TOPIC PROPOSAL
THE TOPOLOGY OF SURFACE BUNDLES

NICK SALTER
AS DISCUSSED WITH BENSON FARB

1. Introduction

Let $M$ be a closed oriented manifold, and let $S_g$ denote the closed surface of genus $g$; unless otherwise specified, assume $g \geq 2$. A surface bundle over $M$ is a fiber bundle with base space $M$ and fiber $S_g$. The study of surface bundles in low dimensions unites ideas in the topology of manifolds of dimensions one, two, three, and four.

The Virtual Fibering Conjecture (recently resolved) asserts that nearly all irreducible three-manifolds have a surface bundle as a finite cover. Work of Donaldson shows that all symplectic four-manifolds admit the structure of a Lefschetz fibration, which is a surface bundle over a surface off of a finite set of points in the base. Work of Farb shows that the family of surface bundles over a surface is large enough to have an unsolvable homeomorphism problem. The study of surface bundles also draws on results from mapping class groups and the group $\text{Homeo}^+(S^1)$, bringing in connections to two and one dimensions, respectively.

A basic problem in the theory of fiber bundles is to understand the family of bundles over a particular base space. Let $F$ be a closed oriented $n$-manifold. The theory of classifying spaces gives the existence of a space $B\text{Diff}^+(F)$ equipped with a “universal oriented $F$ bundle” $\zeta$ and a correspondence

$$\{\text{Isomorphism classes of oriented } F \text{ bundles over } M\} \leftrightarrow \{\text{Homotopy classes of maps } M \to B\text{Diff}^+(F)\}.$$ 

realized by pullback of $\zeta$. In the case where $F$ is a closed surface, there is a second correspondence which brings the mapping class group into play.

Definition 1.1. Let $F$ be a manifold. The mapping class group of $F$ is the quotient

$$\text{Mod}(F) = \text{Diff}^+(F)/\text{Diff}_0^+(F)$$

consisting of isotopy classes of orientation-preserving homeomorphisms.

We can view the above definition as describing an exact sequence of topological groups, and hence a fibration. In the case when $F = S_g$ is a surface of genus $g \geq 2$, the result of Earle-Eells gives that $\text{Diff}_0^+(S_g)$ is contractible, so that by the long exact sequence of a fibration in conjunction with Whitehead’s theorem, there is a homotopy equivalence $\text{Diff}^+(S_g) \sim \text{Mod}(S_g)$. This proves:

Theorem 1. For $g \geq 2$, $B\text{Diff}^+(S_g)$ is a $K(\text{Mod}(S_g), 1)$ space, so that there is a correspondence

$$\{\text{Isomorphism classes of oriented } F \text{ bundles over } M\} \leftrightarrow \{\text{Conjugacy classes of representations } \pi_1(M) \to \text{Mod}(S_g)\}.$$ 

The representation $\pi_1(M) \to \text{Mod}(S_g)$ is called the monodromy of the corresponding bundle, and Theorem 1 shows that it is a complete invariant of surface bundles. It also indicates that the mapping class group plays a key role in the theory of surface bundles. However, the monodromy is difficult to compute in practice, and there has been much interest in developing other, more computable invariants. In this proposal, we discuss several such invariants.
2. The Mapping Class Group

As we have seen in the Introduction, the mapping class group encodes the classification of surface bundles. In this section, we indicate some of the basic results concerning the algebra of the mapping class group, and its action on Teichmüller space. These results can be found in [2].

