

Introduction to Chiral Algebras

Nick Rozenblyum

Our goal will be to prove the fact that the algebra $End(Vac)$ is commutative. The proof itself will be very easy - a version of the Eckmann Hilton argument - once the machinery of chiral algebras is set up.

1 Chiral Algebras

Let X be a smooth curve over \mathbb{C} . A non-unital chiral algebra on X is a D -module \mathcal{A} along with a "chiral bracket" map

$$\mu : j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

where $j : U \rightarrow X^2 \leftarrow Z : \Delta$ are the inclusion of the complement of the diagonal and the diagonal respectively. We require that μ be antisymmetric and satisfy a version of the Jacobi identity:

- Antisymmetry: $\mu = -\sigma_{1,2} \circ \mu \circ \sigma_{1,2}$, where $\sigma_{1,2}$ is induced action on \mathcal{A} by permuting the variables of X^2 .
- Jacobi Identity: we have three maps $\mu_{1(23)}, \mu_{(12)3}$ and $\mu_{2(13)} : j_*j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$ where (somewhat abusing notation) j is the inclusion of the open in X^3 which is the complement of all the diagonals and Δ is the inclusion of X as the diagonal. We have that $\mu_{1(23)}$ is defined as the composition

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_{(x_2=x_3)!}j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!\mathcal{A}$$

and the others are defined similarly. We then demand that

$$\mu_{1(23)} = \mu_{(12)3} + \mu_{2(13)}$$

Example 1. Let $\mathcal{A} = \omega_X$. We then have the canonical exact sequence

$$0 \rightarrow \omega_X \boxtimes \omega_X \rightarrow j_*j^*(\omega_X \boxtimes \omega_X) \rightarrow \Delta_!(\omega_X) \rightarrow 0$$

which gives ω_X a chiral bracket. It is clearly antisymmetric. We need to check that it satisfies the Jacobi identity. To do that, consider the Cousin complex for $\omega_X^{\boxtimes 3}$ on X^3 for the stratification given by the diagonals. It gives the exact sequence

$$0 \rightarrow \omega_X^{\boxtimes 3} \rightarrow j_*j^*(\omega_X^{\boxtimes 3}) \rightarrow \begin{array}{c} \Delta_{x_1=x_2}!j_*j^*(\omega_X^{\boxtimes 2}) \\ \oplus \\ \Delta_{x_1=x_3}!j_*j^*(\omega_X^{\boxtimes 2}) \\ \oplus \\ \Delta_{x_2=x_3}!j_*j^*(\omega_X^{\boxtimes 2}) \end{array} \rightarrow \Delta_!(\omega_X) \rightarrow 0$$

The three maps in the complex

$$j_*j^*(\omega_X^{\boxtimes 3}) \rightarrow \Delta_!(\omega_X)$$

are exactly the maps in the Jacobi identity and the fact that the above Cousin complex is a complex at the term between those is exactly the condition that the Jacobi identity is satisfied.

We can now finish defining a chiral algebra.

Definition 2. A (unital) chiral algebra \mathcal{A} is a non-unital chiral algebra together with a map of chiral algebras

$$\omega_X \rightarrow \mathcal{A}$$

such that the restriction of the chiral bracket

$$\mu : j_*j^*(\omega_X \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

is the canonical map coming from the complex

$$0 \rightarrow \omega_X \boxtimes \mathcal{A} \rightarrow j_*j^*(\omega_X \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A}).$$

As with any kind of algebra, given a chiral algebra \mathcal{A} , we can consider modules over it.

Definition 3. Let \mathcal{A} be a chiral algebra. A chiral \mathcal{A} module is a D -module \mathcal{M} on X together with an action map

$$\rho : j_*j^*(\mathcal{A} \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

satisfying the unit and Lie identity:

- *Unit:* we require that the restriction of ρ to ω_X

$$\rho : j_*j^*(\omega_X \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

be the canonical map.

