \( U_0 \). Suppose \( x \in X_0 - U_0 \) such that \( \mathcal{F} \) is totally ramified at \( x \), then \( H^i_c(U, \mathcal{F}) = 0 \) for all \( i \), except possibly at \( i = 1 \).

**Proof.** Since \( X_0 \) is of dimension 1, we only need to worry about \( i = 0, 1, 2 \). We have the vanishing for \( i = 0 \) since \( U_0 \) is a proper open subscheme. For \( i = 2 \), by Poincare duality, we have
\[ \dim H^2_c(U, \mathcal{F}) = \dim H^0(U, \mathcal{F}) = 0, \]
due to the fact that \( \mathcal{F} \), and hence \( \mathcal{F} \) is totally ramified. \( \square \)

**Theorem 3.2.** The cohomology of \( G_m \) with coefficient in \( \mathcal{F}(\psi \chi^{-1}) \) satisfies the following:

(i) If \( \chi \) is non-trivial, then \( H^*_c \rightarrow H^* \) is an isomorphism.

(ii) \( H^i_c = 0 \) for \( i \neq 1 \) and \( \dim H^1_c = 1 \).

(iii) \( F \) acts on \( H^1_c \) via multiplication by \( \tau(\chi, \psi) \).

**Proof.** Clearly, (iii) is a consequence of (i) and (ii). We will use corollary 2.4 to show (i). To do that, we need to show that \( \mathcal{F}(\psi \chi^{-1}) \) is totally ramified at 0 and \( \infty \) when \( \chi \) is non-trivial.

At 0, \( \mathcal{F}(\psi) \) is unramified, and \( \mathcal{F}(\chi^{-1}) \) is totally tamely ramified (when \( \chi \) is non-trivial). Thus, \( \mathcal{F}(\psi \chi^{-1}) \) is totally ramified at 0. At \( \infty \), we know that \( \mathcal{F}(\chi^{-1}) \) is tamely ramified. If \( \psi \) is trivial, then \( \mathcal{F}(\psi) \) is non-ramified and hence, \( \mathcal{F}(\chi^{-1}\psi) \) is also totally tamely ramified. If \( \psi \) is non-trivial, then \( \mathcal{F}(\psi) \) is totally wildly ramified (with Swan conductor 1). Thus, \( \mathcal{F}(\chi^{-1}\psi) \) is also totally ramified at \( \infty \). This implies that \( j_* \mathcal{F} \cong j_* \mathcal{F} \) and hence, \( H^*_c \rightarrow H^* \) is an isomorphism.

The vanishing of (ii) is guaranteed by lemma 3.1 above. For the second part, we recall that \( \mathcal{F}(\psi \chi^{-1}) \) is tamely ramified at 0 and wildly ramified at \( \infty \) with Swan conductor 1. Therefore, the Grothendieck-Ogg-Shafarevich formula says that \( \dim H^1_c = 1 \). \( \square \)

**Proof of 1.2** From theorem 3.2 above, we know that
\[ \tau(\chi, \psi) = -\text{Tr}(F, H^1_c(G_m, \mathcal{F}(\psi \chi^{-1}))). \]

The isomorphism \( H^i \cong H^i_c \) implies that the weight must be precisely 1, and hence, we have
\[ |\tau(\chi, \psi)| = q^{1/2}. \]
Proof of 1.3 The Hasse-Davenport's identity is derived easily from the following
\[ \tau(\chi N, \psi \text{Tr}) = \text{Tr}(F^n, H^1_c(G_m, \mathcal{F}(\psi \chi^{-1}))) \]
\[ = \text{Tr}(F, H^1_c(G_m, \mathcal{F}(\psi \chi^{-1})))^n \]
(due to 1-dimensionality)
\[ = \tau(\chi, \psi)^n. \]

Remark 3.3. The following theme is similar to the Kloosterman sums case:

(i) Cohomology concentrates at one middle degree. This suggests a link to perverse sheaves.

(ii) Purity follows from isomorphism \( H^i_c \cong H^i \).

Remark 3.4. In the cohomology study above, we can replace \( \mathcal{F}(\chi^{-1}) \) by any \( \mathcal{K}_n(\chi) \) as long as \( n \) is relative prime to \( p \) (to ensure tameness).