2.1. Finite Generation and Finite Presentation. As \( \text{Mod}(S) \) is a discrete group, our first concern will be to establish its finite generation and finite presentation. Both of these results are proved by studying the action of the mapping class group on certain simplicial complexes. The generating set that we will consider will consist of \textbf{Dehn twists}. The Dehn twist about a curve \( \gamma \) is the homeomorphism that cuts the surface along \( \gamma \), applies a full twist to one of the ends, and then reglues. As a mapping class this only depends on the isotopy class of \( \gamma \). Our proof of finite generation will be inductive, and proceeds by cutting the surface open to reduce the genus and increase the number of punctures. As such, we will need to study the relationship between \( \text{Mod}(S_g) \) and \( \text{Mod}(S_{g,1}) \), which is the content of the Birman exact sequence:

\[
\begin{align*}
1 & \longrightarrow \pi_1(S) \longrightarrow \text{Mod}(S_{g,1}) \longrightarrow \text{Mod}(S_g) \longrightarrow 1.
\end{align*}
\]

**Proposition 2.1.** (Birman) Let \( g \geq 1 \), and identify \( \pi_1(S_g) \) with the “point-pushing” subgroup of \( \text{Mod}(S_{g,1}) \). Then the following sequence is exact:

\[
1 \longrightarrow \pi_1(S) \longrightarrow \text{Mod}(S_{g,1}) \longrightarrow \text{Mod}(S_g) \longrightarrow 1.
\]

\textbf{Proof.} (sketch) Using the contractibility of \( \text{Diff}^+(S) \), this reduces to a portion of the long exact sequence for the fiber bundle \( \text{Diff}^+(S) \to S \) sending a diffeomorphism \( f \) to the image of the marked point \( f(p) \). \( \square \)

**Theorem 2.** Let \( g \geq 1 \) and \( n \geq 0 \) be given. Then \( \text{PMod}(S_{g,n}) \) is generated by a finite number of Dehn twists about nonseparating simple closed curves.

\textbf{Proof.} (sketch) We first induct on the number of punctures using the Birman sequence. For the induction on genus, we study the action of \( \text{Mod}(S) \) on the \textbf{curve complex} \( \mathcal{C}(S) \). This is the simplicial flag complex whose vertices are isotopy classes of simple closed curves, with two vertices joined by an edge if they have disjoint representatives. A surgery argument shows that the curve complex is connected, and then a geometric group theory lemma provides a generating set in terms of an isotropy subgroup. We identify this with the mapping class group of the surface obtained by cutting along the fixed curve, which completes the induction on genus. \( \square \)

The proof that \( \text{Mod}(S) \) is finitely presented is similar in spirit: one finds a nice complex (the \textbf{arc complex}) on which the mapping class group acts in order to apply techniques from geometric group theory.

2.2. Homological Properties. Since \( \text{BDiff}^+(S) \) is a \( K(\text{Mod}(S),1) \) space, the cohomology of the mapping class group can be identified with characteristic classes of surface bundles. This is why we are interested in the homology of \( \text{Mod}(S) \). Here, we compute \( H_1 \) and \( H_2 \).

**Theorem 3.** Let \( g \geq 3 \). Then \( H_1(\text{Mod}(S_g);\mathbb{Z}) \) is trivial.

\textbf{Proof.} (sketch) By definition,

\[
H_1(G;\mathbb{Z}) = G/[G,G].
\]

Let \( T_a \) denote the Dehn twist about the isotopy class of curves \( a \). By Theorem 2, it suffices to study what happens to \( T_a \) for \( a \) nonseparating. By change of coordinates, all \( T_a \) are conjugate, so that for \( h \) the class of any \( T_a \), we have

\[
H_1(\text{Mod}(S);\mathbb{Z}) = \langle h \rangle.
\]

When \( g \geq 3 \), the \textbf{lantern relation} says that a certain product of four twists is equal to the product of three others, so that \( h^4 = h^3 \) and so \( H_1(\text{Mod}(S);\mathbb{Z}) = 1 \) as claimed. \( \square \)
Theorem 4. For $g \geq 4$ we have

i. $H_2(\text{Mod}(S_g); \mathbb{Z}) \cong \mathbb{Z}$

ii. $H_2(\text{Mod}(S_{g,1}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. (sketch) We apply Hopf’s formula, which applies to a finitely-presented group $G = F/K$, where $K$ is normally generated by some finite set $R$ of relators:

$$H_2(G; \mathbb{Z}) \cong \frac{K \cap [F,F]}{[K,F]}.$$ 

Wajnryb gave explicit presentations for $\text{Mod}(S_g)$ and $\text{Mod}(S_{g,1})$, which can be plugged into Hopf’s formula. This gives a system of linear equations from which an upper bound on $H^2(\text{Mod}(S_g); \mathbb{Z})$ can be extracted. We can then explicitly construct nontrivial (co)-cycles to finish the proof; see Section 3 below.

