Realization Problems in the Mapping Class Group

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Our fundamental object of study:

\[ 1 \rightarrow \text{Diff}_0^+ \Sigma \rightarrow \text{Diff}^+ \Sigma \rightarrow \text{Mod} \Sigma \rightarrow 1 \]

\(*\)

Q: Given \( \Gamma \leq \text{Mod} \Sigma \), is there a lift \( \Gamma \hookrightarrow \text{Diff}^+ \Sigma \)?
Thm (Nielsen) $\Gamma \cong \mathbb{Z}/n$, \\
$\Gamma \leq \text{Mod } \Sigma$ lifts.

(Pf) Study action $\Gamma \curvearrowright \text{Teich} \Sigma$.

- $K(\Gamma, 1)$-theory: Some $f^k$ has fixed pt.
A fixed pt is a hyp. surf. \((X, g)\) for which \(\exists\) isom. \(I\) s.t. \(\sum \rightarrow (x, g) \downarrow \begin{array}{c} \exists f \\ \rightarrow \end{array} \rightarrow (x, g) \downarrow \begin{array}{c} f \cdot f^n \rightarrow (x, g) \end{array}\). Then \([I] = f^n\), and \(I^n = \text{id.}\).

This does it for \(n = p\ prime.\)
General case: \( n = P_1 \cdots P_l \). Study \( f^P \). Induct: \( \exists (X, g) \in \mathcal{I} \) for which \([I] = f^P\).

Take \( Y = X / \langle I \rangle \); set

\[ \Sigma' = \left( \text{underlying surface} \right) \quad \text{minus exceptional pts.} \]
Claim: \( \text{Teich}(\Sigma') \cong \text{Fix}(\Gamma^p) \)

Now \( f \not\in \text{Fix}(\Gamma^p) \), so fixed a pt (as \( \text{Teich}(\Sigma') \downarrow \)).

Kerckhoff: All finite \( \Gamma \) are realizable.
(Bold) Question: Does all of Mod(2) lift?
I.e. does (*) split?

What CONSEQUENCES would such a lifting entail?
Let's think about Surface Bundles.

**Monodromy**

Given \( \Sigma \to E \overset{\pi}{\to} M \), follow the fiber around loops in \( M \); get a map

\[
\Pi_1(M) \overset{p}{\to} \text{Mod} \Sigma
\]
Fun-fact (Earle-Eells):
\[ \text{Diff}^+ \Sigma \rightarrow \text{contractible.} \]

(Ref.) \[ \text{Diff}^+ \Sigma \rightarrow M(\Sigma) \rightarrow \overline{\text{Teich}}(\Sigma). \]

- Show this is a fiber bundle.
- Show \( M(\Sigma) \) (complex structures on \( \Sigma \)) is contractible.
Consequence: $\text{Diff}^+\Sigma \to \text{Mod}\Sigma$

from $(\ast)$ is htpy equiv. and so

\[
\left\{ \Sigma\text{-bundles over } M \right\} \leftrightarrow \left\{ \text{conj. classes of } \rho: \pi_1(M) \to \text{Mod}\Sigma \right\}
\]

Surface bundles are specified by their monodromy.
Flat Bundles

\[ F \rightarrow E \xrightarrow{\pi} M \]

- \( \exists \, \rho : \pi_1(M) \rightarrow G, \ E \cong \tilde{M} \times F / \pi_1(M) \)
- \( G \) can be taken w. discrete topology.
- \( \exists \) foliation \( F \) of codim. \( n \) on \( E \) transverse to fibers.
So $(\ast)$ splits $\Rightarrow$

All $\Sigma$-bundles are flat.

Is this true???
Bott's Vanishing Thm.

Let $\mathcal{F}$ be a codim-$g$ foliation on $\mathbb{C}^n$-manifold $M$. Then

$$Q[P_1(V(\mathcal{F})), P_2(V(\mathcal{F})), \ldots] \in H^*(M; \mathbb{Q})$$

vanishes in deg $i > 2g$.

