ON THE NON-REALIZABILITY OF BRAID GROUPS BY DIFFEOMORPHISMS

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Abstract. For every compact surface $S$ of finite type (possibly with boundary components but without punctures), we show that when $n$ is sufficiently large there is no lift $\sigma$ of the surface braid group $B_n(S)$ to $\text{Diff}(S, n)$, the group of diffeomorphisms preserving $n$ marked points and restricting to the identity on the boundary. Our methods extend to the more general setting of spaces of codimension-2 embeddings, and we obtain corresponding results for spherical motion groups, including the string motion group.

1. Introduction

Let $N^k$ and $M^{k+2}$ be smooth manifolds. For any $n \geq 1$ the symmetric group $S_n$ acts on the space $\text{Emb}_n(N, M)$ of $C^1$ embeddings $\bigsqcup_n N \to M$ by permuting the components of $\bigsqcup_n N$. The quotient $\text{Conf}_n(N, M) = \text{Emb}_n(N, M)/S_n$ is the configuration space. The most familiar setting is for $k = 0$, so that $M = S$ is a surface and $N = \{\ast\}$ is a point. In this case $\text{Conf}_n(\{\ast\}, S) = \text{Conf}_n(S)$ is the configuration space of $n$-tuples of distinct, unordered points on $S$, and $\pi_1(\text{Conf}_n(M)) =: B_n(S)$ is a surface braid group.

The diffeomorphism group $\text{Diff}(M)$ acts on $\text{Conf}_n(N, M)$ with stabilizer denoted $\text{Diff}(M, \bigsqcup_n N)$. Associated to this action is a homomorphism

$$\mathcal{P} : \pi_1(\text{Conf}_n(N, M)) \to \pi_0(\text{Diff}(M, \bigsqcup_n N))$$

generalizing the point-pushing map $\mathcal{P} : B_n(S) \to \text{Mod}(S, n)$ in the surface braid group setting. See Theorem 2.2 and Proposition 4.2 for detailed constructions. This note focuses on the non-realizability of $\mathcal{P}$ by diffeomorphisms. We say that $\mathcal{P}$ is realized by diffeomorphisms if there exists a homomorphism $\sigma : \pi_1(\text{Conf}_n(N, M)) \to \text{Diff}(M, \bigsqcup_n N)$ such that the composition

$$\pi_1(\text{Conf}_n(N, M)) \xrightarrow{\sigma} \text{Diff}(M, \bigsqcup_n N) \to \pi_0(\text{Diff}(M, \bigsqcup_n N))$$

is equal to $\mathcal{P}$. Such a $\sigma$, if it exists, is called a lift of $\mathcal{P}$.

Bestvina–Church–Souto [BCS13] show by a cohomological argument that $B_n(S)$ is not realized by diffeomorphisms when $S$ is closed, genus$(S) \geq 2$, and $n \geq 1$ (note that $B_1(S) \cong \pi_1(S)$). It

1 All diffeomorphisms considered in this paper will be orientation-preserving.

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does not seem that their methods extend to surfaces with boundary or to surfaces of low genus. In particular, this leaves the case of the classical braid group $B_n = B_n(D^2)$ unresolved.

Morita’s non-lifting theorem [Mor87] shows that there is no lift $\text{Mod}_g = \pi_0\left(\text{Diff}(\Sigma_g)\right) \rightarrow \text{Diff}(\Sigma_g)$ by showing that $H^*(\text{Mod}_g) \rightarrow H^*(\text{Diff}(\Sigma_g))$ fails to be injective for $g$ sufficiently large. It is tempting to try and follow this strategy for $B_n$, exploiting the fact that $B_n = \pi_0\left(\text{Diff}(D^2, n)\right)$.

However, Nariman [Nar14] has shown that $H^*(B_n; \mathbb{Z})$ is a direct summand of $H^*(\text{Diff}(D^2, n); \mathbb{Z})$, so no argument in the spirit of Morita will work. By using more geometric methods, we are able to sidestep these difficulties.

**Theorem 1.1.** Fix $n \geq 5$. Let $S$ be a compact surface of genus $g$ with $b$ boundary components. If $g + 2b + n \geq 7$ then $\mathcal{P} : B_n(S) \rightarrow \text{Mod}(S, n)$ is not realized by diffeomorphisms.

**Remark 1.2.** Much less is understood about realizing $B_n(S)$ by homeomorphisms. Thurston showed that $B_3$ is realized by homeomorphisms [Thu11]. In contrast, $B_6(S^2)$ is not realized by homeomorphisms (for otherwise, one could lift this realization to the branched cover $\Sigma_2 \rightarrow S^2$ to obtain a realization of $\text{Mod}(\Sigma_2)$ by homeomorphisms, and this is impossible by work of Markovic–Šarić [MS08], building on the ideas of Markovic [Mar07]).

Along with surface braid groups, we will also be concerned with the space $\text{Conf}_n(S_k, \mathbb{R}^{k+2})$ of configurations of unlinked, codimension-2 spheres in Euclidean space for $k \geq 1$; see Section 4. The fundamental group $SM_{nk} = \pi_1\left(\text{Conf}_n(S_k, \mathbb{R}^{k+2})\right)$ is called the spherical motion group. In the case $k = 1$, this group is closely related to the ring group studied by Brendle and Hatcher in [BH13] (see Section 6). Here is the result.

**Theorem 1.3.** Fix $n \geq 6$. Fix an unlinked embedding $\phi : \bigsqcup_n S^k \hookrightarrow \mathbb{R}^{k+2}$, and let $[\phi] \in \text{Conf}_n(S_k, \mathbb{R}^{k+2})$ denote the corresponding configuration. Let $\text{Diff}_c(\mathbb{R}^{k+2}, \phi)$ be the group of compactly supported $C^1$ diffeomorphisms $f : \mathbb{R}^{k+2} \rightarrow \mathbb{R}^{k+2}$ such that $[f \circ \phi] = [\phi]$. Then the “spherical push map” $\mathcal{P} : SM_{nk} \rightarrow \pi_0\left(\text{Diff}_c(\mathbb{R}^{k+2}, \phi)\right)$ is not realized by diffeomorphisms.