2.3. Moduli Space. Groups are understood via the spaces on which they act. In our setting, the mapping class group acts on Teichmüller space $\text{Teich}(S_g)$, which parameterizes isotopy classes of homeomorphisms $\psi : S_g \to X$ where $X$ is endowed with a hyperbolic metric. The mapping class group has a natural action via precomposition, and it is possible to introduce the Teichmüller metric on $\text{Teich}(S_g)$ in such a way that this action is by isometries.

Theorem 5. For $g \geq 2$, $\text{Teich}(S_g) \cong \mathbb{R}^{6g-6}$.

Proof. (sketch) We introduce the Fenchel-Nielsen coordinates on Teichmüller space. A pants decomposition of $S_g$ consists of a collection of $3g-3$ disjoint isotopy classes of simple closed curves. To each such curve we associate a length and a twist parameter, and show that these $6g-6$ real numbers determine a unique hyperbolic structure on $S_g$.

Theorem 6. (Fricke) The action of $\text{Mod}(S_g)$ on $\text{Teich}(S_g)$ is properly discontinuous.

Proof. (sketch) By considering the properly-discontinuous action of $\pi_1(X)$ on $\mathbb{H}^2$, we see that the length spectrum on a hyperbolic surface $X$ is discrete. Wolpert’s lemma describes how the length of a curve changes as the marking of the surface moves through Teichmüller space. We apply these results to a pair of filling curves and then use the Alexander method.

The quotient $\text{Teich}(S_g)/\text{Mod}(S_g)$ is known as the moduli space $\mathcal{M}(S_g)$. In the classification of surface bundles, moduli space is of interest because it is a rational classifying space:

Theorem 7. For $g \geq 2$, we have

$$H^*(\mathcal{M}(S_g); \mathbb{Q}) \cong H^*(\text{BDiff}^+(S_g); \mathbb{Q}).$$

Proof. (sketch) By studying the symplectic representation, one can find a finite-index torsion-free subgroup $\Gamma \leq \text{Mod}(S_g)$. The result follows by the Borel construction and a transfer argument.

3. Characteristic Classes of Surface Bundles

3.1. The Euler Class. A central ingredient in the discussion to follow is the Euler class, which will show up in several guises, drawing on various areas of mathematics.

Definition 3.1. The Euler class is any one of the following cohomology classes, which will all be denoted as $eu \in H^2(M; \mathbb{Z})$ in the appropriate space:

(1) The cocycle in $H^2(\text{Homeo}^+(S^1); \mathbb{Z})$ corresponding to the central extension

$$0 \to \mathbb{Z} \to \widetilde{\text{Homeo}}^+(S^1) \to \text{Homeo}^+(S^1) \to 1,$$

where $\widetilde{\text{Homeo}}^+(S^1)$ is the group of all lifts of $\text{Homeo}^+(S^1)$ to $\text{Homeo}^+(\mathbb{R})$. 

(2) For an oriented flat circle bundle $E \to M$ (i.e. one specified by a monodromy map $h : \pi_1(M) \to \text{Homeo}^+(S^1)$), we may pull back the Euler class of (1) to obtain a class

$$h^*(eu) \in H^2(\pi_1(M); \mathbb{Z}) = H^2(K(\pi_1(M), 1), \mathbb{Z}).$$

We can then pull $h^*(eu)$ back along the classifying map $M \to K(\pi_1(M), \mathbb{Z})$ to obtain an Euler class in $H^2(M, \mathbb{Z}).$

(3) For a general oriented circle bundle $E \to M$, we define the Euler class as the cocycle in $H^2(M; \pi_1(S^1))$ measuring the obstruction to extending a section from the one-skeleton to the two-skeleton.

