Induced representations

1. Setup - we have a finite group $G$ and a subgroup $H \leq G$.

   Given a representation of $H$ $\rho: H \to \text{GL}(V)$, we want to find/construct from it a $G$-rep in a canonical way.

   This canonical $G$-rep will be called the induced representation $(\text{Ind}_H^G \rho): G \to \text{GL}(V)$.

2. Definition: $\rho_G: G \to \text{GL}(V)$ is said to be induced by $\rho_H: H \to \text{GL}(W)$ if:
   
   1. $W \leq V$ an $H$-invariant subspace of $V$.
   
   2. For every $H$-coset in $G$, $\sigma = gH \in G/H$, there exists a vector subspace $W_{\sigma} \leq V$ s.t. $V = \bigoplus_{\sigma \in G/H} W_{\sigma}$ and $W_{\sigma} \cdot W_{\sigma'} = 0$.

   3. For every $g \in G$, we have $g \cdot (W_{xH}) = W_{g^0xH}$, i.e., $\rho(g)$ acts on these subspaces by sending $W_{\sigma}$ to $W_{g^0\sigma}$.

   Note: $G$ acts on $G/H$ by left multiplication $g \cdot (xH) = (gx)H$, and it acts on the subspaces $W_{xH}$ as well.

   Requirement (c) asks for the two actions to be equivariant, i.e., $G/H \xrightarrow{\rho} G/H \quad \text{commutes}$.

   \[ \{W_{\sigma}\} \xrightarrow{\rho} \{W_{\sigma}\} \quad \text{for } \sigma \in G/H \]

   \[ \{W_{xH}\} \xrightarrow{\rho} \{W_{xH}\} \quad \text{for } x \in G \]

3. Universal property

   Recall that if an object satisfies some universal property, the object characterizes the object uniquely!

   A Universal property of the induced representation:

   Let $W$ be an $H$-rep, and $V$ a $G$-rep.

   We say that $W \xrightarrow{\alpha} V$ is the induction if it satisfies the following universal property:

   $\alpha$ is an $H$-linear map $W \xrightarrow{\alpha} V$ such that for every $G$-rep $Z$ and a $H$-linear map $W \xrightarrow{\beta} Z$ there exists a $G$-linear map $V \xrightarrow{\beta} Z$ such that $V \xrightarrow{\beta} Z$ commutes, i.e., $\beta \circ \alpha = \beta$.

In diagrams -

\[ \xymatrix{ W \ar[r]^-{\alpha} & V \ar[dr]_-{\beta} & Z \ar[l]^-{\beta} \ar@/_1pc/[ll]^-{\exists!} } \]

Note: This essentially says that $V$ is the universal $G$-rep associated to the $H$-rep $W$.

Any other $G$-rep which $W$ maps into, must factor uniquely through $V$.  

3) \( \exists \bar{\beta} \): Define \( \bar{\beta} \) by the formula \( \circledast \):

\[
\bar{\beta}(w) = g \cdot \beta(g' \cdot u) \quad \forall \text{ } w \in W_{g'H}
\]

and extend linearly to \( V = \bigoplus_{w \in W_0} W_{g'H} \). We need to show that \( \beta = \beta_0 \alpha \) and \( \bar{\beta} \) is \( G \)-linear.

Indeed, \( \forall \text{ } w \in W_{g'H} \) we have

\[
\bar{\beta}_0 \alpha(w) = \bar{\beta}(w) = \circledast \text{ by def.}
\]

and \( \forall \text{ } w \in W_{g'H} \) and \( x \in G \) we have

\[
\Rightarrow \beta(x.w) = x.(g \cdot \beta(g' \cdot (x' \cdot u)))
\]

This shows that \( \bar{\beta}_0 x = x \cdot \bar{\beta} \) on a generating set of \( V \) it’s true on every element of \( V ! \)

4) Explicit construction of \( \text{Ind}_H^G W = V \).

(or alternatively, \( \text{Ind}_H^G W \) exists!)

Given an \( H \)-rep. \( W \), we will construct \( \text{Ind}_H^G W \).

First, pick representatives e.g. \( g_1, g_2, \ldots, g_n \in G \) for all the cosets in \( G/H \), i.e. \( g_1H \neq g_2H \forall i \neq j \) and \( \forall \text{ } g \in G \exists \text{ } i \text{ st. } gh = g_iH \).

Define a vector space \( V = W \oplus W_0 \oplus \ldots \oplus W = W^n \) n times

where the subspaces are labeled by the cosets \( W_{g_1H}, W_{g_2H}, \ldots, W_{g_nH} \).
Define a $G$-action on $V$ in the following way:

For every $g \in G$ and $w \in W_{g,h}$, first find the unique $h'$ s.t.
$$g(g_i h') = g g_i h = g_i h$$
i.e. $f = h$ s.t. $g g_i = g_i h$.