3.3 Kloosterman Sums

The following identity serves as the inspiration for our cohomological study of Kloosterman Sums:

\[ K_{n,a} = \sum_{x_1, x_2, \ldots, x_n = a} \psi(x_1 + x_2 + \cdots + x_n) \]
\[ = \sum_{x_1} \psi(x_1) \sum_{x_2 \cdots x_n = a/x_1} \psi(x_2 + x_3 + \cdots + x_n) \]
\[ = \sum_{x \in \mathbb{F}_q} \psi(x) K_{n-1,a/x}. \]

Recall: denote \( \pi, \sigma : \mathbb{A}_0^n \to \mathbb{A}_0^1 \) defined by the product and sum of the coordinates respectively. Let \( \psi \) be an additive character, then the sheaf \( \mathcal{F}(\sigma \psi) \) restricted to \( V_a^{n-1} = \pi^{-1}(a) \) is the geometrization of our Kloosterman sum \( K_{n,a} \).

As in the case of Gauss sums, the main estimate for \( K_{n,a} \) comes from the following cohomology result.

Theorem 3.5. The cohomology of \( V_a^{n-1} \) with coefficient in \( \mathcal{F}(\psi \sigma) \) satisfies the following:

(i) \( H^i_c = 0 \) for all \( i \neq n-1 \).

(ii) \( H^* \cong H^s \).

(iii) For \( a \neq 0 \), \( \dim H^{n-1}_c = n \).

(iv) For \( a = 0 \), \( \dim H^{n-1}_c \) is canonically isomorphic to \( \overline{\mathbb{Q}}_l \).

We will prove this theorem alongside with its global analogue.

Theorem 3.6. We have the following global analog of 3.5
(i) The sheaf $R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)$ is smooth of rank $n$ over $\mathbb{A}^1 - \{0\}$.

(ii) The extension by $0$ of $R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)$ from $\mathbb{A}^1$ to $\mathbb{P}^1$ is the same as the direct image to $\mathbb{P}^1$ of the restriction to $\mathbb{A}^1 - \{0\}$.

(iii) At $0$, the monodromy tame and unipotent, with exactly 1 Jordan block.

(iv) At $\infty$, the wild inertia acts without non-zero fixed point, and the Swan conductor is equal to $1$.

(v) We have $R^{n-1} \pi_1 \mathcal{F}(\psi \sigma) \cong R^i \pi_* \mathcal{F}(\psi \sigma)$.

We will use the following notation: $3.5(n)$ and $3.6(n)$ are used to denote theorem $3.5$ and respectively $3.6$ for case $n$. The general strategy is an induction argument (the case $n = 1$ is trivial):

$\text{Theorem } 3.5(n) \xrightarrow{(\ast)} \text{Theorem } 3.6(n) \xrightarrow{(\ast\ast)} \text{Theorem } 3.5(n+1)$.

One of the points of globalizing is that even though each fiber is not nice, together they form a very nice family. For instance, we can take advantage of the vanishing theorem $2.5$.

Proof of $(\ast)$. By the proper base change theorem, we know that $R^i \pi_1 \mathcal{F}(\psi \sigma)_a \cong H^i_c(V_a, \mathcal{F}(\psi \sigma))$. We also know that $R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)$ is a constructible sheaf, and hence, there exists an open dense subscheme of $\mathbb{A}^1$ such that $R^i \pi_* \mathcal{F}(\psi \sigma)_a \cong H^i_c(V_a, \mathcal{F}(\psi \sigma))$. Thus, theorem $3.5(n)$ gives us

Lemma 3.7.

(i) $R^i \pi_1 \mathcal{F}(\psi \sigma) = 0$, for $i \neq n - 1$.

(ii) For $i = n - 1$, the stalks of this sheaf at all points $a \neq 0$ is of constant rank $n$. At $0$, it's of rank $1$.

(iii) On an open dense subscheme $U$, we have $R^i \pi_1 \mathcal{F}(\psi \sigma) \cong R^i \pi_* \mathcal{F}(\psi \sigma)$.

The lemma above shows that $H^q_c(\mathbb{A}^1, R^n \pi_1 \mathcal{F}(\sigma \psi)) = 0$, except possibly when $q = n - 1$. Thus, the Leray spectral sequence for $\pi$ collapse. But from theorem $2.5$, we know that $H^q_c(\mathbb{A}^n, \mathcal{F}(\psi \sigma)) \cong H^q(\mathbb{A}^n, \mathcal{F}(\psi \sigma)) = 0$. We must therefore get the vanishing of the whole 2nd page of the spectral sequence.