(4) For an oriented rank-two real vector bundle $\eta$ over $M$, the Euler class is defined as the class in (3) of the associated unit circle bundle (relative to any Riemannian metric).

(5) Also in the case of an oriented rank-two real vector bundle, we may take the Euler class as the Poincaré dual of the zero locus of a section that is transverse to the zero section.

(6) In the case of a surface bundle $\eta : S_g \to E \to M$ with section, there is an Euler class in $H^2(M; \mathbb{Z})$ as follows. Since $\eta$ has a section, the monodromy map

$$h : \pi_1(M) \to \text{Mod}(S_{g,1})$$

has image in $\text{Mod}(S_{g,1})$. Using hyperbolic geometry, Nielsen constructed a faithful representation

$$\text{Mod}(S_{g,1}) \to \text{Homeo}^+(S^1)$$

by lifting a mapping class $f$ to a diffeomorphism $\phi$ which then lifts to the hyperbolic plane; because $f$ fixes a marked point there is a canonical lift $\tilde{\phi}$ fixing a basepoint. This is a quasi-isometry and so it induces a homeomorphism of the boundary circle, giving rise to an element of $\text{Homeo}^+(S^1)$. In other words, Nielsen’s construction associates a flat circle bundle over $M$ to every surface bundle with section, and the Euler class is defined as the class from (2) of this circle bundle.

(7) In the case of a general surface bundle $\eta : S_g \to E \to M$, there is an Euler class in $H^2(E, \mathbb{Z})$ which is the Euler class in the sense of (4) of the 2-plane bundle $T\eta$ whose fiber is the kernel of the differential of the projection $TE \to TM$; alternatively $T\eta$ assigns to a point $x \in E$ the tangent space to the fiber surface through $x$ of $\eta$.

**Theorem 8.** For an oriented rank-2 vector bundle $\eta$ over $M$, definitions (4) and (5) yield the same classes in $H^2(M; \mathbb{Z})$, and in the case where the associated circle bundle is flat, these are both equal to the definition given in (2). In the case of a surface bundle $\eta$ given by a map $\pi : E \to M$, let $\pi^*(\eta)$ denote the bundle over $E$ obtained by pullback along $\pi$. Then this has a section and the Euler class of $\pi^*(\eta)$ in the sense of (6) agrees with the Euler class of $\eta$ in the sense of (7).

**Proof. (sketch)** For the equivalence of (4), and (5), we make use of the fact

$$H^2(\text{Homeo}^+(S^1); \mathbb{Z}) \cong H^2(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}$$

to see that it suffices to compare our two definitions on a single bundle. A lemma of Milnor gives the equivalence of (2) and (4). For the equivalence of (6) and (7), we show directly that the relevant bundles are isomorphic, following Morita’s proof in [4]. □
3.2. Morita-Mumford-Miller Classes. Let $\pi : E \rightarrow M$ be a surface bundle with Euler class $eu(\pi) \in H^2(E;\mathbb{Z})$ as in Definition 3.1.7. We define the $i^{th}$ Morita-Mumford-Miller class (MMM class, for short) as follows:

$$e_i = \pi_*(eu(\pi)^{i+1}) \in H^{2i}(M;\mathbb{Z}),$$

where $\pi_* : H^{2i+2}(E;\mathbb{Z}) \rightarrow H^{2i}(M;\mathbb{Z})$ is the Gysin map associated to $\pi$. It is then easy to see from the properties of the constructions involved that these give well-defined characteristic classes of surface bundles, in the sense that they are functorial under bundle maps. The discussion in this section follows S. Morita in [5], except where otherwise mentioned.

**Theorem 9.** For each $n \geq 0$, there exists $g(n)$ such that for $g \geq g(n)$, the map

$$\mu : \mathbb{Q}[e_1, e_2, \ldots] \rightarrow H^n(\text{Mod}(S_g);\mathbb{Q})$$

is injective in degrees $\leq 2n$.

