Basic operations on G-reps (and their effect on characters)

1) Direct sum: Suppose V and W are G-reps.
   We define $V \oplus W$ to be the G-rep with the G-action
   $g \cdot (u \oplus w) = (g \cdot u \oplus g \cdot w)$
   i.e. $\rho_{V \oplus W}(g)(u \oplus w) = (\rho_V(g)u, \rho_W(g)w)$
   $\rho_{V \oplus W}(g)$ is a basis with the property
   that $\rho_{V \oplus W}(g)$ has the block form
   \[
   \begin{pmatrix}
   \rho_V(g) & 0 \\
   0 & \rho_W(g)
   \end{pmatrix}
   \]
   in this basis.

2) Computing the trace of this block matrix, we find
   $\chi_{V \oplus W}(g) = \text{Tr}(\rho_{V \oplus W}(g)) = \text{Tr}(\rho_V(g)) + \text{Tr}(\rho_W(g))$
   $\chi_V(g) + \chi_W(g)$

So characters are additive w.r.t. direct sums.

2) Dual space: Suppose $V$ is a G-rep.
   We wish to define a natural G-action on the dual $V^* = \text{Hom}_C(V, C)$.
   There is a natural way in which a linear map $T: V \to W$ induces a linear map on the dual spaces
   $V^* \to W^* : T^*$
   (going in the opposite direction)
   namely - if $\alpha \in W^*$ is a linear functional on $W$, define $T^*\alpha \in V^*$ to be the
   linear functional $\alpha \circ T : V \to C$.
   $T^*$ is called the dual map.

- How does $()^*$ react to composition?
   If $V \xrightarrow{T} W \xrightarrow{S} U$ is dualized, we get
   $V^* \xrightarrow{T^*} W^* \xrightarrow{S^*} U^*$
   i.e. $(S \circ T)^* = T^* \circ S^*$ - reverses the direction of composition!

- Given a G-rep $\rho_V$ on the space $V$ we can try defining
   $V^* \xleftarrow{\rho_V^*} V^*$
   for all $g \in G$, but because of the arrow reversal we would get
   $\rho_V^*(gh)^* = (\rho_V^*(g)\rho_V^*(h))^* = \rho_V^*(h)\rho_V^*(g)^*$
   and this does not satisfy the composition rule.

   To get the arrows back in the right direction, we replace $g$ by $g^{-1}$:
   $V \xrightarrow{\rho_V(h)} V^* \xleftarrow{\rho_V(h)^*} V^* \xrightarrow{\rho_V(g)} V^*$
   $V \xrightarrow{\rho_V(g^{-1})} V^* \xleftarrow{\rho_V(g^{-1})^*} V^*$

   and this diagram has the form of a G-rep.

- We therefore define
   $\rho_V^*(g) := \rho_V(g^{-1})^*$
   the dual rep. to $V$. 
8) Let's compute the character of the dual rep.

For some $g \in G$. Since $g^* = 1$ we have $\rho(g^*)$ is a diagonalizable transformation, and all its eigenvalues are $n$-th roots of unity. In particular they all lie on the unit circle $\mathbb{C}^*$.

Let $u_1, ..., u_n \in V$ be a diagonalizing basis for $\rho(g^*)$, and

$$\rho(g) u_i = \lambda_i u_i.$$  

In matrix form $\rho(g)$ is

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

and $x_V(g) = \text{Tr}(\rho(g)) = \lambda_1 + \cdots + \lambda_n$.

The diagonal coefficient of $g$ is the complex conjugate of that of $g^*$.

Ex: If $T : V \to V$ has matrix form $\begin{pmatrix} a_{ij} \end{pmatrix}$ w.r.t. a basis $(u_i)$, and $(u^*_i)$ is the dual basis (i.e. $f_i(u_j) = \delta_{ij}$) then $T^* : V^* \to V^*$ has matrix form $\begin{pmatrix} a_{ji} \end{pmatrix}$ in the dual basis.

In particular $\text{Tr}(T) = \text{Tr}(a) = \text{Tr}(a^*) = \text{Tr}(T^*)$.

Cor. $x_{V^*}(g) = \text{Tr}(\rho(g)^*) = \overline{x_V(g)}$ and we find:

$\overline{x_V} = x_{V^*}$ complex conjugates.

3) Tensor products:

If $V$ and $W$ are $G$-reps. then $V \otimes W$ is a $G$-rep via the induced action $\rho_{V \otimes W}(g) = (\rho_V(g) \otimes \rho_W(g))$. If $u_1, ..., u_n \in V$ and $w_1, ..., w_m \in W$ are bases, then $(u_i \otimes w_j)_i$ is a basis for $V \otimes W$.

If $\rho_V(g) u_i = \sum a_{ij} u_j$

$\rho_W(g) w_k = \sum b_{ik} w_k$

then $\rho_{V \otimes W}(g) (u_i \otimes w_k) = \sum a_{ij} b_{kl} u_j \otimes w_k$.

The diagonal coefficient of $(l, l)$-th coefficient is $a_{ll} b_{ll}$.

$$x_{V \otimes W}(g) = \sum_{i,l} a_{ll} b_{ll} = (\sum_{i,l} (\xi_i, b_{ll})).$$

$\implies x_{V \otimes W} = x_V \cdot x_W$ the product of characters.

4) Homomorphism space: Let $V$ and $W$ be $G$-reps.

By similar considerations to the one made for $V^*$, there is a natural $G$-action on the space $\text{Hom}_G(V, W) = \{T : V \to W : T \text{ linear} \}$ given by $g \cdot (T) = gTg^*$ i.e.

$$\rho_{\text{Hom}}(g)(T) = \rho_g(g) \circ T \circ \rho_g(g)^* \in \text{Hom}_G(V, W).$$

We described an isomorphism of vector spaces $V^* \otimes W \cong \text{Hom}_G(V, W)$.