- *Lie action:*

$$\rho(\mu \boxtimes id) = \rho(id \boxtimes \rho) - \sigma_{12} \circ \rho((id \boxtimes \rho) \circ \sigma_{12})$$

as maps

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M}).$$

Example 4. Let \mathcal{A} be a chiral algebra. Then \mathcal{A} is canonically a chiral \mathcal{A} module.

Let \mathcal{M} be a D -module on X . Then the canonical map

$$j_*j^*(\omega \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

makes \mathcal{M} into a chiral ω module. In fact, because of the unit axiom, this is the unique structure of a chiral ω module on \mathcal{M} . Thus we have an equivalence of categories

$$\{D\text{-modules}\} \xrightarrow{\sim} \{\text{chiral } \omega \text{ modules}\}.$$

For our purposes, we will assume that \mathcal{A} is flat as an \mathcal{O}_X module. However, we will not make a similar assumption on \mathcal{M} . For instance, we will often be interested in modules supported at a point $x \in X$. A typical example of this is the vacuum module $\mathcal{M} = i_{x!}i_x^!(\mathcal{A})[1]$.

Recall that for D -modules we have the deRham functor

$$h : D\text{-mods} \rightarrow \{\text{sheaves}\}$$

given by modding out by the action of vector fields:

$$h(\mathcal{M}) = \mathcal{M}/\mathcal{M} \cdot \Theta.$$

Now, for a chiral algebra \mathcal{A} , consider the composition

$$\mathcal{A} \boxtimes \mathcal{A} \rightarrow j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A}).$$

Applying the deRham functor, we get

$$h(\mathcal{A}) \boxtimes h(\mathcal{A}) \rightarrow \Delta_*(h(\mathcal{A}))$$

which by adjunction gives the map

$$h(\mathcal{A}) \otimes h(\mathcal{A}) \rightarrow h(\mathcal{A})$$

which makes $h(\mathcal{A})$ into a sheaf of Lie algebras.

Now suppose that \mathcal{M} is a chiral \mathcal{A} module supported at a point $x \in X$. Let $M = i_x^!(\mathcal{M})$ be the underlying vector space. Pushing forward the action map

$$j_* j^*(\mathcal{A} \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

along the first projection we get

$$DR(X - x, \mathcal{A}) \otimes M \rightarrow M$$

which is an action of the Lie algebra $DR(X - x, \mathcal{A})$ on M . In fact, we can shrink the curve X to get an action of the topological Lie algebra $DR(D_x^\circ, \mathcal{A})$ on M where

$$DR(D_x^\circ, \mathcal{A}) = \varprojlim_{\leftarrow} i_x^!(j_{x*} j_x^*(\mathcal{A})/\mathcal{A}_\xi)$$

where the inverse limit is taken over submodules $\mathcal{A}_\xi \subset j_{x*} j_x^*(\mathcal{A})$ such that the quotient $j_{x*} j_x^*(\mathcal{A})/\mathcal{A}_\xi$ is supported at x and j_x is the inclusion the open set $X - x$.

2 Factorization

There is another equivalent description of chiral algebras in terms of factorization which given what we've been doing in the seminar might be more

familiar.

For the moment, let X be a topological space. We can then consider the Ran space of X defined as

$$\text{Ran}(X) = \{\text{nonempty finite subsets of } X\}$$

It is topologized so that the maps $X^n \rightarrow \text{Ran}(X)$ are continuous. There is a very important fact which will not be relevant for now, but is very important when dealing with homology of chiral algebras.

Theorem 5. *If X is connected, the topological space $\text{Ran}(X)$ is weakly contractible.*

We will be interested in doing algebraic geometry on $\text{Ran}(X)$ for X an algebraic curve. Unfortunately, it is not possible to define $\text{Ran}(X)$ as any kind of algebraic space but we will be able to make sense of quasi-coherent sheaves on $\text{Ran}(X)$. So let's return to X being an algebraic curve over \mathbb{C} .