Define $g u$ by setting
$$g u = h u$$
in the $W_{g,h}$ copy,

i.e. make $u$ move from $W_{g,h} \rightarrow W_{g,h}$

and act on it by $h: W \rightarrow W$.

Example. This is a $G$-action on $V$
and it coincides with the $H$-action
on $W = W_{g,h}$.

In particular,
$$gu = (g)^{-1} (g_i u) = (g_i h)(g_i u)$$

in the $W_{g,h}$ copy, on which $H$-acts as
prescribed by $W$.

$$= g h (g_i u)$$

$h$ acts on $W$,

and $g$ moves the result

to the $W_{g,h}$ copy!

The resulting $G$-rep $V$ satisfies
the requirements of $\text{Ind}_H^G W$
and we are done.

Remark: Another way to get
$$V = W \otimes \iota W = W$$
is by looking at the space of functions
$$\{ f: G_H \rightarrow W \}$$
this is like saying $C[G]$ is
the space of functions on $G$.

$H$ acts on $f: G_H \rightarrow W$ by acting on its

values $(h, f(a)) = h \cdot (f(a))$,

and $G$ acts on $f$ by acting on $G_H$
before applying, and at the same time
acting by $H$:

For every $g \in G$ find the unique $h'$
and $h$ s.t.
$$g g_i = g_i h$$

Define,
$$g w = h \cdot (f(g_i, o)) = h \cdot f(g_i o)$$
($g_i$ and $g$ act identically on $G_H$)

Note: $W_0 = \{ f \in W: f(r) = 0 \forall r \in o \}$
i.e. function that take values $0$
only on $o$.

$g W_0$ takes values only on
the coset $g o \Rightarrow g W_0 = W_0$.

$$(g f)(g o) = h f(g g o) = h f(o)$$

The space $W = W_{g,h}$ is
$H$ invariant, and on it the
two actions of $H$ coincide.

Cor. $\{ f: G_H \rightarrow W \}$ with this
$G$-action is the induced rep.

(5) Examples:

1. $\text{Ind}_H^G (C_{\text{triv}}) = C[G_H]$.

Proof. The $H$-invariant subspace $W$ is
the 1-dim space $C \cdot e_H$.

$W$ is $H$ invariant, since for all
$$h \in H, e_H = e_{oH} = e_{oH} e \in W$$

The $H$-action restricted to $W$
is trivial.

$\Rightarrow W = C_{\text{triv}}$ as an $H$-rep.

$C[G_H] = \bigoplus_{o \in O_H} W_o = \bigoplus_{o \in O_H} C e_o$. 

$(g f)(g o) = h f(g g o) = h f(o)$.
and the $G$-action on $\{W_\alpha\}_{\alpha \in \mathcal{G}_H}$ is precisely the action on $\mathcal{G}_H$:

$g\cdot e_\alpha = e_{g\alpha} \in W_{g\alpha}.$

Thus $C[\mathcal{G}_H] = \bigoplus_{\alpha \in \mathcal{G}_H} W_\alpha$ in the right way, with $W=W_{eH}$ the trivial $H$-rep

$\implies C[\mathcal{G}_H] = \text{Ind}_H^G (C_{\text{triv}}).$

(1) In particular,

$\text{Ind}_H^G (C_{\text{triv}}) = C[\mathcal{G}_H] \cdot C[G]$.

the standard rep. is induced from the trivial rep. of $\{e\}$ of $G$.

(2) $\text{Ind}_H^G (C[H]) = C[G]$.

Ps: Let $W = C \langle e_h : h \in H \rangle$

$= C e_1 \otimes C e_2 \otimes \ldots \otimes C e_n$

the space spanned be the elements of $H$.

(3) $W$ is $H$-invariant - $H e_h e W$

$h \cdot (e_h) = e_{h h^{-1}} \in W$

(4) The $H$-action restricted to $W$

is precisely the regular rep. $C[H]$.

(5) Let $\sigma = g H \in \mathcal{G}_H$ be a coset.

and define $W_\sigma = C \langle e_{g \alpha} : \alpha \in \mathcal{G}_H \rangle$

$C \langle e_x : x \in \sigma = g H \rangle$.

Then $g' e_{g \alpha} = e_{g' g \alpha} \in W_{g' g H}$

$\implies g'(W_{g H}) = W_{g' g H}$

and $C[G] = \bigoplus_{\alpha \in \mathcal{G}_H} C \langle e_{g \alpha} : x \in \mathcal{G}_H \rangle$

$= \bigoplus_{\alpha \in \mathcal{G}_H} W_{g \alpha}$

as required from the induced representation.

(6) This gives another proof for

$\text{Ind}_H^G (C_{\text{triv}}) = C[G]$.

since $C_{\text{triv}} = C[\{e\}]$ the regular rep. of the group with 1 element.