Ex. Prove that this is an iso. of $G$-representations!

8) Computing the character,

$$x_{\text{Hom}_G(V, W)} = x_{V^* \otimes W} = x_V \cdot x_W = \overline{x_V} \cdot x_W.$$ 

Rem: $G$-homomorphisms $\text{Hom}_G(V, W)$ are precisely the maps fixed by all $g \in G$. 
Properties of the character table

Rem 1: By the fundamental thm. of character theory,
\[ V \text{ is irreducible} \iff \frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2 = 1. \]

Since \( |\chi_V(g)| = |\chi_V(g)| \), we find that \( \chi^* \) is irreducible \( \iff \chi \) is real.

\[ \implies \] For every irrep, we find, if \( \chi \) is not a real function, then \( \chi^* \) is a distinct irrep of \( G \).

Rem 2: For every vec. space \( V \),
\[ \text{Tr}(\text{id}_V) = \dim V. \]
- Therefore, if \( \chi \) is the character of some \( G \)-rep. \( W \), then
\[ \chi(1) = \dim W. \]
- \( \forall g \in G \), \( \chi(g) = \frac{1}{|G|} \sum_{i=1}^n \lambda_i \) - the eigenvalues of \( \rho_g(g) \).

\[ \implies |\chi(g)| \leq \frac{1}{|G|} \sum_{i=1}^n |\lambda_i| = \frac{1}{|G|} \dim W \]
and equality holds \( \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n \)
i.e., \( \rho(g) = \chi(g) \text{id}_W \)

So the character tells us some direct information about the rep.

Example: Character table for \( D_4 \).
The Dihedral group of order 8 \( \text{is the symmetry group of a square} \)
\[
\begin{array}{c|c|c|c|c}
& e & r & r^2 & r^3 \\
\hline
\sigma & 1 & 1 & 1 & 1 \\
\tau & 1 & -1 & 1 & -1 \\
\end{array}
\]
it's generated by:
- a rotation by 90° - \( \tau \)
  \( (\tau^4 = 1) \)
- a reflection along the horizontal axis - \( \sigma \)
  \( (\sigma^2 = 1) \)

and these are subject to the relation \( \sigma \tau \sigma = \tau^{-1} \)
(rotation in the opposite direction)

\( * \) The conjugacy classes are:
\[ \{e\}, \{\sigma, \tau^3\sigma\}, \{\sigma \tau, \tau^3 \sigma^2\}, \{\sigma, \tau^3\}, \{\tau^3\} \]
so we will have 5 irreps!

\( * \) We have a relation between \( |G| \) and the dimensions of the irreps.
\[ 8 = |G| = \sum_{i=1}^5 (\dim V_i)^2 \]
and since these are integers, the only possibility is \((\dim V_i) = (1,1,1,1,2)\).

\( * \) Let's find the 1-dim. reps.

Note: If \( V_i \) is 1-dim, then
\[ \rho_i(g) = \text{multiplication by some scalar } \lambda_i \in \mathbb{C}^x \]

In particular,
\[ \rho_i(gh) = \rho_i(g)\rho_i(h) = \rho_i(g) \rho_i(h) = \rho_i(hg). \]
For \( D_4 \), this means,
\[ \rho_i(r^2 \sigma) = \rho_i(\sigma r^2 \sigma) = \rho_i(\sigma) \rho_i(r^2) = \rho_i(r^2) \]
\[ \implies \rho_i(r^2) = 1. \]

Thus \( \rho_i \) factors through the quotient \( D_4 \rightarrow D_4/_{r^2} \rightarrow \text{GL}(V) \)

Ex: \( D_4/_{r^2} \)
\[ \begin{array}{c|c|c|c}
\sigma & 1 & 1 & 1 \\
\tau & -1 & -1 & 1 \\
\end{array} \]

by \( \sigma \rightarrow (1,0) \)
\( \tau \rightarrow (-1,0) \).

And we know exactly 4 non-iso.
1-dim. representations of \( \mathbb{Z}_2^2 \):
(1) \( V_{\text{triv}} \in V_{\text{triv}} \otimes V_{\text{triv}} \)
(2) \( V_{\text{sign}} \in V_{\text{sign}} \otimes V_{\text{sign}} \)
(3) \( V_{\text{triv}} \in V_{\text{triv}} \cdot V_{\text{sign}} \)
(4) \( V_{\text{sign}} \in V_{\text{sign}} \cdot V_{\text{sign}} \)
\[ \tau^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau^3 = \tau^* = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \]

and \( \sigma \tau \) switches \( V_i \) and \( V_{-i} \).

so always have trace 0.

\[ \text{Tr}(\tau^2) = 1 - 1 = 0 \]
\[ \text{Tr}(\tau^3) = -1 - 1 = -2 \]
\[ \text{Tr}(\tau^*) = -i + i = 0 \]

and we have completed our character table:

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<th>( C_7^2 )</th>
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</tbody>
</table>

Ex. Verify that the rows and columns of this 5\times5 table are orthogonal w.r.t. the inner product:

\[ \frac{1}{16} \sum_{c \in C} \chi_c(\tau) \chi_c(\sigma) \]

\[ = \frac{1}{16} \sum_{g \in G} \chi_{1}(g) \chi_{2}(g) \]

\[ = \frac{1}{16} \sum_{g \in G} \chi_{(c,1)}(g) \chi_{(c,i)}(g) \]