Definition 6. *A quasi-coherent sheaf \mathcal{F} on $\text{Ran}(X)$ is a collection of quasi-coherent sheaves \mathcal{F}^I for each finite set I together with isomorphisms*

$$\nu^{(\pi)} : \Delta^{(J/I)*} \mathcal{F}_J \xrightarrow{\sim} \mathcal{F}_I$$

for every surjection $\pi : J \rightarrow I$, where $\Delta^{J/I} : X^I \rightarrow X^J$ is the corresponding diagonal. We require that the $\nu^{(\pi)}$ be compatible with composition of surjections. Moreover, we demand that the $\mathcal{F}^{(I)}$ have no sections supported on the diagonals.

Remark 7. *Because of the condition requiring no sections supported on the diagonals, quasi-coherent sheaves on $\text{Ran}(X)$ do not form an abelian category.*

Definition 8. *A non-unital factorization algebra \mathcal{B} is a quasi-coherent sheaf on $\text{Ran}(X)$ along with isomorphisms*

$$c_\alpha : j_\alpha^*(\boxtimes \mathcal{B}^{(I_i)}) \xrightarrow{\sim} j_\alpha^*(\mathcal{B}^{(I)})$$

for a partition $\alpha : I = I_1 \sqcup \dots \sqcup I_n$ a partition of I and j_α is the inclusion of the open set

$$U = \{x_i \neq x_j \text{ if } i \text{ and } j \text{ are in different } I_j\}.$$

We require that the c_α be compatible with subpartitions and with the $\nu^{(\pi)}$.

Example 9. Let \mathcal{O} be the non-unital factorization algebra given by $\mathcal{O}^{(I)} = \mathcal{O}_{X^I}$. This is a factorizable algebra in the obvious way. It is the unit factorization algebra.

Definition 10. A (unital) factorization algebra \mathcal{B} is a non-unital factorization algebra equipped with a map of non-unital factorization algebras

$$\mathcal{O} \rightarrow \mathcal{B}$$

such that locally for every section $b \in \mathcal{B}^{(1)}$, $1 \boxtimes b \in j_*j^*(\mathcal{B}^{(1)} \boxtimes \mathcal{B}^{(1)})$ lies in $B^{(2)} \subset j_*j^*(B^{(1)} \boxtimes B^{(1)})$ and $\Delta^*(1 \boxtimes b) = b$.

Remark 11. In the definition of a factorization algebra, we required that the unit give a map

$$\mathcal{B}^{(1)} \boxtimes \mathcal{O}_X \rightarrow \mathcal{B}^{(2)}$$

compatible with restriction to the diagonal. In fact, we leave it to the reader to check that this implies that we have canonical maps

$$\mathcal{B}^{(I_1)} \boxtimes \mathcal{O}_{X^{I_2}} \rightarrow \mathcal{B}^{(I_1 \sqcup I_2)}$$

compatible with factorization and restrictions to the diagonals. This follows from the condition requiring no sections supported on the diagonals for quasi-coherent sheaves on $\text{Ran}(X)$. If we consider dg-factorization algebras, then we need to specify all these maps as part of the data of a unital dg-factorization algebra.

Theorem 12. There is an equivalence of categories

$$\{\text{factorization algebras}\} \xrightarrow{\sim} \{\text{chiral algebras}\}$$

given by

$$\mathcal{B} \mapsto \mathcal{B}^{(1)} \otimes \omega_X$$

Proof. Let \mathcal{B} be a factorization algebra. Let's show that each $\mathcal{B}^{(I)}$ has a canonical structure of a left D -module. Giving $\mathcal{B}^{(I)}$ such a structure is equivalent to giving an isomorphism between $\mathcal{B}^{(I)} \boxtimes \mathcal{O}_{X^I}$ and $\mathcal{O}_{X^I} \boxtimes \mathcal{B}^{(I)}$ on the formal completion of the diagonal in $X^I \times X^I$. The unit gives maps

$$\mathcal{B}^{(I)} \boxtimes \mathcal{O}_{X^I} \rightarrow \mathcal{B}^{(I \sqcup I)} \leftarrow \mathcal{O}_{X^I} \boxtimes \mathcal{B}^{(I)}$$

which are isomorphisms on the formal neighborhood of the diagonal X^I giving the canonical connection.