**Remark 1.4.** The arguments of Theorems 1.1 and 1.3 can be extended to certain finite-index subgroups, but do not work, e.g. for the pure braid group $PB_n \leq B_n(D^2)$. It is also not clear whether the bound $n \geq 5, g + 2b + n \geq 7$ of Theorem 1.1 (or the bound $n \geq 6$ of Theorem 1.3) can be improved, although the methods of the current paper do not extend beyond this range. See Remark 1.2 for some related discussions.

In Theorem 1.3 the diffeomorphism groups under consideration are required to fix the image of $\phi$ pointwise up to permutation. In Section 6 we use work of Parwani [Par08] to give an extension of Theorem 1.3 that deals with the possibility of a lift of $\mathcal{P}$ that only fixes the image of $\phi$ setwise, in the case $k = 1$. We also treat a generalization of Theorem 1.1 where the marked points on $S$ are replaced by boundary components.
The proof of Theorems 1.1 and 1.3 involves two main ingredients. The first is the Thurston
stability theorem [Thu74], which can be used to impose restrictions on the homology of finitely-
generated subgroups of diffeomorphisms. The second is the fact that $B_n$ interacts poorly with
these restrictions. The main theorems are proved by exhibiting suitable subgroups isomorphic
to $B_n$ in each of the braid or motion groups under consideration.

The paper is organized as follows. In Section 2 we briefly review Birman’s theory of push
maps for surface braid groups. In Section 3 we prove Theorem 1.1. In Section 4 we develop a
notion of push maps for spherical motion groups. In Section 5 we prove Theorem 1.3. Finally in
Section 6 we prove some strengthenings of Theorems 1.1 and 1.3 in low dimensions.

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2. From configuration spaces to mapping class groups

In this section, we review how surface braid groups give rise to subgroups of mapping class
groups via push maps. Let $S$ be a surface. The pure configuration space of $n$ points in $S$ is
defined as

$$PConf_n(S) = \{(x_1, \ldots, x_n) \in S \times \cdots \times S \mid x_i \neq x_j \text{ if } i \neq j\}.$$ 

The configuration space is defined as the quotient $Conf_n(S) = PConf_n(S)/S_n$ by the (free)
action of the symmetric group on $n$ letters via permutation of coordinates.

**Definition 2.1.** The braid group on $n$ strands in $S$, written $B_n(S)$, is defined to be $\pi_1(Conf_n(S))$.

In the case $S = D^2$, we write $B_n = B_n(\mathbb{D}^2)$.

The following is due to J. Birman. See [FM12, Section 9.1.4].

**Theorem 2.2** (Birman). Let $S$ be a compact surface with possibly nonempty boundary. Let
$X_n = \{x_1, \ldots, x_n\}$ be a set of $n$ distinct points in $S$. There is a homomorphism

$$\mathcal{P} : B_n(S) \to \pi_0(Diff(S, \partial S, X_n));$$

here $Diff(S, \partial S, X_n)$ is the group of $C^1$ diffeomorphisms of $S$ restricting to the identity on $\partial S$
that preserve $X_n$ setwise. The kernel of $\mathcal{P}$ is isomorphic to a quotient of $\pi_1(Diff(S, \partial S))$.

**Remark 2.3.** The condition $\pi_1(Diff(S, \partial S)) = 1$ is satisfied whenever $\chi(S) < 0$, and also when
$S = D^2$ (see [FM12, Theorem 1.14 and Theorem 9.1]). In the exceptional cases, $\pi_1(Diff(S^2)) \cong \mathbb{Z}/2$, and $\pi_1(Diff(T^2)) \cong \mathbb{Z}^2$. It follows that for all $n \geq 5$ (the cases under consideration in this paper), the map $\mathcal{P}$ is nonzero.
3. Proof of Theorem 1.1

The situation can be expressed diagrammatically as follows:

\[
\begin{array}{ccc}
\text{Diff}(S, \partial S, X_n) & \xrightarrow{\sigma} & B_n(S) \\
\downarrow \pi & & \downarrow \pi_0 \\
\pi_0(\text{Diff}(S, \partial S, X_n)) & \rightarrow & \pi_0(\text{Diff}(S, \partial S, X_n))
\end{array}
\]

We seek to obstruct the existence of a homomorphism \( \sigma \) lifting \( P \). Our method will be to reduce to the Thurston stability theorem.

**Step 1: Local indicability and the Thurston stability theorem.** The aim of this section is to show that certain diffeomorphism groups do not contain braid subgroups. We will be concerned with a property of groups known as local indicability.

**Definition 3.1.** A group \( G \) is said to be locally indicable if every nontrivial finitely-generated subgroup \( \Gamma \leq G \) admits a surjection \( \Gamma \rightarrow \mathbb{Z} \). Equivalently, \( G \) is locally indicable if every finitely-generated subgroup \( \Gamma \) has \( H^1(\Gamma, \mathbb{R}) \neq 0 \).

A group \( G \) is said to be strongly non-indicable if there exists a nontrivial finitely-generated subgroup \( \Gamma \) that is perfect, i.e. with \([\Gamma, \Gamma] = \Gamma\).

**Remark 3.2.** Suppose \( G \) is not locally indicable, and let \( H \leq G \) be a subgroup witnessing this fact. If \( N \triangleleft G \) is a normal subgroup with \( H \cap N \neq H \), then \( HN/N \) witnesses the non-indicability of \( G/N \). The same is true for strong non-indicability.

In [Thu74], Thurston showed that certain diffeomorphism groups are locally indicable. While the statement below is concerned only with surfaces, the analogous statement holds for all smooth manifolds \( M \) of dimension \( k \geq 1 \).

