Homework Assignment 3, due Wednesday May 1 and Wednesday May 8

PART 1) (due Wednesday May 1st) Chapter 10 from Rudin: 20-21-22

PART 2)(due Wednesday May 8) Chapter 11 Problems 1-2-3-4-5-6

PART 3)(due Wednesday May 8) We prove that there cannot exist a set function $\Phi : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$ satisfying the three following properties:

1. $\Phi$ is countably additive,
2. $\Phi$ is translation invariant (i.e. if $x \in \mathbb{R}$, $E \subset \mathbb{R}$, then $\Phi(E) = \Phi(E + x)$),
3. $\Phi([0,1]) = 1$.

Define the equivalence relation $\sim$ on $[0,1)$ so that $x \sim y \iff x - y \in \mathbb{Q}$, and let $N$ be a subset of $[0,1)$ containing exactly one member of each equivalence class (You can assume such a set exists. Its existence is actually a direct consequence of the axiom of choice). We then define for each $r \in \mathbb{Q} \cap [0,1)$ the set $N_r$ to be:

$$N_r = \{x + r; x \in [0,1 - r)\} \cup \{x - (1 - r); x \in [1 - r,1)\}.$$ 

a) Prove that $[0,1) \cup \{x \in 0, 1 - r\} \cup \{x - (1 - r); x \in [1 - r,1)\}$. 

b) Prove that $N_r \cap N_s = \emptyset$ if $r \neq s$.

c) Deduce that there exist no function $\Phi$ as described in the beginning of the problem.

Part 4)(due May 8th)a) Let $\mathcal{E}$ be the following subset of $\mathcal{P}(\mathbb{R})$:

$$\mathcal{E} = \{E \subset \mathbb{R}; \chi_E \text{ is Riemann integrable}\},$$

where $\chi_E$ is the indicator function of $E$ given by $\chi_E(x) = 1$ if $x \in E$, 0 otherwise. We define the set function $l : \mathcal{E} \rightarrow [0, +\infty]$ by

$$l(E) = \int \chi_E(t)dt.$$ 

Prove that $l$ satisfies the following properties:

1. $l(\emptyset) = 0$,
2. if $A, B$ are disjoint, $A, B$ are in $\mathcal{E}$, then $l(A \cup B) = l(A) + l(B)$. More generally, if $\{E_i\}_{i=1}^n \subset \mathcal{E}$ are pairwise disjoint then $l(\cup_{i=1}^n E_i) = \sum_{i=1}^n l(E_i)$.
3. if $x \in \mathbb{R}$, then $l(E + x) = l(E)$.

b) Prove statements (12)-(13)-(14)-(15) on page 303 from Rudin