Now, let $\mathcal{A}^{(I)} = \mathcal{B}^{(I)} \otimes \omega_{X^I}$ be the corresponding right D -modules and let $\mathcal{A} = \mathcal{A}^{(1)}$. We then have the Cousin complex for $\mathcal{A}^{(2)}$:

$$0 \rightarrow \mathcal{A}^{(2)} \rightarrow j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A}) \rightarrow 0$$

which gives the chiral bracket. It is clearly antisymmetric and the unit axiom is satisfied. The Cousin complex for $\mathcal{A}^{(3)}$ with stratification given by the diagonals in X^3 gives the Jacobi identity. Thus, we have a functor from factorization algebras to chiral algebras.

Let us now construct the inverse functor. Let \mathcal{A} be a chiral algebra. On X^I consider the Chevalley-Cousin complex:

$$C_I^\bullet = \left(j_* j^* \left(\underbrace{\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A}}_{I\text{-times}} \right) \rightarrow \bigoplus_{\alpha \in \text{Part}_{|I|-1}(I)} \Delta_{\alpha!} (j_* j^* (\mathcal{A} \boxtimes \cdots \boxtimes \mathcal{A})) \rightarrow \cdots \Delta_!(\mathcal{A}) \right)$$

where $\text{Part}_n(I)$ is the set of partitions of I into n subsets and Δ_α for $\alpha \in \text{Part}_n(I)$ is the corresponding n dimensional diagonal. The terms with n copies of \mathcal{A} are in degree $-n$ and the differentials are given by the various chiral brackets. This complex is called the Chevalley-Cousin complex because it is the Chevalley complex from the point of view of the chiral algebra, and it is the Cousin complex (for the stratification given by the diagonals) from the point of view of the factorization algebra.

Evidently, these complexes are factorize and are compatible with restriction to diagonals, i.e. we have isomorphisms

$$\nu^{(\pi)} : \Delta^{(J/I)!} C_J^\bullet \xrightarrow{\sim} C_I^\bullet$$

for surjections $\pi : J \twoheadrightarrow I$ and

$$c_\alpha : j_\alpha^* (\boxtimes C_{I_i}^\bullet) \xrightarrow{\sim} j_\alpha^* (C_I^\bullet)$$

for partitions $I = I_1 \sqcup \cdots \sqcup I_n$.

We will show that $H^n(C_I^\bullet) = 0$ unless $n = -|I|$ by induction on $|I|$. For

$|I| = 1$, the complex is just given by \mathcal{A} and there is nothing to prove. Now, for a general I , consider a codimension one diagonal

$$i : X^{I'} \hookrightarrow X^I \hookleftarrow U : v$$

and a given decomposition $I = I' \sqcup [1]$, with U the complement of $X^{I'}$. The open set U is affine and we have a short exact sequence of complexes

$$0 \rightarrow i_!(C_{I'}^\bullet) \rightarrow C_I^\bullet \rightarrow v_*v^*(C_I^\bullet) \rightarrow 0.$$

By induction, $H^n(i_!(C_{I'}^\bullet)) = 0$ unless $n = -|I'| = -|I| + 1$ and by induction and factorization $H^n(v_*v^*(C_I^\bullet)) = 0$ unless $n = -|I|$ since U is a union of complements of diagonals. Thus we need to show that the map

$$H^{-|I|+1}(i_!(C_{I'}^\bullet)) \rightarrow H^{-|I|+1}(C_I^\bullet)$$

vanishes. Let

$$Z := H^{-|I|+1}(C_{I'}^\bullet) = \text{Ker}(C_{I'}^{-|I'|} \rightarrow C_{I'}^{-|I'|+1})$$