In particular, $H^q_c(\mathbb{A}^n, R^{n-1} \pi_1 \mathcal{F}(\sigma \psi)) = 0$. This means that $R^{n-1} \pi_1 \mathcal{F}(\sigma \psi)$ doesn’t have any isolated support at a point (no punctual support). But since the rank of $R^{n-1} \mathcal{F}(\psi \sigma)$ is constant on $\mathbb{A}^1 - \{0\}$, $R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)$ must be locally constant on $\mathbb{A}^1 - \{0\}$, and this finishes (i). This argument is very nice! Motto: use cohomology with compact support to detect punctual support on an open curve.

Next we will show (ii). First we worry about the point $0$. Let $\mathcal{G}$ be the direct image of the restriction of $R^{n-1} \pi_1 \mathcal{F}(\sigma \psi)$ to $\mathbb{A}^1 - \{0\}$. Then, we have the following exact sequence

$$
\begin{array}{c}
0 \longrightarrow R^{n-1} \pi_1 \mathcal{F}(\psi \sigma) \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0,
\end{array}
$$

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where \( \mathcal{O} \) has support only at 0. Note that the injectivity comes from the fact that \( R^{n-1} \pi_1 \mathcal{F} \) doesn’t have any punctual support. This gives
\[
0 = H^0_c(A^1, \mathcal{G}) \longrightarrow H^0_c(A^1, \mathcal{O}) \longrightarrow H^1_c(A^1, R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)) = 0,
\]
where the first equality is from the fact that \( A^1 \) is an open curve, and the second equality is from what we said above. Thus, \( H^0_c(A^1, \mathcal{O}) = 0 \), and hence, \( \mathcal{O} = 0 \), which implies \( R^{n-1} \pi_1 \mathcal{F}(\psi \sigma) \cong \mathcal{G} \). This concludes (ii) for the point 0: \( R^{n-1} \pi_1 \mathcal{F}(\psi \sigma) \) is the direct image of its restriction to \( A^1 - \{0\} \).

For the point \( \infty \), let \( j : A^1 \rightarrow \mathbb{P}^1 \) and let \( \Delta \) be the mapping cone of \( j_! R \pi_1 \mathcal{F}(\psi \sigma) \rightarrow Rj_* R \pi_1 \mathcal{F}(\psi \sigma) \). Then, from lemma [3.7] we know that the cohomology sheaves of \( \Delta \) has finite support. But observe that
\[
\mathcal{H}^n(\mathbb{P}^1, j_! R \pi_1 \mathcal{F}(\psi \sigma)) \cong \mathcal{H}^n_c(A^1, R \pi_1 \mathcal{F}(\psi \sigma)) \cong H^n_c(A^n, \mathcal{F}(\psi \sigma)) = 0
\]
and
\[
\mathcal{H}^n(\mathbb{P}^1, Rj_* R \pi_1 \mathcal{F}(\psi \sigma)) \cong \mathcal{H}^n(A^1, R \pi_1 \mathcal{F}(\psi \sigma)) \cong H^n(A^n, \mathcal{F}(\psi \sigma)) = 0.
\]
This means \( \mathcal{H}^n(\mathbb{P}^1, \Delta) = 0 \). Using a spectral sequence for hypercohomology, we see that \( H^0(\mathbb{P}^1, \mathcal{H}^n(\Delta)) = 0 \), and hence, \( \Delta = 0 \) and we get
\[
j_! R \pi_1 \mathcal{F}(\psi \sigma) \cong Rj_* R \pi_1 \mathcal{F}(\psi \sigma).
\]

Thus, in particular, \( R \pi_1 \mathcal{F}(\psi \sigma) \cong R \pi_1(\psi \sigma) \) which concludes (v), and both concentrate at 1 degree, \( n-1 \). Therefore,
\[
j_! R \pi_1 \mathcal{F}(\psi \sigma) \cong Rj_* R \pi_1 \mathcal{F}(\psi \sigma) \cong j_! R \pi_1 \mathcal{F}(\psi \sigma) \cong j_* R \pi_1 \mathcal{F}(\psi \sigma).
\]
This concludes (ii).

**Note that this kind of argument applies whenever the base is a curve and the top space has no cohomology.**

We have seen above (at the beginning of this proof) that \( H^c(A^1, R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)) = 0 \), and hence, by Grothendieck-Ogg-Shafarevich formula (note that there is one dimensional stalk at 0), we have
\[
\text{Swan}_0(R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)) + \text{Swan}_\infty(R^{n-1} \pi_1 \mathcal{F}(\psi \sigma)) = 1. \tag{1}
\]
Thus, this sheaf is wildly ramified at exactly one point 0 or \( \infty \).