($P_i$: Poincaré classes,
$V(\mathcal{F})$: Normal bundle of $\mathcal{F}$.)
For all \( k \in \mathbb{Z} \), \( \psi^k \neq 0 \) if the vertical tangent bundle.

Both: \( p_k^*(T\pi) = e^y(T\pi) = 0 \).

(\( e \): \textit{Euler Class} )

So: \( (\star) \) splits \( \Rightarrow e^y = 0 \) A bundles
(Def) $i^{th}$ MMM class: $e_i \in H^{2i}(M; \mathbb{Z}) = \Pi_k^*(e^{i+1})$

(Def) (Gysin Map)

Thm (Morita): \forall \: n, \: \exists \: g \circ (n)

for which

$Q[e_1, \ldots, e_n] \to H^*(\text{Mod}_{2g}; \mathbb{Q})$

injects in degree $\leq n$ (g2 g(n)).
Cor.: (*) doesn't split!

(P4) Sketch $e_i \neq 0$.

One idea: $\langle P_1(E), [E] \rangle = 3\text{Sig}(E)$
for 4-manifolds (Hirzebruch)
and $P_1(E) = P_1(T_{\pi}) = c^2(T_{\pi})$.

So let's find $\Sigma g \rightarrow E$ w. $\text{Sig}(E) \neq 0$. 

$\Sigma l_
Our proof is different, and uses **Cyclic Ramified Covers**.

- Local model $\mathbb{C} \times \mathbb{R}^{n-2}$ w. mnp 
  
  $$(z, x_3, ..., x_n) \mapsto (z^m, x_3, ..., x_n).$$

As $H^2(M; \mathbb{Z}) = [M; \mathbb{C}P^\infty]$, given $D \in M_{\text{codim. 2}},$ 

$\exists$ C-bundle η, $c_1(\eta) = [D]^*$. 

Claim: If \([\mathcal{D}]^* = 0\) in \(H^2(M; \mathbb{Z}/m)\), then \(\tilde{\mathcal{M}} \to M\) \(m\)-fold cyclic ram along \(D\) exists.

\[(\text{Pf})\]

\[
\begin{align*}
\tilde{E}(\eta') & \overset{\nu \to \nu^m}{\longrightarrow} E(\eta) \\
\pi' & \downarrow \pi \\
\tilde{\mathcal{M}} & \to M
\end{align*}
\]

\(\tilde{\mathcal{M}} = f^{-1}(S(M))\) works.
Atiyah-Kodaira:

\[(**\)]\quad E \rightarrow M_3 \times M_2 \rightarrow M_2 \times M_2 \rightarrow M_1 \times M_1 \]

\[M_3 = M_3 \rightarrow M_2 \rightarrow M_1\]

\[\pi_1(M_3) = \ker(\pi_1(M_2) \rightarrow H_1(M_2) \otimes \mathbb{Z}/m)\]

\[M\text{-fold ramified cover along } D\text{: other covers constructed so } [D]^x m = 0.\]
Lemma

\[ D \xrightarrow{\sim} E \xrightarrow{f} E \]

\[ M = M \]

Then \( \tilde{e} = f^*(e - (1-\lambda_m)[D]^*) \).

(Pf) Thom isomorphism.
\[ \langle e_1, [M_3] \rangle = \langle e^2, [E] \rangle \]

Lemma: \[ e = (2-2g)[M_2]^* - (1-\lambda_m)[D]^* \]

Move down to \[ M_2 \times M_2 \ (\text{deg} \ m^{2g+1}) \]:

\[ = m^{2g+1} \langle (2-2g)[M_2]^* - (1-\lambda_m)[D]^* \rangle^2 \]

[\langle [M_2 \times M_2] \rangle]
Compute \( (M_2^*)^2 \), \( M_2^* \cdot [D_2]^* (D_2^*)^2 \)

via transversality.