We have a canonical exact sequence with respect to the decomposition $X^I = X^{I'} \times X$

$$0 \rightarrow Z \boxtimes \omega_X \rightarrow v_*v^*(Z \boxtimes \omega_X) \rightarrow i_!(Z) \rightarrow 0.$$

Furthermore, we have a commutative diagram

$$\begin{array}{ccc} v_*v^*(Z \boxtimes \omega) & \longrightarrow & C_I^{-|I|} \\ \downarrow & & \downarrow d \\ i_!(Z) & \longrightarrow & C_I^{-|I|+1} \end{array}$$

where the top map is given by the unit. It follows that the map

$$H^{-|I|+1}(i_!(C_{I'}^\bullet)) \rightarrow H^{-|I|+1}(C_I^\bullet)$$

is indeed zero.

Now, let $\mathcal{B}^{(I)} := H^{-|I|}(C_I^\bullet) \otimes \omega_{X^I}^{-1}$. By above, it is a factorization algebra. Furthermore, it gives the inverse functor

$$\{\text{chiral algebras}\} \rightarrow \{\text{factorization algebras}\}.$$

□

The factorization perspective on chiral algebras is a very useful one. Suppose that \mathcal{B}_1 and \mathcal{B}_2 are factorization algebras. Then $\mathcal{B}^{(I)} = \mathcal{B}_1^{(I)} \otimes \mathcal{B}_2^{(I)}$ is also a factorization algebra in the obvious way. In this way, we can consider tensor products of chiral algebras.

Another important aspect of factorization algebra is that they admit a non-linear analogue of factorization spaces. This will allow us to construct chiral algebras from geometry.

Definition 13. *A factorization space \mathcal{G} is a collection of (ind-) schemes \mathcal{G}_I over X^I for each finite set I along with isomorphisms*

$$(\mathcal{G}_J)|_{X^I} \simeq \mathcal{G}_I$$

for every diagonal embedding $X^I \hookrightarrow X^J$ and for every partition $I = I_1 \sqcup \dots \sqcup I_n$ factorization isomorphisms

$$\mathcal{G}_I|_U \simeq (\times \mathcal{G}_{I_i})|_U$$

where U is the open set corresponding to the partition. We require that these isomorphisms be compatible in the usual way.

A factorization space is unital if in addition we have maps

$$X^{I_1} \times \mathcal{G}_{I_2} \rightarrow \mathcal{G}_{I_2 \sqcup I_1}$$

compatible with factorization and restrictions to diagonals.

We have already seen an example of a unital factorization space, namely the Beilinson-Drinfeld Grassmannian. Let G be an algebraic group. Recall that the Beilinson-Drinfeld Grassmannian $Gr_I(G) \rightarrow X^I$ is defined as the moduli space of the following triples (in what follows we omit reference to G in the notation)

$$Gr_I = \{\mathcal{P} \in Bun_G(X), (x_1, \dots, x_I) \in X^I, \phi : \mathcal{P}|_{X - \{(x_i)\}} \simeq \mathcal{P}^{triv}|_{X - \{(x_i)\}}\}$$

where $\mathcal{P}^{triv} \in Bun_G(X)$ is the trivial G -bundle.

Now, suppose we are given a unital factorization space \mathcal{G} and a "linearization functor", i.e. a rule for obtaining a sheaf on X^I from each \mathcal{G}_I which preserves factorization then we can obtain a factorization algebra. Examples

of such "linearization functors" are global sections and pushforward of some canonically defined sheaf.