Now, we use the following nice lemma (which is just linear algebra) to detect which one is which.

**Lemma 3.8.** Let \( D = \text{Gal} K^{\text{sep}}/K \) be the Galois group of a local field \( K \), whose residue field is finite. Let \( I \) and \( P \) be the inertia, and wild inertia groups respectively. Let \( V \) be a \( \overline{\mathbb{Q}}_l \)-representation of \( D \). Then

(i) If \( V^I = 0 \) and \( (V \otimes \chi)_I = 0 \) for all characters \( \chi \) of \( I \) that factors through \( I/P \), then \( V^P = 0 \). In particular, \( V \) is wildly ramified.
(ii) If \( V \) is tamely ramified, \( \dim V^i = 1 \) and \( (V \otimes \chi)^{!p} = 0 \) for all \( \chi \) as above. Then, the representation of \( I \) has to be unipotent with uniquely one Jordan block.

Proof. For (i), suppose \( V^p \neq 0 \), then \( V^p \) is a representation of \( D/P \). We know that \( I \) acts quasi-unipotently, and thus, there is a character \( \chi \) of \( I/P \) that makes the first entry of a Jordan block of \( V^p \otimes \chi \) one. This means that \( (V^p \otimes \chi)^{!p} \neq 0 \), and hence, \( (V \otimes \chi)^{!} \neq 0 \), which contradicts the hypothesis.

For (ii), we can argue in a similar way as above. \( \square \)

Let \( \mathcal{G} \) be a Kummer sheaf of rank 1 over \( \mathbb{G}_m \), i.e. \( \mathcal{G} = \mathcal{K}_n(\chi) \), such that \( \chi \neq 1 \). Using the computation in the Gauss sum section (cf. remark 3.4 as well) and Künneth formula, we get the following isomorphism

\[
H^*_c(V^*, \pi^* \mathcal{G} \otimes \mathcal{F}(\psi \sigma)) \cong H^*(V^*, \pi^* \mathcal{G} \otimes \mathcal{F}(\psi \sigma)),
\]

where \( V^* = \pi^{-1}(\mathbb{G}_m) = \mathbb{G}_m^n \). Using Leray spectral sequence for \( \pi \) and projection formula, we get

\[
\mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma) \cong \mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma).
\]

Let \( \Delta \) be the mapping cone of \( i_!(\mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma)) \to R\pi_!(\mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma)) \), where \( i \) is the inclusion \( \mathbb{G}_m \to \mathbb{P}^1 \). Then, a similar argument as earlier implies that \( \Delta = 0 \).

We can now conclude that \( i_!(\mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma)) \cong i_!(\mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma)) \). In particular, we have

(i) At 0 and \( \infty \), \( \mathcal{G} \otimes R^{n-1} \pi_! \mathcal{F}(\psi \sigma) \) is totally ramified.

(ii) From the lemma, we know that \( R^{n-1} \pi_! \mathcal{F}(\psi \sigma) \) is wildly ramified. The equality (1) then implies that \( R^{n-1} \pi_! \mathcal{F}(\psi \sigma) \) is totally wildly ramified, with Swan conductor 1.

(iii) Equality (1) then implies that \( R^{n-1} \pi_! \mathcal{F}(\psi \sigma) \) is tamely ramified (Swan_0 = 0). The lemma then implies that the action of \( I/P \) is unipotent, with a unique Jordan block.

This concludes the proof of (**). \( \square \)

**Proof of (**)**. The case where \( a = 0 \) is treated separately in a simple way using a spectral sequence argument. We will now deal with the case where \( a \neq 0 \).