**Proof.** (sketch) The first step is to show that each $e_i$ is nonzero. We construct surface bundles for which the MMM classes are computable via Morita’s $m$-construction, which is a generalization of Atiyah and Kodaira’s surface bundle over a surface. The $m$-construction combines four operations on an $S_g$-bundle $\pi : E \rightarrow M$.

1. Pull back along $\pi$ to obtain a surface bundle $\tilde{\pi} : \tilde{E} \rightarrow E$, which has a diagonal section.
2. Let $S_{g'}$ be an $m$-fold cover of $S_g$, and take a finite cover $f : M' \rightarrow M$ such that the pullback of $\pi$ to $M'$ admits a fiberwise $m$-fold covering; i.e. there is an $S_{g'}$-bundle $\pi' : E' \rightarrow M'$ and a map $E' \rightarrow f^*(E)$ that is an $m$-fold cover.
3. Given a codimension-two submanifold $D \subset E$ with Poincaré dual $\nu \in H^2(E;\mathbb{Z})$, take a finite cover $f : M^* \rightarrow M$ such that the class $f^*(\nu) \in H^2(f^*(E);\mathbb{Z})$ Poincaré dual to $f^{-1}(D)$ vanishes mod $m$.
4. Given a codimension-two submanifold $D \subset E$ with Poincaré dual $\nu \in H^2(E;\mathbb{Z})$ such that $\nu$ vanishes mod $m$, take an $m$-fold cyclic ramified covering of $E$ branched along $D$.

We then find a formula relating the MMM classes of the input and output of the $m$-construction which allows us to inductively construct surface bundles for which the $e_i$ do not vanish. As for algebraic independence, the connect-sum operation gives a map

$$\iota : \text{Mod}(S_{g_1,0,1})^{d_1} \times \cdots \times \text{Mod}(S_{g_n,0,1})^{d_n} \rightarrow \text{Mod}(S_{g,0,1})$$

for all $g \geq \sum d_i g_i$. Select $g_i$ so that $e_i$ is nonzero in $H^*(\text{Mod}(S_{g_i,0,1}))$. Then if $id_i \geq n$ for all $i$, an application of the Künneth formula shows that $\iota^* \circ \mu$ is an injection in degrees $\leq 2n$. 

3.3. (Non-)Lifting Theorems. Given a subgroup $H \leq \text{Mod}(S_g)$ of mapping classes, it is natural to ask whether $H$ can be lifted to a subgroup of $\text{Diff}^+(S_g)$; this is known as the Nielsen realization problem. In the affirmative direction, we have Nielsen’s original result on finite cyclic subgroups; we present the proof given in [2]:

**Theorem 10.** (Nielsen) Assume $g \geq 2$ and let $f \in \text{Mod}(S_g)$ have order $n$. Then there exists $\phi \in \text{Diff}^+(S_g)$ with $[\phi] = f$ of order $n$; furthermore, $\phi$ can be taken to be an isometry of $S_g$ for some hyperbolic metric.

**Proof.** (sketch) $\mathbb{Z}/n$ acts on $\text{Teich}(S_g)$ via $\langle f \rangle$. By $K(\pi,1)$-theory, some $f^k$ must have a fixed point, and an induction argument handles the case when $k$ and $n$ are coprime.
Theorem 11. (Morita) Let $\pi : \text{Diff}^+(S_g) \to \text{Mod}(S_g)$ denote the natural projection, and let $\pi^* : H^*(\text{Mod}(S_g); \mathbb{Q}) \to H^*(\text{Diff}^+_S(S_g); \mathbb{Q})$ be the induced map on cohomology (here $\text{Diff}^+_S(S_g)$ carries the discrete topology). Then for $i \geq 3$, we have $\pi^*(e_i) = 0$.

Proof. (sketch) It suffices to show that $\text{eu}(\xi)^4 = 0$ in $H^8(E; \mathbb{Q})$ for all flat $S_g$ bundles $\xi : E \to M$. An application of the Bott Vanishing Theorem shows that $p^2_1(\ker d\xi) = 0$, and the result then follows from the relation $p^2_1(\ker d\xi) = \text{eu}(\xi)^2$.