In the case of the Beilinson-Drinfeld Grassmannian, consider for each I , \mathcal{S}_I the sheaf of D -module δ functions along the unit section $X^I \rightarrow Gr_I$. Now let V_I be the pushforward of \mathcal{S}_I to X^I as a quasi-coherent sheaf. These form a factorization algebra with fibers given by the vacuum module for every $x \in X$. We could also consider δ functions as a twisted D -module and in the same way obtain a factorization algebra with fibers given by the vacuum module at the corresponding level.

In addition to describing chiral algebras in factorization terms, we can also describe modules.

Definition 14. *Let \mathcal{B} be a factorization algebra. A factorization \mathcal{B} module \mathcal{M} is a collection of D -modules $M^{\tilde{I}}$ on $X^{\tilde{I}}$ for each finite set I , where $\tilde{I} = I \sqcup \{\star\}$. For every surjection $\pi : \tilde{J} \rightarrow \tilde{I}$ such that $\pi(\star) = \star$, we have the corresponding diagonal $X^{\tilde{I}} \hookrightarrow X^{\tilde{J}}$ and we are given isomorphisms*

$$\mathcal{M}^{(\tilde{J})}|_{X^{\tilde{I}}} \simeq \mathcal{M}^{(\tilde{I})}$$

and for every partition $I = I_1 \sqcup \dots \sqcup I_n$, we are given factorization isomorphisms

$$\mathcal{M}^{\tilde{I}}|_U \simeq \left(\bigotimes_{0 < i < n} \mathcal{B}^{I_i} \otimes \mathcal{M}^{\tilde{I}_n} \right)|_U$$

where $U \subset X^{\tilde{I}}$ is the open set corresponding to the partition $\tilde{I} = I_1 \sqcup \dots \sqcup I_{n-1} \sqcup \tilde{I}_n$. We require that the isomorphisms be mutually compatible.

Theorem 15. *Let \mathcal{A} be a chiral algebra and \mathcal{B} be the corresponding factorization algebra. There is an equivalence of categories*

$$\{\text{factorization } \mathcal{B}\text{-modules}\} \simeq \{\text{chiral } \mathcal{A}\text{-modules}\}$$

As a consequence, we can consider tensor products of chiral modules: let \mathcal{B}_1 and \mathcal{B}_2 be factorization algebras and \mathcal{M}_1 and \mathcal{M}_2 respective factorization modules. Then $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$ is a factorization module for the factorization algebra $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$.

3 Lie-* algebras

A Lie-* algebra L is a D -module on X along with a "Lie-*" bracket

$$\mu : L \boxtimes L \rightarrow \Delta_!(L)$$

satisfying the Jacobi identity (in the same way as in the definition of a chiral algebra).

Remark 16. *Suppose that a Lie-* algebra L is holonomic as a D -module. In this case, we have the functor Δ^* which is left adjoint to $\Delta_! = \Delta_*$. In this case, we can consider the *-tensor product $L \otimes^* L := \Delta^*(L \boxtimes L)$ and being a Lie-* algebra is equivalent to being a Lie algebra*

$$\mu : L \otimes^* L \rightarrow L$$

*in the tensor category of holonomic D -modules with the *-tensor product.*

An advantage of considering Lie-* algebras is that it is relatively easy to construct examples. Suppose \mathfrak{g} is a Lie algebra. Then $\mathfrak{g} \otimes D_X$ is a Lie-* algebra. In fact, we could implement this construction for any Lie algebra in the category of quasi-coherent sheaves on X by instead tensoring over \mathcal{O}_X .

Let \mathcal{A} be a chiral algebra. Then the composition

$$\mathcal{A} \boxtimes \mathcal{A} \rightarrow j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

makes \mathcal{A} into a Lie-* algebra. In fact this functor has a left adjoint.

Theorem 17. *The above functor*

$$\{\text{chiral algebras}\} \rightarrow \{\text{Lie-* algebras}\}$$

has a left adjoint $L \mapsto \mathcal{A}(L)$ called the "chiral envelope".