**Theorem 3.3** (Thurston stability theorem). Let \( S \) be a surface, and let \( x \in S \) be given. For a diffeomorphism \( g \) of \( S \) fixing \( x \), we write \( (Dg)_x \in GL(T_x S) \) for the derivative. Then the group

\[
\mathcal{G} = \{ g \in \text{Diff}(S) \mid g(x) = x, \ (Dg)_x = I \}
\]

is locally indicable (and hence any subgroup of \( \mathcal{G} \) is locally indicable as well).

The strategy for the remainder of the proof is to argue that a lift \( \sigma \) of \( P \) would force \( \mathcal{G} \) to contain a non-locally-indicable subgroup. We will show that \( B_n \) is a suitable group.

**Step 2: Braid groups are strongly non-indicable.**

**Theorem 3.4.**

(i) (Gorin–Lin) For \( n \geq 5 \), the commutator subgroup of the braid group \( B_n \) is perfect, i.e.

\[
[B_n, B_n] = [[B_n, B_n], [B_n, B_n]].
\]

(ii) For \( n \geq 2 \), \( [B_n, B_n] \) is finitely generated.
Consequently $B_n$ is strongly non-indicable for $n \geq 5$.

Proof. We omit the proof of (i), as it was established in [GL69, Corollary 2.2]. The result of (ii) is also well-known. One approach is to study the Milnor fibration $\Delta : \text{Conf}_n(C) \to C^*$ provided by the discriminant polynomial, viewing $\text{Conf}_n(C)$ as the space of monic, square-free polynomials of degree $n$ over $C$. Milnor [Mil68, Theorem 5.1] has shown that the fiber $\Delta^{-1}(1)$ has the homotopy type of a finite CW complex. Via the long exact sequence of a fibration, one verifies that $F$ is a $K(G,1)$, for $G = [B_n, B_n]$. Alternatively, one can proceed group-theoretically, by appealing to the Reidemeister-Schreier method (see, eg. [LS77, II.4]). \qed

Step 3: Produce $B_m \leq B_n(S)$. The following is a special case of a theorem of Paris-Rolfsen [PR00, Corollary 4.2(iii)].

Theorem 3.5 (Paris-Rolfsen). If $S \neq S^2$, an inclusion $(D, X_n) \hookrightarrow (S, X_{n+1})$ induces an injection $B_n \hookrightarrow B_{n+1}(S)$. If $S = S^2$, an inclusion $(D, X_n) \hookrightarrow (S, X_{n+2})$ induces an injection $B_n \hookrightarrow B_{n+2}(S^2)$.

Remark 3.6. By construction, these subgroups $B_n$ stabilize $X_{n+k} \setminus X_n$ (for $k = 1, 2$ as appropriate). More precisely, if $\tau \in B_n \leq B_{n+k}(S)$ and $\phi \in \Diff(S, \partial S, X_{n+k})$ is any representative of $P(\tau) \in \pi_0(\Diff(S, \partial S, X_{n+k}))$, then $\phi$ fixes each element of $X_{n+k} \setminus X_n$.

Step 4: Reduction to Thurston stability. We will employ two obstructions for realizing a finitely-generated group $\Gamma$ as a subgroup of $\Diff(S)$. The case where $\partial S \neq \emptyset$ is simpler. Proposition 3.7 below is a special case of [FH09, Proposition 3.1].

Proposition 3.7 (Franks-Handel). Let $S$ be a surface with $\partial S \neq \emptyset$. Then $\Diff(S, \partial S)$, the group of $C^1$ diffeomorphisms of $S$ restricting to the identity on the boundary, is locally indicable.

Combining Theorem 3.5 with Proposition 3.7, we arrive at the desired contradiction to the existence of $\sigma$ in the case where $\partial S \neq \emptyset$. The case where $\partial S = \emptyset$ will require a different criterion.

Lemma 3.8. For $n \geq 5$, every homomorphism $f : B_n \to GL_2^+(\mathbb{R})$ has abelian image.

This is a consequence of the following more general criterion (which we will employ again in Section 6).

Lemma 3.9. Let $G$ be a group generated by elements $\tau_1, \ldots, \tau_n$ that satisfy the following properties:

1. The elements $\tau_i$ are all mutually conjugate.
2. There exists $k \geq 1$ such that $[\tau_i, \tau_j] = 1$ for $|j - i| \geq k$ (here we mean distance in $\mathbb{R}/n\mathbb{Z}$).

Then for $n \geq 2k + 1$, every homomorphism $f : G \to GL_2^+(\mathbb{R})$ has abelian image.
Proof. It suffices to show that the projection \( \bar{f} : G \to \text{GL}_2^+(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \) has image contained in a one-parameter subgroup. This is because the preimage in \( \text{GL}_2^+(\mathbb{R}) \) of any one-parameter subgroup in \( \text{PSL}_2(\mathbb{R}) \) is abelian. For convenience, we will write \( \bar{\tau}_i \) in place of \( \bar{f}(\tau_i) \). By condition (1) above, if \( \bar{f} \) is a nontrivial homomorphism, then each \( \bar{\tau}_i \neq I. \)

If the image of \( \bar{f} \) is not contained in some one-parameter subgroup, then in particular, there must be some pair of elements \( \bar{\tau}_i \) and \( \bar{\tau}_j \) that do not commute. By relabeling if necessary, we may assume \( i = 1 \) and \( 2 \leq j \leq k \). Furthermore, we may assume \( j \) is the smallest integer between \( 2 \) and \( k \) for which \( \bar{\tau}_1 \) and \( \bar{\tau}_j \) do not commute.