Corollary 3.1. For $g$ sufficiently large, the sequence

$$1 \to \text{Diff}^+_S(S_g) \to \text{Diff}^+(S_g) \to \text{Mod}(S_g) \to 1$$

does not split.

Proof. (sketch) A splitting would yield a section $s : \text{Mod}(S_g) \to \text{Diff}^+(S_g)$, and the composition $s^*\pi^*$ would equal the identity on $H^2(\text{Mod}(S_g); \mathbb{Q})$. Combining Theorem 11 with Theorem 9 then yields a contradiction.

4. Thurston and Gromov Norm

The final type of surface bundle invariant that we consider is that of a norm on the (co)homology of the total space. The norm structure can make homology sensitive to information that can’t otherwise be detected homologically. The first of these, the Thurston norm, provides a striking description of the totality of ways that a given three-manifold can fiber, while the Gromov norm and its dual theory of bounded cohomology provide a bridge between topology and geometry.

4.1. Thurston Norm. The results in this section are taken from Thurston’s original paper [6]. Let $M$ be a compact oriented three-manifold, and for simplicity assume that $M$ is without boundary, irreducible and atoroidal. The Thurston norm is a norm on $H_2(M; \mathbb{R})$ that measures the minimal complexity of an embedded surface $S$ representing a given homology class. For a connected surface $S$, define $\xi(S) = \max\{-\chi(S), 0\}$. For $S = S_1 \cup \cdots \cup S_n$ a disjoint union of connected surfaces, define

$$\xi(S) = \sum_{i=1}^n \xi(S_i).$$

For a homology class $a \in H_2(M; \mathbb{Z})$, define

$$\xi(a) = \inf\{\xi(S) : S \text{ embedded}, [S] = a\}.$$

Theorem 12. (Thurston)

i. The definition of $\xi$ extends to real coefficients so that $\xi$ is convex and linear on rays.

ii. The unit ball is a rationally-defined polyhedron, and so $H_2(M; \mathbb{R})$ decomposes as a union of cones on the top-dimensional faces.

iii. Let $M \to S^1$ be a fibration, and let $a \in H_2(M; \mathbb{Z})$ be the class of the fiber. Then $a$ is contained in the interior of a cone on a top-dimensional face.

iv. Let $C$ be a cone on a top-dimensional face in $H_2(M; \mathbb{R})$, and suppose $a \in C$ corresponds to a fibering. Then every integral class in the interior of $C$ is the class of the fiber of a fibration; such a cone is called a cone on a fibered face. In other words, a class $a \in H_2(M; \mathbb{Z})$ corresponds to a fibration if and only if it is contained in the interior of a cone on a fibered face.

Proof. (sketch) For (i.), we use surgery to resolve intersections between embedded representatives in order to establish the triangle inequality. The result in (ii.) is a general but surprisingly tricky fact about norms that are $\mathbb{Z}$-valued on lattice points. For (iii.) and (iv.), we work in $H^1(M; \mathbb{R})$: a class $\alpha$ corresponds to a fibration if it is nondegenerate and its periods are integral. We obtain the local statement of (iii.) by showing that a small rational perturbation of a fibration class lies in the same supporting hyperplane of the unit ball. To get the global statement of (iv.), we use some
results on foliations to find a vector field $X$ on a surface $S$ for which $[S]^*$ lies in the same cone as $\alpha$, such that $(s\alpha + t[S]^*)(X) > 0$ for $s, t > 0$.

One application of the Thurston norm is to the study of link complements. Alexander duality asserts that the homology of a link complement is insensitive to linking data, but Thurston shows that distinct link complements may have non-isometric homologies.

**Corollary 4.1.** The image of $\text{Diff}^+(M)$ in $\text{Aut}(H_2(M; \mathbb{R}))$ is finite.

*Proof. (sketch)* Diffeomorphisms of $M$ act as isometries of $H_2(M; \mathbb{R})$, and there are finitely many isometries of a space with a polyhedral unit ball. □

**Corollary 4.2.** Let $M$ be a three-manifold fibering over $S^1$ with $b_1(M) > 1$. Then:

i. There exist classes $a \in H_2(M; \mathbb{Z})$ that do not correspond to the fiber of a fibration.

ii. $M$ fibers as an $S_g$ bundle over $S^1$ for infinitely many genera $g$.

