History

Group characters first appeared long before representations!

In analytic number theory, series were used. The most famous of which is the Riemann zeta function
\[ \zeta(s) = \prod_{\text{prime}} (1 - p^{-s})^{-1} \]
but also functions of the form
\[ \zeta(s) = \prod_{\text{prime}} (1 - \frac{\chi(p)}{p^s})^{-1} \]
for some multiplicative function \( \chi : \mathbb{N} \rightarrow \mathbb{C}^* \)
(\( \chi(nm) = \chi(n)\chi(m) \).
(The zeta function corresponding to the case \( s = 1 \)).

Their importance fueled the study of characters:

homomorphisms \( \chi : \mathbb{Z}/N \rightarrow \mathbb{C}^* \)

and more generally, homomorphisms \( \chi : G \rightarrow S \leq \mathbb{C}^* \).

\( \otimes \) Dedekind disclosed that for an abelian group \( G \), one can write the multiplication table of \( G \) as

\[
\begin{array}{ccc}
g_1 & g_2 & \cdots & g_n \\
g_1 & & & \\
g_2 & & & \\
\vdots & & & \\
g_n & & & \\
\end{array}
\]

and taking the determinant
\[ P(x_1, \ldots, x_n) = \det (x_{g_i g_j}) = \text{polynomial in } g_1, \ldots, g_n \]
one finds that \( P \) splits as a product of linear factors

\[ P = \prod_{\chi} (\sum_{x \in G} \chi(x) x^*)^{-1} \]
where \( \chi \) are all the irreducible characters of \( G \) (meaning that they can't be realized as a character of a quotient of \( G \)).

But when taking \( G \) non-abelian, Dedekind couldn't generalize his formula.

He sent this puzzle to Frobenius to solve.

\( \otimes \) Frobenius realized that there is on orthonormal basis of class functions \( \chi_1, \ldots, \chi_k \)
s.t. \( \omega_i = \chi_i(1) \) and \( P \) splits as a product
\[ P = \prod F_i^{\chi_i(1)} \]
where the coeff. \( \frac{\chi_i(1)}{x_i} \) of \( x_i \) in \( F_i \) is \( \chi_i(1) \).

\( \otimes \) Still very mysterious, he found an explicit construction:

If \( \rho : G \rightarrow GL_d(\mathbb{C}) \) is a homomorphism, and there is no basis in which \( p(g) = \begin{pmatrix} A & 0 \\ \# & B \end{pmatrix} \)
[i.e. no invariant subspace - irrep.]
then \( \chi_i(\rho(g)) = \text{Tr}(\rho_i(g)) \).

- But still he cared only about the functions \( \chi_i \) and not about \( \rho_i \).

\( \otimes \) Later, Dedekind and Schur descided to study the \( \rho_i \)'s in their own right and made them the focus of the theory.
The center \( Z(G) \) in the character table

1. In your HW you defined
   \[ \ker (x) = \{ g \in G : x(g) = \dim V \} \]
   and worked out that
   \[ \ker (x) = \ker (\phi) \triangleleft G \text{ a normal subgroup.} \]
   \[ \Rightarrow \text{ The character table detects normal subgroups of } G. \]
   
2. **Claim:** \( Z(G) = \bigcap \{ g \in G : \dim V = \chi(g) \} \) for \( \chi \) irreps.
   
   **PF.** We showed in a previous problem session (2) that
   \[ |\chi(g)| = \dim V \iff \chi(g) = \chi_1 \]
   for \( \chi_1 \in \chi \) some root of 1.
   Thus \( |\chi(g)| = \dim V \iff \chi(g) \) commutes with any other matrix.
   In particular \( \chi(g)\chi(h) = \chi(h)\chi(g) \forall g,h \in G \).
   But if this is the case on every irrep, it will be true on any \( G \)-rep.
   In particular on \( CG \):
   \[ e_{gn} = \phi(g)e_1 = \phi(g)\phi(h)e_1 = \phi(h)\phi(g)e_1 = e_{ng} \]
   \[ \Rightarrow gh = hg \forall g,h \in G, \]
   i.e. \( g \in Z(G) \).

Conversely, if \( g \in Z(G) \) then the map \( \chi(g) : V \to V \) is \( G \)-linear
\[ \chi(g)\chi(h) = \chi(h)\chi(g) \]
   i.e. \( T \circ \chi(h) = \chi(h) \circ T \)
   where \( T = \chi(g) \).

   and thus by Schur's lemma, \( \chi(g) = \chi_1 \) on every irrep.

**Algebraic integers**

1. In a general setting:
   \[ S \leq R \text{ two integral domains.} \]
   We proved in class that if \( \alpha \in R \) satisfies
   \[ \alpha^n + s_1 \alpha^{n-1} + \ldots + s_n = 0 \]
   then \( S[\alpha] \) is finitely generated over \( S \), e.g. by \( 1, \alpha, \ldots, \alpha^n \).
   **Claim:** The converse is also true.
   **PF.** Suppose \( \alpha_1, \ldots, \alpha_n \in S[\alpha] \) are a generating set. Then \( \forall r_i, \alpha_i \in S[\alpha] \Rightarrow r_i S_i \subseteq S \) st.
   \[ \alpha_i = \sum S_i r_i \]
   or, in matrix form \( A = (S_i) \)
   \[ (\alpha_i) = A (r_i) \Rightarrow (dI - A) (r_i) = 0 \]

Let \( \text{adj}(dI - A) \) be the adjoint matrix
(this can be defined over any commutative ring, and has the same properties).

Multiplying by it: \( \text{adj}(A) \cdot \alpha = \text{det}(A)I \)
\[ 0 = \text{det}(A) (r_i) \]

But since \( R \) is a domain,
\[ \text{det}(A) \neq 0 \]
But \( \text{det}(dI - A) = \alpha^n - \text{tr}(A) \alpha^{n-1} + \ldots \]
i.e. \( \alpha \) is integral over \( S \).

2. **Application of \( \mathbb{Z} \cap \mathbb{Q} = \mathbb{Z} \).**
   **Claim:** \( \alpha \in \mathbb{Q} \) is an algebraic integer (in \( \mathbb{Z} \)) if it's minimal polynomial over \( \mathbb{Q} \) has integer coeff.
   **PF.** (\( \Rightarrow \)) is clear by def.
   (\( \Leftarrow \)) Let \( \alpha, \alpha_1, \ldots, \alpha_n \in \mathbb{Q} \) be all the conjugates of \( \alpha \) over \( \mathbb{Q} \)
i.e. the roots of the minimal polynomial of \( \alpha \).

\( \alpha \) is integral \( \Leftrightarrow \exists q \in \mathbb{Z}[\alpha] \) s.t.

\[ q(\alpha) = 0. \]

But then \( \forall i = 1, 2, \ldots, n \)

\[ q(\alpha_i) = 0 \text{ as well} \Rightarrow \{\alpha_i\} \text{ are all algebraic integers.} \]

Since the algebraic integers are closed under addition and multiplication, all symmetric functions

\( s_i(\alpha_1, \ldots, \alpha_n) \) are algebraic integers.

But \( p_i = s_i(\alpha_1, \ldots, \alpha_n) \) is the coefficient of \( \alpha^i \) in the minimal polynomial of \( \alpha \)

\[ \Rightarrow p_i \in \mathbb{Q}. \]

Since \( p_i \in \mathbb{Z} \cap \mathbb{Q} = \mathbb{Z} \) we conclude that \( \min(\alpha) \in \mathbb{Z}[\alpha] \) a monic integer polynomial.

\( \square \)