We wish to show \( j = 2 \). Suppose \( j > 2 \). If \( \bar{\tau}_{j-1} \) and \( \bar{\tau}_j \) do not commute, then by relabeling again, we may assume that \( \bar{\tau}_1 \) and \( \bar{\tau}_2 \) do not commute. If, on the other hand, \( \bar{\tau}_{j-1} \) commutes with \( \bar{\tau}_j \), then both \( \bar{\tau}_1 \) and \( \bar{\tau}_j \) are contained in \( \text{Cent}_{\text{PSL}_2(\mathbb{R})}(\bar{\tau}_{j-1}) \). As the latter is a one-parameter subgroup, necessarily \( \bar{\tau}_1 \) and \( \bar{\tau}_j \) commute, contrary to assumption. We conclude that up to a cyclic relabeling of the generators \( \tau_i \), we must have \( \bar{\tau}_1 \) and \( \bar{\tau}_2 \) noncommuting elements of \( \text{PSL}_2(\mathbb{R}) \).

By condition (2) above and the assumption \( n \geq 2k + 1 \), the element \( \bar{\tau}_{k+2} \) commutes with both \( \bar{\tau}_1 \) and \( \bar{\tau}_2 \). Therefore, \( \bar{\tau}_1 \) and \( \bar{\tau}_2 \) are contained in the abelian subgroup \( \text{Cent}_{\text{PSL}_2(\mathbb{R})}(\bar{\tau}_{k+2}) \), contrary to assumption.

Proof. (of Lemma 3.8) We show that \( B_n \) satisfies the hypotheses of Lemma 3.9 for \( k = 2 \). Indeed, for \( 1 \leq i \leq n \), let \( \tau_i = \sigma_i \) the \( i^{th} \) standard generator of \( B_n \). We interpret \( \sigma_n \) to be the element crossing the \( n^{th} \) strand over the first, under a cyclic ordering of the strands. As the elements \( \sigma_i \) are mutually conjugate and \( [\sigma_i, \sigma_j] = 1 \) for \( |j - i| \geq 2 \), the result follows.

Remark 3.10. The assumption \( n \geq 5 \) in Lemma 3.8 cannot be relaxed: it is well-known that there is a homomorphism \( B_3 \to \text{SL}_2 \mathbb{Z} \) with nonabelian image. The case \( n = 4 \) follows from the existence of an exceptional surjective homomorphism \( B_4 \to B_3 \).

To complete the proof of Theorem 1.1 in the case \( \partial S = \emptyset \), we suppose, for a contradiction, that a lift \( \sigma : B_n(S) \to \text{Diff}(S, \partial S, X_n) \) is given. By Theorem 3.5 there is a subgroup \( B_{n-k} \leq B_n(S) \) (where \( k = 2 \) if \( S = S^2 \) and \( k = 1 \) otherwise). By Remark 3.6 the lift \( \sigma(B_{n-k}) \) fixes some point \( x \in X_n \setminus X_{n-k} \). Let \( D : B_{n-k} \to \text{GL}_2^+(\mathbb{R}) \) denote the derivative mapping at \( x \). Via Lemma 3.8 \( [B_{n-k}, B_{n-k}] \leq \text{ker} D \). Thurston stability (Theorem 3.3) then asserts that \( [B_{n-k}, B_{n-k}] \) must be locally indicable, but this contradicts Theorem 3.4.

4. Push maps for spherical motion groups

We turn now to Theorem 1.3. It is first necessary to establish the existence of the push homomorphisms \( P \) that are the higher-dimensional analogues of the homomorphism in Theorem 2.2 Fix \( k, n \geq 1 \). Consider the space \( \text{Emb}_n(S^k, \mathbb{R}^{k+2}) \) of \( C^k \) embeddings \( \coprod_n S^k \to \mathbb{R}^{k+2} \). The symmetric group \( S_n \) acts on \( \coprod_n S^k \) by permuting the components, and this induces an action on \( \text{Emb}_n(S^k, \mathbb{R}^{k+2}) \) by precomposing an embedding by a permutation. Fix an embedding \( \phi \) that is
We seek to obstruct the existence of a lift \( \sigma \) of \( P \). Define the configuration space

\[
\text{Conf}_n(S^k, \mathbb{R}^{k+2}) = \text{Emb}_n(S^k, \mathbb{R}^{k+2}; \phi)/S_n.
\]

An element of \( \text{Conf}_n(S^k, \mathbb{R}^{k+2}) \) is a collection of disjoint, unordered, unlinked spheres, each of which comes with a parameterization.

**Definition 4.1.** Let \([\phi] \in \text{Conf}_n(S^k, \mathbb{R}^{k+2})\) denote the equivalence class of the embedding \( \phi \).

The group \( SM^k_n := \pi_1(\text{Conf}_n(S^k, \mathbb{R}^{k+2}), [\phi]) \) is a spherical motion group\(^2\).

In order to state the analog of Theorem 2.2 for \( SM^k_n \), let \( \text{Diff}_c(\mathbb{R}^{k+2}, \phi) \leq \text{Diff}_c(\mathbb{R}^{k+2}) \) be the subgroup of compactly supported \( C^1 \) diffeomorphisms of \( \mathbb{R}^{k+2} \) which satisfy \([f \circ \phi] = [\phi]\).

By viewing \( \phi \) as defining a parameterization on its image \( \text{Im}(\phi) \subset \mathbb{R}^{k+2} \), diffeomorphisms of \( \text{Diff}_c(\mathbb{R}^{k+2}, \phi) \) preserve \( \text{Im}(\phi) \) together with the parameterization on each sphere, up to permutations. In particular, \( f \in \text{Diff}_c(\mathbb{R}^{k+2}, \phi) \) fixes pointwise any component of \( \text{Im}(\phi) \) taken to itself.

**Proposition 4.2.** Fix \( n \geq 1 \). There is a homomorphism \( \mathcal{P} : SM^k_n \rightarrow \pi_0(\text{Diff}_c(\mathbb{R}^{k+2}, \phi)) \). The kernel of \( \mathcal{P} \) is abelian.

**Proof.** Let \( m = k+2 \). For the proof, it will be convenient to view our configurations as being contained in a ball \( \mathbb{R}^m \cong \text{int}(\mathbb{D}^m) \).

