Homework Assignment 4, due Wednesday May 15

PART 1) Chapter 11 Problems 1-2-3-4-5-6

The goal of the next problem (in three parts) is to prove that: 1) there exist measurable, non-Borel sets and 2) there exist a measurable function $F$ and a continuous function $G$ such that $F \circ G$ is not measurable.

PART 2a) Let $E$ be a set in $\mathcal{M}(m)$ where $m$ is Lebesgue measure.

1. Prove that if $E \subset N$ (where $N$ is the non-measurable set from last week’s problem set), then $m(E) = 0$.
2. Deduce that if $m(E) > 0$ then $E$ contains a non-measurable set.

PART 2b) Prove that every subset $A$ of $\mathbb{R}^p$ such that $m^*(A) = 0$ (where $m$ is Lebesgue measure) is measurable.

PART 2c) Let $f : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function, defined last quarter, and let $g(x) = f(x) + x$. Prove the following three statements:

1. $g$ is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.
2. If $C$ is the Cantor set, $m(g(C)) = 1$.
3. Let $A$ be a non-measurable subset of $g(C)$ and $B = g^{-1}(A)$. Then $B$ is measurable but not Borel.
4. There exist a measurable function $F$ and a continuous function $G$ on $\mathbb{R}$ such that $F \circ G$ is not measurable.