An introduction to surface bundles
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(I) - Some key examples
- A surface bundle is a fiber bundle $\Sigma \to E$ with fiber $\Sigma$, usually of finite type. Also typically $K(E) < 0$. Also typically $\Sigma$ (and so $E$) is a smooth manifold.

First example: Take $q \in \text{Diff}^+(\Sigma)$, form mapping torus $M_q = \Sigma \times [0,1] / (x,0) = (q(x),1)$. Then $\Sigma \to M_q \to S^1$.

Thurston studied these in great depths. He gave a characterization of the diffeos of that give rise to $M_q$ admitting a hyperbolic metric (the so-called "pseudo-Anosov" ends $q$). He also showed that when $b_2(M_q) > 2$, then such an $M$ admits infinitely many different fiberings over $S^1$, with arbitrarily large genus (this many fiberings in each genus).

Second example Consider the family $E$ of curves $y^2 = (x^2+t)(x-1)$, $t \in \mathbb{C} \setminus \mathbb{R}$.

Nonsingular for $t \not\in \{0,1\}$. So there is fibration $E \to \mathbb{C} \setminus \{0,1\}$.

$y^2 = (x^2+t)(x-1) \mapsto t$. (This can also be generalized to many bundles of hyperelliptic curves of higher genus.) This construction has a cool property: $E$ carries a natural complex structure.
In fact, when the base is a compact Riemann surface, the "geometric Shafarevich" asserts that there are only finitely many bundle-isomorphism classes of surface bundles over surfaces with a complex structure. We'll see an example soon.

Problem: Which SBS can carry complex structures?

- Third example: Take any symplectic 4-manifold, \( M^4 \). Donalson \( \Rightarrow \)
  - A map \( M^4 \rightarrow S^2 \) giving that is a so-called Lefschetz fibration.
  - For our purposes, this simply means the following: by deleting \( f^{-1}(z_1), \ldots, f^{-1}(z_n) \), the remaining \( M' \rightarrow S^2 - \{ z_1, \ldots, z_n \} \) is a surface bundle!

- Fourth example: (Agol) Take any closed hyperbolic 3-manifold \( M^3 \).
  - Then there is a finite cover \( M' \) of \( M \) that is a surface bundle over \( S \). So this is a (virtual) converse to example 1.

- Fifth example \( \text{Moduli space of genus } g \text{ Riemann surfaces.} \)
  - Let \( x \in \text{H}_e(M_g, \mathbb{Q}) \) be given. Thom asserts that some multiple of \( x \) is realizable as a map \( B^n \rightarrow M_g \). Pull back the universal bundle over \( M_g \); this yields \( SB \rightarrow E^{2n} \).
  - Often profitable to understand the (co)homology \( \hat{H}^* \) of \( M_g \) by understanding the topology of the corresponding bundle.
Example 6: (Atiyah-Kodaira construction)

Starts with theorem of Chern-Hirzebruch-Serre (C.H.S.)
Fiber bundle $F \to E \to B$ of closed oriented manifolds. Signatures $\sigma(E), \sigma(F), \sigma(B)$ defined.

Fact: $\chi(E) = \chi(F) \chi(B)$. Qn: Is $\sigma(E) = \sigma(F) \sigma(B)$? 

Theorem (C.H.S.): Suppose the monodromy action $\pi_1(B) \to Aut(H^*F)$ is trivial. Then $\sigma(E) = \sigma(F) \sigma(B)$.

Can this assumption be lifted?

Yes (Atiyah-Kodaira) gives construction $\Sigma_{6} \to E \to \Sigma_{12g}$, $\sigma(E) = 256$.

The idea: take an "interesting" fiberwise branched covering of a product of surfaces.

Want to branch-cover the fiber over $z$ at $z_1, z_2$, making a bundle with fiber of genus $G$ (by Riemann-Hurwitz).

Problem: there's a (finite) obstruction to doing this. Pull back to $\Sigma_{12g}$ to resolve this: $E \to \Sigma_{12}$

$
\downarrow$

Double branch cover.

One can compute $\sigma(E) = 256$ (eg by G-sig theorem).

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This manifold has an array of awesome properties:
- Nonzero signature
- $E$ can be made into an algebraic variety. So it is one of finitely many examples (in genera $6, 12g$).
- $E$ is an important example in symplectic geometry:
  Thin (thudton): Every SBS admits a symplectic structure built from data of area forms on base, fiber.

A question in symplectic geometry: how many families of symplectic structures on a manifold carry? LeBrun observed the following fundamental fact:
- $E$ has two distinct surface bundle structures. 

He then showed that the symplectic structures associated to the fibrations are "equivalent" in a strong sense.

(II) - Characteristic Classes of Surface Bundles.

**Def.** A characteristic class is an assignment $(E \rightarrow B) \rightarrow c(B) \in H^*(B)$ that is functorial under pull back.

By general nonsense, this is the same thing as a class $c \in H^*(BDiff \Sigma_g)$.