Letting \( \text{Diff}_c(\mathbb{D}^m \text{ rel } \partial) \) denote the group of diffeomorphisms with compact support in \( \text{int}(\mathbb{D}^m) \), we have an evaluation map \( \eta : \text{Diff}_c(\mathbb{D}^m \text{ rel } \partial) \rightarrow \text{Conf}_n(S^k, \mathbb{D}^m) \) sending \( f \) to \([f \circ \phi]\).

By Palais [Pal60] this map determines a fibration

\[
\text{Diff}(\mathbb{D}^m, \phi \text{ rel } \partial) \rightarrow \text{Diff}_c(\mathbb{D}^m \text{ rel } \partial) \xrightarrow{\eta} \text{Conf}_n(S^k, \mathbb{D}^m),
\]

where \( \text{Diff}(\mathbb{D}^m, \phi \text{ rel } \partial) \leq \text{Diff}_c(\mathbb{D}^m \text{ rel } \partial) \) is the stabilizer of \([\phi]\).

The long exact sequence of homotopy groups of this fibration gives an exact sequence

\[
\pi_1\left(\text{Diff}_c(\mathbb{D}^m \text{ rel } \partial)\right) \rightarrow SM^k_n \xrightarrow{\mathcal{P}} \pi_0\left(\text{Diff}(\mathbb{D}^m, \phi \text{ rel } \partial)\right).
\]

This defines \( \mathcal{P} \). Note that as \( \text{Diff}_c(\mathbb{D}^m \text{ rel } \partial) \) is a topological group, \( \pi_1\left(\text{Diff}_c(\mathbb{D}^m \text{ rel } \partial)\right) \) is abelian, from which it follows that \( \ker \mathcal{P} \) is as well. \( \square \)

5. **Proof of Theorem 1.3**

Once again, the situation can be expressed diagrammatically as follows:

\[
\begin{array}{ccc}
\text{Diff}_c(\mathbb{R}^{k+2}, \phi) & \xrightarrow{\sigma} & SM^k_n \\
\downarrow \pi & & \downarrow \mathcal{P}
\end{array}
\]

We seek to obstruct the existence of a lift \( \sigma \) of \( \mathcal{P} \). The outline of the proof is essentially the same as for Theorem 1.1. Step 2 of Theorem 1.1 which concerns the group theory of \( B_n \), needs no modification and so will not be reproduced below.

\(^2\)These groups were first studied by Dahm [Dah62].
Step 1: Local indicability and the Thurston stability theorem. As noted above, the Thurston stability theorem (Theorem 3.3) holds for any smooth manifold $M$; in particular $M = \mathbb{R}^{k+2}$ as is under consideration here.

Step 2: Produce $B_n \leq SM^k_n$. In this section we prove the following proposition.

Proposition 5.1. There is an embedding $B_n \leq SM^k_n$.

Proof. This proof was suggested to the authors by A. Hatcher. In order to show that $SM^k_n = \pi_1(\text{Conf}_n(S^k, \mathbb{R}^{k+2}))$ contains a braid group, we will use a finite dimensional sub-space $R\text{Conf}_n(S^k, \mathbb{R}^{k+2}) \subset \text{Conf}_n(S^k, \mathbb{R}^{k+2})$, defined as follows. First give $S^k$ and $\mathbb{R}^{k+2}$ the standard round and Euclidean metrics, respectively. Define $R\text{Emb}_n(S^k, \mathbb{R}^{k+2})$ as the space of embeddings $\phi : \bigvee S^k \to \mathbb{R}^{k+2}$ that are isometries, up to scaling. In analogy to the definition of $\text{Conf}_n(S^k, \mathbb{R}^{k+2})$ the space $R\text{Conf}_n(S^k, \mathbb{R}^{k+2})$ is defined as the quotient $R\text{Emb}_n(S^k, \mathbb{R}^{k+2})/S_n$.

Lemma 5.2. There is an embedding $B_n \hookrightarrow \pi_1(R\text{Conf}_n(S^k, \mathbb{R}^{k+2}))$.

Proof. Let $[\phi] \in R\text{Conf}_n(S^k, \mathbb{R}^{k+2})$ be the configuration defined by an embedding

$$\phi : \bigvee S^k \to \mathbb{R}^{k+2},$$

where the $i^{th}$ sphere is mapped to the equator of the sphere of radius $1/4$ centered at $(i, 0, \ldots, 0) \in \mathbb{R}^{k+2}$. For $i = 1, \ldots, n - 1$, let $\rho_i \in \pi_1(R\text{Conf}_n(S^k, \mathbb{R}^{k+2}), \phi)$ be the motion that exchanges the $i^{th}$ and $(i+1)^{st}$ spheres of $\phi$, passing the $(i+1)^{st}$ sphere through the $i^{th}$ sphere. See Figure 1.

![Figure 1](image)

The motions $\rho_1, \ldots, \rho_{n-1}$ generate a braid subgroup $B_n \leq \pi_1(R\text{Conf}_n(S^k, \mathbb{R}^{k+2}))$ by Brendle–Hatcher [BH13, Proposition 4.3]. (Although they focus on the case $k = 1$, their argument generalizes verbatim to the case $k \geq 1$).

It is not apparent that $SM^k_n = \pi_1(\text{Conf}_n(S^k, \mathbb{R}^{k+2}))$ contains a braid group without further investigation of the homomorphism $\pi_1(R\text{Conf}_n(S^k, \mathbb{R}^{k+2})) \to \pi_1(\text{Conf}_n(S^k, \mathbb{R}^{k+2}))$. For this, we will study the action of the motions $\rho_i$ on the fundamental group of $\mathbb{R}^{k+2} \setminus \bigvee S^k$.

Lemma 5.3. Fix $k \geq 2$. Let $\bigvee S^k \hookrightarrow \mathbb{R}^{k+2}$ be an unlinked embedding. Then $\pi_1(\mathbb{R}^{k+2} \setminus \bigvee S^k)$ is isomorphic to the free group $F_n$. 