*Proof. (sketch)* The first statement follows from Theorem 12.iii, while the second follows from Theorem 12.iv and the fact that primitive lattice points in the cone on a fibered face have connected fiber. □

4.2. **Gromov Norm.** Let $X$ be a topological space and take $k \geq 0$. The Gromov (semi)-norm on $H_k(X; \mathbb{R})$ is induced from the $\ell^1$-norm on $C_k(X; \mathbb{R})$, treated as a real Banach space generated by the singular chains on $X$. One of the primary applications of Gromov norm is to the hyperbolic setting, where it is proved that the Gromov norm of the fundamental class of a closed hyperbolic manifold (the simplicial volume, which we write as $||M||$) equals (up to a universal constant) the hyperbolic volume.

However, we concern ourselves here with some applications to the Euler class and to surface bundles. Treating $(C_k(X; \mathbb{R}), ||\cdot||_1)$ as a Banach space, it is natural to consider the dual space $(C^*_b(X; \mathbb{R}), ||\cdot||_\infty)$ of bounded cochains. The homology of the resulting chain complex is the bounded cohomology of $X$. The theory of bounded cohomology provides a nice proof of the Milnor-Wood inequality, which restricts the possible values of the Euler class of a flat $S^1$-bundle over a surface $S_g$.

**Theorem 13.** (Milnor-Wood) Let $\zeta$ be a flat $S^1$-bundle over $S_g$ with $g \geq 2$. Then the Euler class $eu \in H^2(S_g; \mathbb{Z}) \cong \mathbb{Z}$ (in the sense of Definition 3.1.2) satisfies the following inequality:

$$|\langle eu, [S_g] \rangle| \leq |\chi(S_g)| = 2g - 2.$$

*Proof. (sketch)* One shows that $||S_g|| = 2|\chi(S_g)|$ and that the Euler class is bounded with norm at most $1/2$ by constructing an explicit cocycle. □

The Gromov norm of a fiber bundle is a subtle invariant. On the one hand, we have the following result of Gromov, which computes the norm of a product:

**Theorem 14.** (Gromov) Let $M^m$ and $N^n$ be closed manifolds of the indicated dimensions. We have the following:

$$||M|| \cdot ||N|| \leq ||M \times N|| \leq c_{m,n} ||M|| \cdot ||N||,$$

with $c_{m,n} = \binom{m+n}{m}$.

*Proof. (sketch)* Taking products of chains representing fundamental classes for $M$ and $N$ provides the upper bound. The lower bound is attained using bounded cohomology. □

On the other hand, consider the case of a hyperbolic $M^3$ fibering over the circle. As $M$ is hyperbolic, the Gromov norm is proportional to volume and hence nonzero, however the simplicial volume of $S^1$ is zero, showing that the upper bound in Theorem 14 cannot hold for general fiber
bundles. However, a result of Hoster-Kotschick shows that the lower bound still holds in the case of surface bundles:

**Theorem 15.** (Hoster-Kotschick [3]) Let \( \pi : E \to M \) be a surface bundle with fiber a closed surface \( S \). Then we have

\[
||S|| M \leq ||E||.
\]

**Proof.** (sketch) The inequality is nontrivial only when \( S = S_g \) with \( g \geq 2 \) and \( ||M|| \neq 0 \). Integration along the fiber then shows that the class \( eu \cup \pi^* \omega_M \), with \( eu(\pi) \) the Euler class in the sense of Definition 3.1.7 and \( \omega_M \) dual to the fundamental class, is equal to \( \chi(S_g) \omega_E \). The unit circle bundle of \( T\pi \) is flat (cf. the proof of Theorem 8), and so we can take \( eu \) as in Definition 3.1.2. The theorem then follows from the Milnor-Wood inequality (Theorem 13).

\[ \square \]

**References**