We will give a cohomological reflection of the identity

\[
K_{n+1,a} = \sum_{x \in \mathbb{P}^n_{/q}} \psi(x) K_{n,a/\chi}.
\]

Denote \( x_0, x_1, \ldots, x_n \) the coordinates of \( \mathbb{A}^{n+1} \) and \( V_a^r \subset \mathbb{A}^{n+1} \) defined by \( x_0 x_1 \cdots x_n = a \). Let \( g : \mathbb{A}^{n+1} \to \mathbb{A}^1 \) be defined by the projection onto the first coordinate \( x_0 \). By abuse of notation, we will write \( g|_{V_a^r} = g \) as well, and note that \( g|_{V_a^r} : V_a^r \to \mathbb{G}_m \). Let \( \tau : \mathbb{G}_m \to \mathbb{G}_m \) be an involution defined by \( x \mapsto ax^{-1} \), and let \( \pi : \mathbb{A}^{n+1} \to \mathbb{A}^1 \) by \( \pi(x_0, \ldots, x_n) = x_1 x_2 \cdots x_n \), then \( g|_{V_a} = \tau \pi|_{V_a} \).

\[\text{1} \]

The long exact sequence of hyper-cohomology has consecutive isomorphic terms due to \( \Delta \) has punctual support due to \( \square \)
As suggested by the formula above, we will use the Leray spectral sequence for \( g \). We write \( A_{n+1} = A_1 \times A_n \), and note that the Artin-Shreier sheaf of interest on \( V_a \) is \( \mathcal{F}(\psi \sigma) \cong \mathcal{F}(\psi) \boxtimes \mathcal{F}(\psi \sigma') \), where \( \sigma \) is sum of all \( n + 1 \) coordinates, and \( \sigma' \) is sum of the last \( n \) coordinates. By projection formula, we have (everything restricted to \( V_a^n \))

\[
Rg_*(\mathcal{F}(\psi) \boxtimes \mathcal{F}(\psi \sigma')) \cong \mathcal{F}(\psi) \otimes Rg_* \mathcal{F}(\psi \sigma') \cong \mathcal{F}(\psi) \otimes \tau^s R\pi_* \mathcal{F}(\psi \sigma'). \tag{4}
\]

and

\[
Rg_!(\mathcal{F}(\psi) \boxtimes \mathcal{F}(\psi \sigma')) \cong \mathcal{F}(\psi) \otimes \tau^s R\pi_! \mathcal{F}(\psi \sigma'). \tag{5}
\]

Observe the following commutative diagram

\[
\begin{array}{ccc}
V_a & \cong & G_m \\
\downarrow \pi & & \downarrow \pi \\
G_m & \\ & \lllt \subset A_n
\end{array}
\]

and the sheaf \( \mathcal{F}(\psi \sigma) \) on \( A^n \) is the same as the sheaf \( \mathcal{F}(\psi \sigma') \) on \( V_a^n \). Thus, we can use results in \([3.6]\) here.

Using (4) and (5), we have the following spectral sequences:

\[
'E^{pq}_{2} = H^p(G_m, \mathcal{F}(\psi) \otimes \tau^s R\pi_! \mathcal{F}(\psi \sigma')) \Rightarrow H^{p+q}(V_a, \mathcal{F}(\psi \sigma)).
\]

From theorem \([3.6]\), we see that \( 'E^{pq}_{2} \cong ''E^{pq}_{2} \). Hence,

\[
H^r(V_a, \mathcal{F}(\psi \sigma)) \cong H^r(V_a, \mathcal{F}(\psi \sigma)),
\]

and this finishes (ii) of theorem \([3.5]\) (n+1).

By Poincaré duality and cohomological dimension of affine schemes, we get (i) for \([3.5]\) (n+1) as well.

For (iii), we first note that \( R^{n-1} \pi_! \mathcal{F}(\psi \sigma') \) (all other ones vanish) is tamely ramified at 0 and wildly ramified at \( \infty \) with Swan conductor 1, by theorem \([3.6]\) (n). Thus, \( \tau^s R^{n-1} \pi_! \mathcal{F}(\psi \sigma') \) is tamely ramified at \( \infty \) and wildly ramified at 0, with Swan conductor 1. But we know that \( \mathcal{F}(\psi) \) is not ramified at 0, and wildly ramified at \( \infty \) with Swan conductor 1. Thus, \( \mathcal{F}(\psi) \otimes \tau^s R^{n-1} \pi_! \mathcal{F}(\psi \sigma') \) is totally wildly ramified at both 0 and \( \infty \) with Swan conductor 1 and \( n \) (the rank of \( \tau^s R^{n-1} \pi_! \mathcal{F}(\psi \sigma') \) respectively.

Observe that \( ''E^{pq}_{2} = 0 \) unless \( p = 1 \) and \( q = n - 1 \). Moreover, by Grothendieck-Ogg-Shafarevich, we know that \( ''E^{pq}_{2} = n + 1 \) and we are done.

**Remark 3.9.** We didn't really use the thing about Jordan block.

### 4 Reference

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