Proof. For definiteness, we work with the embeddings \( \phi \) from equation (2). We proceed by induction on \( n \). For the base case \( n = 1 \), first note that

\[
S^{k+2} \cong \partial(D^{k+1} \times D^2) = S^k \times D^2 \bigcup_{S^k \times S^1} \mathbb{D}^{k+1} \times S^1.
\]

It follows that \( \pi_1(S^{k+2} \setminus S^k) \cong \pi_1(D^{k+1} \times S^1) \cong \mathbb{Z} \). Then also \( \pi_1(R^{k+2} \setminus S^k) \cong \mathbb{Z} \), since removing a single point from a \((m \geq 3)\)-manifold does not change the fundamental group.

For the inductive step, take \( \phi \) as above and decompose \( R^{k+2} \) into open sets

\[
U = \{(x_1, \ldots, x_{k+2}) : x_1 < n - \frac{1}{2} + \varepsilon\} \quad \text{and} \quad V = \{(x_1, \ldots, x_{k+2}) : x_1 > n - \frac{1}{2} - \varepsilon\}
\]

for any small positive \( \varepsilon \). By construction \( U \) contains the first \( n - 1 \) spheres and \( V \) contains the \( n^{th} \) sphere. Since \( U \cap V \) is contractible, by Seifert–van Kampen, we have

\[
\pi_1(R^{k+2} \setminus \bigcup_n S^k) \cong \pi_1(R^{k+2} \setminus \bigcup_{n-1} S^k) \ast \pi_1(R^{k+2} \setminus S^k) \cong F_{n-1} \ast \mathbb{Z} \cong F_n.
\]

The second isomorphism uses the inductive hypothesis and the base case. \( \square \)

Next we determine how \( B_n \leq \pi_1(R\text{Conf}_n(S^k, R^{k+2})) \) acts on \( \pi_1(R^{k+2} \setminus \bigcup_n S^k) \). The homomorphism \( P \) of Proposition 4.2 gives a homomorphism \( \pi_1(R\text{Conf}_n(S^k, R^{k+2})) \to \pi_0(\text{Diff}_c(R^{k+2}, \phi)) \). The latter group acts on \( \pi_1(R^{k+2} \setminus \phi) \cong F_n \), so we have a homomorphism

\[
\beta : B_n \to \pi_1(R\text{Conf}_n(S^k, R^{k+2})) \to \text{Aut}(F_n). \tag{3}
\]

Lemma 5.4. The homomorphism \( \beta \) is injective.

Proof. There is another homomorphism \( \beta' : B_n \to \text{Aut}(F_n) \) induced by the action of the mapping class group \( \text{Mod}(\mathbb{D}, n) \cong B_n \) on \( \pi_1(\mathbb{D} \setminus \{n \ \text{points}\}) \cong F_n \). It is well-known that \( \beta' \) is injective (see [FM12 Chapter 9]). We prove the lemma by showing that \( \beta \) and \( \beta' \) coincide after making the right identifications.

Choose a configuration \( Y = \{y_1, \ldots, y_n\} \subset \mathbb{D} \) as in Figure 2. Let \( \{\sigma_1, \ldots, \sigma_{n-1}\} \) be the standard generating set for \( B_n \) (c.f. Lemma 3.8). The isomorphism \( B_n \sim \text{Mod}(\mathbb{D}, n) \) is defined by sending \( \sigma_i \) to the mapping class that exchanges \( y_i \) and \( y_{i+1} \) by moving them counterclockwise around their midpoint. We choose generators \( \eta_i \) for \( \pi_1(\mathbb{D} \setminus Y, \ast) \cong F_n \) as in Figure 2. It is easy to compute (c.f. Figure 2)

\[
\beta'(\sigma_i) : \begin{cases}
\eta_j &\mapsto \eta_j \\
\eta_i &\mapsto \eta_{i+1} \\
\eta_{i+1} &\mapsto \eta_{i+1} \eta_i \eta_{i+1}^{-1}
\end{cases}
\]

On the other hand, the inclusion \( B_n \hookrightarrow \pi_1(R\text{Conf}_n(S^k, R^{k+2})) \) sends \( \sigma_i \) to the motion \( \rho_i \) defined in Lemma 5.2. We identify \( \pi_1(R^{k+2} \setminus \bigcup_n S^k) \cong F_n \) as follows. Fix a basepoint \( \ast \in R^{k+2} \setminus \bigcup_n S^k \), and choose an embedding \( \bigcup S^k \to R^{k+2} \) such that the boundary of the \( i^{th} \) disk \( D_i \) is the \( i^{th} \) sphere. Then \( \pi_1(R^{k+2} \setminus \bigcup_n S^k, \ast) \) is generated by loops \( \gamma_1, \ldots, \gamma_n : [0,1] \to R^{k+2} \setminus \bigcup_n S^k \) such that \( \gamma_i \cap D_j = \emptyset \) for \( i \neq j \) and \( \gamma_i \) has a single, positive transverse
intersection with $D_i$. Then for any $\gamma \in \pi_1(R^{k+2} \setminus \bigsqcup S^k, *)$, expressing $\rho_i(\gamma) \in F_n$ as a word in $\gamma_1, \ldots, \gamma_n$ reduces to computing the intersection of $\rho_i(\gamma)$ with the disks $D_1, \ldots, D_n$. From this it is easy to see $\rho_i$ sends $\gamma_i$ to $\gamma_{i+1}$, sends $\gamma_{i+1}$ to $\gamma_{i+1} \gamma_i \gamma_{i+1}^{-1}$, and fixes $\gamma_j$ for $j \neq i, i + 1$; see Figure 3. Since $\rho_i = \beta(\sigma_i)$, this shows that $\beta$ and $\beta'$ agree, as desired. 

We can now complete the proof of Proposition 5.1. By definition, the homomorphism $\beta$ in (3) factors $\beta : B_n \to \pi_1(RConf_n(S^k, R^{k+2})) \to SM_n \to \text{Aut}(F_n)$. By Lemma 5.4, $\beta$ is injective. Thus $SM_n$ contains a braid subgroup. 