\( (M_2^*)^2 = 0 \)

\( M_2^* \cdot [D_2]^* = m \cdot [M_2 \times M_2]^* \)

\( (D_2^*)^2 = M (2 - 2g), [M_2 \times M_2]^* \)
Can compute:

\[ \langle e_1, [H_3] \rangle = (2g-2)M^2 (m^2 - 1) \neq 0. \]
The M-Construction

Gist of (**): Given bundle $E \to M$ w. section, take covers of base, fiber; so that we can take ramified cover along section.
Operations on bundles:

1) Pull back along projection:

\[ \pi^*(b) \rightarrow E \]
\[ \downarrow \]
\[ E \xrightarrow{\pi} M \]

This has a natural diagonal section.
2) Take a cover of the base that admits a fiberwise covering:

\[ E'' \rightarrow E' \rightarrow E \]

\[ \Sigma' \rightarrow \Sigma \rightarrow \Sigma \]

\[ M' = M' \rightarrow M \]

Here, \( \Sigma' \rightarrow \Sigma \) is m-fold cover.
3) Given $E \to M$ with $D \subseteq E$ codim. 2, there cover $\mu^* \xrightarrow{\approx} M$ s.t. $f^{-1}(D)$ is ramification less of $m$-fold cyclic ram. cover.

Need hypotheses; e.g. $M = S^n$ has no covers.
Given $\eta: E^{2n+2} \xrightarrow{\pi} M^{2n}$

with $\langle e^n(\eta), [M] \rangle \neq 0$, observe steps 1, 2, 3 to get

\[ \xymatrix{ \tilde{E} \ar[d] \ar[r] & \pi^*(E) \ar[d] \ar[r] & E \ar[d] \ar[r]^{\pi} & \tilde{E} \ar[d] \ar[r] & E \ar[d] \ar[r]^{\pi} & M } \]

Relate $\langle e^{n+1}(\tilde{\eta}), [E'] \rangle$, $\langle e^n(\eta), [M] \rangle$. 
Non-lifting of small subgroups.

(B) \[ 1 \rightarrow \pi_1(\Sigma, z) \rightarrow Mod(\Sigma, z) \rightarrow Mod(\Sigma) \rightarrow 1. \]

\[ \text{Thm (Bestvina-Church-Souto): } \]

\[ \pi_1(\Sigma, z) \leq Mod(\Sigma, z) \text{ doesn't lift } (g \geq 2). \]
The idea:

$\text{Mod}(\Sigma, \mathbb{Z})$ is all about bundles w. section, and there have some naturally-associated $S^1$-bundles.
Nielsen Map

\[ \text{Mod}(Σ, 2) \rightarrow \text{PSL}_2 \mathbb{R} \text{ Home}^+(S^1) \]

Pick rep of f, lift to \( \tilde{\phi} : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \)
fixing \( \tilde{z} \), look at action on \( \partial \mathbb{H}^2 \).

Well-defined, as isotopies move by bold distance.
Nielsen Bundle

Given $\Sigma \rightarrow E$, take monodromy $\pi_1(M) \rightarrow \text{Mod}(\mathbb{Z}, \mathbb{Z})$.

Then consider the flat $S^1$-bundle over $M$ given by

$$\pi_1(M) \rightarrow \text{Mod}(\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Homeo}^+(S^1).$$
BCS observation:

When $\text{Br}_1(M) \rightarrow \text{Mod}(\mathbb{Z}/2) \rightarrow \text{Diff}(\mathbb{Z}/2)$ lifts, there is a second bundle: $\text{Br}_1(M)$ acts on $T\Sigma$ by $GL_2^+(\mathbb{R})$, giving a flat linear $\mathbb{Z}$-bundle.

Restrict to Spec. of directions.
These are isomorphic, as they co-bound an annulus.

So their Euler numbers agree.
Consider $\Sigma \rightarrow E \xrightarrow{\pi} \Sigma$

with monodromy $\pi_1(\Sigma) \xrightarrow{id} \pi_1(\Sigma) \subset \text{Mod}(\Sigma, \mathbb{Z})$.

Observe: $E = \Sigma \times \Sigma$, $\sigma$ is diag. section.

Moriya: Nielsen bundle $\cong UT(\Sigma)$, hence $e = 2 - 2g$. 
Milnor: $|e| \leq g-1$ for $GL_2(R)$ bundles.

So no lifting!