**Problem:** This is only as good as our understanding of $BDiff$.

**Amazing Theorem:** (Earle-Eells): $Diff^+ \Sigma_g \subset \ast$.

Basic alg. topology $\Rightarrow BDiff^+ \Sigma_g \cong B\pi_0(Diff^+ \Sigma_g) = K(\pi_0(Diff^+ \Sigma_g), 1)$.

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**Def:** $\Pi^0(Diff^+\Sigma_g) = \text{Mod } \Sigma_g$, the mapping class group. Elements: diffeos up to isotopy.

The upshot:

\[
\begin{cases}
\text{ Iso. classes of}
\begin{cases}
\text{ genus- } g \text{ bundles over } B
\end{cases}
\rightarrow 1-1 \rightarrow \left\{ \text{ Conj. classes of reps } \pi_1 B \rightarrow \text{ Mod } \Sigma_g \right\}
\end{cases}
\]

(Such a $\rho$ is called the "monodromy" of the bundle.)

**Basic Problem:** Relate topology/geometry/etc. of the LHS to the algebra/geometric group theory/etc. of the RHS.

* Ex: $H_\infty$ per bolic $Map \ \Sigma_3 \rightarrow S^1$ $\longmapsto \rho$ "Pseudo-Anosov" (Thurston)

- $\Sigma_3 \rightarrow E \leftarrow \text{ Complex}$ $\longmapsto ?? $
  \downarrow \quad \times \quad \text{ Riem. svt.}$
- $E \rightarrow \text{ has 72 fibrings } \longmapsto ?? ?? $
  B_1 \rightarrow B_2 \ (B_3?)$
- $\Sigma_3 \rightarrow E \rightarrow B$ has $\text{imp } \leq \Sigma_g$ the "Torelli group"
  \[ H^*_E \cong H^*B \otimes H^*\Sigma_g \text{ (comp) } \]
- Same, but $H^*_E \cong H^*B \otimes H^*\Sigma_g$ $\text{imp } \rho \leq \Sigma_g$ the "Johnson kernel".

(5)
Back to char. classes:

\[ H^\bullet(\text{BDiff}) \cong H^\bullet(\text{Mod}_g). \] Computing low-dim homology in the presence of a tractable presentation can be done!

- \( H^1(\text{Mod}_g, \mathbb{Z}) = \text{Mod}_g^{(ab)} = (1) \). So no 1-dim char. classes.
  
  \text{(NB: No longer true for } \Gamma \text{. If } \text{Mod}_g \text{ for } \Sigma \text{, } \Gamma \text{ there is a very rich theory.)}

- \( H_2(\text{Mod}_g, \mathbb{Z}) \) is computable from "Hopf's formula", using a presentation of Wajnryb. He finds \( H_2(\text{Mod}_g, \mathbb{Z}) \cong \mathbb{Z} \).

Cool fact: Any two 2-cochains give proportional cohomology classes, even when they seem to be measuring very different things!

- Example: Atiyah-Kodaira revisited. A 2-cochain on \( \text{BDiff} \) eats a surface in \( \text{BDiff} \), returns a number.

  Surface in \( \text{BDiff} \mapsto \text{SBS} \mapsto \sigma(\text{SBS}) \in \mathbb{Z} \).

  Meyer (via Novikov additivity) showed \( \sigma \) was a cocycle. So \( E \) represents a nontrivial cycle in \( H_2(\text{BDiff}) \).

- Here's a second construction of a char. class \( e \in H^2(\text{Mod}_g) \).

\[
\begin{array}{ccc}
\Sigma \rightarrow & T\Sigma \rightarrow & TE \\
\downarrow & \downarrow & \downarrow \\
B & TB & \\
\end{array}
\]

Associate the Euler class of \( T\Sigma : e \in H^2(E, \mathbb{Z}) \).

When \( B = B^2 \) a surface, \( e^2 \in H^4E \cong H^2B \cong \mathbb{Z} \), and it's easy to see that this is a char. class.

This is known as \( e_1 \), the first MMM class.
More generally, can look at $e^{i\varepsilon_1} \in H^{2k+2}(E^{2k+2}) \cong H^{2k}(B^{2k})$.

We call this $e_I$.

Hugely important thm of Madsen-Weiss: The "Stable" cohomology of the mapping class group is a polynomial algebra on the $e_I$.

However, then almost all of the cohomology is "unstable," but we know only a tiny number of actual examples, and not in an "intrinsic" way like $e_I$. They are as ubiquitous as they are mysterious.

Characteristic classes interact with multiply-fibered tangent bundles in a neat way.

Defn. (Church- Farb-Thiebault) A $k$-diml class $\alpha$ is geometric if $\alpha \left( \frac{E^{2k+2}}{X_k} \right)$ depends only on $E$, so that if also $\alpha \left( \frac{E^{2k+2}}{Y_k} \right)$, then $\alpha \left( \frac{E}{X} \right) = \alpha \left( \frac{E}{Y} \right)$.

Thm (C-F-T) All odd MMM classes exist are geometric.

Problem: How ubiquitous is multiple fibering?

Open problem: Does there exist a SBS with 7/3 fiberings? (We know there are only finitely many).

Thm (S--): No, when the monodromy lies in $\pi_1 S^3$.