**Step 3: Reduction to Thurston stability.** For spherical motion groups, there is one additional step that is required in the reduction process. Below, $\text{Diff}_c(R^{k+2}, S^k)$ denotes the group of compactly supported diffeomorphisms of $R^{k+2}$ that restrict to the identity on the image of a fixed embedding $S^k \to R^{k+2}$.

**Proposition 5.5.** Let $\Gamma \leq \text{Diff}_c(R^{k+2}, S^k)$ be finitely generated. If $\Gamma$ is strongly non-indicable, then there is a homomorphism $f : \Gamma \to GL_2^+(R)$ with nonabelian image.

**Proof.** Choose $x \in S^k$. Then there are coordinates in which any $g \in \text{Diff}_c(R^{k+2}, S^k)$ has derivative given by

$$(Dg)_x = \begin{pmatrix} I_{k-2} & V_g \\ 0 & A_g \end{pmatrix}.$$
In this setting, \( V_g \in M_{k-2,2}(\mathbb{R}) \) is a \((k - 2) \times 2\) matrix, and \( A_g \in GL_2^+(\mathbb{R}) \). Denote by \( p : \text{Diff}_c(\mathbb{R}^{k+2}, S^k) \to GL_2^+(\mathbb{R}) \) the homomorphism given by \( p(g) = A_g \).

Let \( \Gamma' \leq \text{Diff}_c(\mathbb{R}^{k+2}, S^k) \) be strongly non-indicable, and let \( \Gamma' \leq \Gamma \) be a finitely-generated perfect subgroup. We claim that \( p : \Gamma \to GL_2^+(\mathbb{R}) \) has nonabelian image. If not, then \( \Gamma' \leq \ker p \). In this case, there is a map \( V : \Gamma' \to M_{k-2,2}(\mathbb{R}) \) defined by \( V(g) = V_g \). As \( M_{k-2,2}(\mathbb{R}) \) is abelian and \( \Gamma' \) is perfect, \( V \) must be trivial. But then Thurston stability implies that \( \Gamma' \) is locally indicable, a contradiction.

To complete the proof of Theorem 6.1, suppose \( \sigma : SM^k_n \to \text{Diff}(\mathbb{R}^{k+2}, \phi) \) is a lift of \( \mathcal{P} \). By Proposition 5.1 there is a subgroup \( B_n \leq SM^k_n \). There is a further subgroup \( B_{n-1} \leq B_n \) such that \( \sigma(B_{n-1}) \) fixes some component of \( \text{Im}(\phi) \) pointwise. By Proposition 4.2 the image of \( B_{n-1} \) in \( \text{Diff}_c(\mathbb{R}^{k+2}, \phi) \) is nontrivial, and \( \sigma([B_{n-1}, B_{n-1}]) \) is a nontrivial finitely-generated perfect subgroup. Consequently \( \sigma(B_{n-1}) \) is strongly non-indicable. By Proposition 3.5 there is a homomorphism \( f : \sigma(B_{n-1}) \to GL_2^+(\mathbb{R}) \) with nonabelian image, but this contradicts Lemma 3.8.

6. Extensions of the main theorems

In this section we give a strengthening of Theorems 1.1 and 1.3 using a result of Parwani [Par08, Theorem 1.4] building off of work of Deroin-Kleptsyn-Navas [DKN07].

**Theorem 6.1** (Parwani). Let \( G \) and \( H \) be two finitely generated groups such that \( H_1(G; \mathbb{Z}) = 0 = H_1(H; \mathbb{Z}) \). Then for any \( C^1 \) action of \( G \times H \) on \( S^1 \), either \( G \times 1 \) or \( 1 \times H \) acts trivially.

6.1. **Surfaces.** Let \( S \) be a closed surface and let \( X \subset S \) finite. Let \( S' \) be the compact surface obtained by replacing each marked point \( x \in X \) with a boundary component. In what follows, \( \text{Diff}(S') \) denotes the group of diffeomorphisms of \( S' \) where the boundary components of \( S' \) are not required to be fixed pointwise. It is well-known that \( \pi_0 \text{Diff}(S, X) \cong \pi_0 \text{Diff}(S') \). Therefore, one can ask whether the homomorphism

\[
\mathcal{P} : B_n(S) \to \pi_0 \text{Diff}(S, X) \cong \pi_0 \text{Diff}(S') \tag{4}
\]

lifts to a homomorphism \( B_n(S) \to \text{Diff}(S') \).

**Theorem 6.2.** Fix \( n \geq 11 \). Then \( \mathcal{P} : B_n(S) \to \pi_0 \text{Diff}(S') \) is not realized by diffeomorphisms. That is, there does not exist a homomorphism \( B_n(S) \to \text{Diff}(S') \) such that the composition \( B_n(S) \to \text{Diff}(S') \to \pi_0 \text{Diff}(S') \) is equal to \( \mathcal{P} \).

**Proof.** Suppose for a contradiction that \( \sigma : B_n(S) \to \text{Diff}(S') \) is a lift of \( \mathcal{P} \). By passing to a finite-index subgroup of \( B_n(S) \) we may assume one component \( C \subset \partial S' \) is fixed. By the assumption \( n \geq 11 \), this finite-index subgroup contains \( B_5 \times B_5 \). We may therefore take \( G = [B_5, B_5] \times 1 \) and \( H = 1 \times [B_5, B_5] \) in Theorem 6.1 to conclude that, without loss of generality, \( G \) acts trivially on \( C \). As \( G \) is non-locally indicable (Theorem 3.4), this contradicts Proposition 3.7. \( \square \)
6.2. Spheres. Let $\text{Emb}_n(S^k, \mathbb{R}^{k+2}; \phi)$ be the embedding space defined in Section 4. Define the (unparameterized) configuration space $\overline{\text{Conf}}_n(S^k, \mathbb{R}^{k+2})$ as

$$\overline{\text{Conf}}_n(S^k, \mathbb{R}^{k+2}) = \text{Emb}_n(S^k, \mathbb{R}^{k+2}; \phi)/\text{Diff}(\prod_n S^k).$$

Note that $\overline{\text{Conf}}_n(S^k, \mathbb{R}^{k+2})$ is a quotient of $\text{Conf}_n(S^k, \mathbb{R}^{k+2})$, since $\text{Diff}(\prod_n S^k)$ is isomorphic to the wreath product $\text{Diff}(S^k) \wr S_n$. An element of $\overline{\text{Conf}}_n(S^k, \mathbb{R}^{k+2})$ is a collection of disjoint, unordered, unlinked spheres (with no additional information about the parameterization).

Fix $X \in \overline{\text{Conf}}_n(S^k, \mathbb{R}^{k+2})$, and let $\overline{SM}_n^k = \pi_1(\overline{\text{Conf}}_n(S^k, \mathbb{R}^{k+2}), X)$. In the case $k = 1$, this group coincides with the ring group studied by Brendle and Hatcher in [BH13]. By the argument in Proposition 4.2 there is a homomorphism

$$\mathcal{P} : \overline{SM}_n^k \to \pi_0(\text{Diff}_c(\mathbb{R}^{k+2}, X)),$$

where $\text{Diff}_c(\mathbb{R}^{k+2}, X) \leq \text{Diff}_c(\mathbb{R}^{k+2})$ is the subgroup of diffeomorphisms that preserve $X$ as a set.

We have the following strengthening of Theorem 1.3 in the case $k = 1$.

**Theorem 6.3.** Fix $n \geq 15$. Then the homomorphism $\mathcal{P} : \overline{SM}_n^1 \to \pi_0(\text{Diff}_c(\mathbb{R}^3, X))$ is not realized by diffeomorphisms.

**Proof.** Suppose for a contradiction that $\sigma : \overline{SM}_n^1 \to \text{Diff}_c(\mathbb{R}^3, X)$ is a lift of $\mathcal{P}$. By passing to a finite-index subgroup of $\overline{SM}_n^1$, we may assume that one component $C \cong \mathbb{S}^1 \subset X$ is fixed. By the assumption $n \geq 15$, this finite-index subgroup contains $B_7 \times B_7$ and a fortiori contains $[B_7, B_7] \times [B_7, B_7]$. Taking $G = [B_7, B_7] \times 1$ and $H = 1 \times [B_7, B_7]$ in Theorem 6.1 it follows that (without loss of generality) $G$ fixes $C$ pointwise.

For the remainder of the argument, we follow the strategy in Step 3 of Theorem 1.3. In order to be able to derive a contradiction from Proposition 5.5, we must have that every homomorphism $f : [B_7, B_7] \to \text{GL}_2(\mathbb{R})$ has abelian image.

**Lemma 6.4.** For $n \geq 5$, the set

$$S = \{\sigma_i\sigma_{i+1}^{-1} \mid 1 \leq i \leq n-2\}$$

generates $[B_n, B_n]$. Moreover, the elements of $S$ are all mutually conjugate within $[B_n, B_n]$.

**Proof.** For $1 \leq i \leq n$, let $\sigma_i \in B_n$ denote the braid that passes the $i$th strand over the $(i + 1)$st, with subscripts interpreted mod $n$. The elements $\sigma_1, \ldots, \sigma_n$ are all mutually conjugate, and the abelianization map $A : B_n \to \mathbb{Z}$ is given by the total exponent sum of all the generators. Consequently, the set

$$S = \{\sigma_i\sigma_{i+1}^{-1} \mid 1 \leq i \leq n-1\}$$

normally generates $[B_n, B_n]$ inside $B_n$. 
To prove the claim, it therefore suffices to show that the subgroup \( \langle S \rangle \) of \( B_n \) generated by \( S \) is normal, which in turn reduces to showing that \( \sigma_j (\sigma_i \sigma_{i+1}^{-1}) \sigma_j^{-1} \in \langle S \rangle \) for any \( 1 \leq j \leq n \). As \( n \geq 5 \), the generator \( \sigma_i+3 \) commutes with \( \sigma_i \) and \( \sigma_{i+1} \), from which
\[
\sigma_j (\sigma_i \sigma_{i+1}^{-1}) \sigma_j^{-1} = (\sigma_j \sigma_{i+3}^{-1})(\sigma_i \sigma_{i+1}^{-1})(\sigma_j \sigma_{i+3}^{-1})^{-1}.
\]
The right-hand side exhibits \( \sigma_j (\sigma_i \sigma_{i+1}^{-1}) \sigma_j^{-1} \) as a product of elements of \( \langle S \rangle \), and the result follows.

The next step is to show that the elements of \( S \) are all conjugate within \([B_n, B_n]\). Via the braid relations,
\[
(\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+2} \sigma_i \sigma_{i+3}^{-1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+2}^{-1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+2}^{-1} = \sigma_{i+1} \sigma_{i+2}^{-1}.
\]
As above, the element
\[
\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+2} \sigma_i \sigma_{i+3}^{-1} \in [B_n, B_n]
\]
also conjugates \( \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+2} \sigma_i \sigma_{i+3}^{-1} \) to \( \sigma_{i+1} \sigma_{i+2}^{-1} \).

The generating set \( S \) of Lemma 6.4 satisfies the hypotheses of Lemma 3.9 for \( k = 3 \). It follows that every homomorphism \( f : [B_7, B_7] \to \text{GL}_2^+(\mathbb{R}) \) has abelian image as desired. The argument in step 3 of Theorem 1.3 can now be carried out, showing that \([B_7, B_7] \times 1 \leq \text{SM}_n^1 \) lies in the kernel of any homomorphism \( \sigma : \text{SM}_n^1 \to \text{Diff}_c(\mathbb{R}^3, X) \). Therefore \( \text{SM}_n^1 \) cannot be realized by diffeomorphisms.

\[\square\]

References


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