Proof (Sketch). The statement is local on the curve so we can assume without loss of generality that X is affine.

Given a Lie-* algebra, we will construct a factorization algebra using auxiliary Lie algebras. For a finite set I , consider the space $X^I \times X$ and let

p_i for $i = 1, 2$ be the two projection maps to X^I and X respectively. Let $j : U \hookrightarrow X^I \times X$ be the open subset given by

$$U = \{((x_i), x) \in X^I \times X \mid x_i \neq x \text{ for } i \in I\}.$$

Now, consider

$$\tilde{L}_0^{(I)} = p_{1*}p_2^*(L) \otimes \omega_{X^I}^{-1} \quad \text{and} \quad \tilde{L}^{(I)} = p_{1*}j_*j^*p_2^*(L) \otimes \omega_{X^I}^{-1}.$$

These are Lie algebras in the category of left D -modules on X^I . We have that the fibers of $\tilde{L}_0^{(I)}$ and $\tilde{L}^{(I)}$ are given by

$$\tilde{L}_0^{(I)}(x_1, \dots, x_I) = H_{dR}(X, L) \quad \text{and} \quad \tilde{L}^{(I)}(x_1, \dots, x_I) = H_{dR}(X - \{x_1, \dots, x_I\}, L).$$

By construction, we have that

$$\mathcal{B}(L)^{(I)} = U(\tilde{L}^{(I)})/U(\tilde{L}_0^{(I)}) = \text{Ind}_{\tilde{L}_0^{(I)}}^{\tilde{L}^{(I)}} \mathcal{O}_{X^I}$$

is a factorization algebra (here $U(L)$ denotes the universal enveloping algebra of the Lie algebra L). Let $\mathcal{A}(L)$ be the corresponding chiral algebra.

Note that we have an exact sequence of D -modules on X

$$0 \rightarrow \tilde{L}_0^{(1)} \rightarrow \tilde{L}^{(1)} \rightarrow L \otimes \omega_X^{-1} \rightarrow 0$$

which gives a map $L \rightarrow \mathcal{A}(L)$. The fact that it's a map of Lie-* algebras follows from a similar exact sequence on X^2 .

Now suppose \mathcal{A} is a chiral algebra and we have a map of Lie-* algebras $L \rightarrow \mathcal{A}$. Taking deRham cohomology of the action map

$$j_*j^*(L \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

along the first component makes $\mathcal{B}^{(1)} := \mathcal{A} \otimes \omega_X^{-1}$ into a Lie $\tilde{L}^{(1)}$ module. The unit section of $\mathcal{B}^{(1)}$ gives a map of left D -modules

$$U(\tilde{L}^{(1)}) \rightarrow \mathcal{B}^{(1)}.$$

Furthermore, the following diagram commutes

$$\begin{array}{ccc} \tilde{L}^{(1)} & \longrightarrow & \tilde{\mathcal{A}}^{(1)} \\ \downarrow & & \downarrow \\ L \otimes \omega_X^{-1} & \longrightarrow & \mathcal{A} \otimes \omega_X^{-1} \end{array}$$

where $\tilde{A}^{(1)} = p_{1*}j_*j^*p_2^*(A) \otimes \omega_{X^1}^{-1}$. It follows that $\tilde{L}_0^{(1)}$ kills the unit section in $\mathcal{B}^{(1)}$. This gives us the desired map $\mathcal{A}(L) \rightarrow \mathcal{A}$. A similar argument on X^2 shows that it's a map of chiral algebras. \square

Consider the case where $L = \mathfrak{g} \otimes D_X$. In this case, we have that the fibers $\mathcal{A}(L)_x$ at $x \in X$ are given by

$$\mathcal{A}(L)_x = \text{Ind}_{H_{dR}^0(X, \mathfrak{g} \otimes D_X)}^{H_{dR}^0(X-x, \mathfrak{g} \otimes D_X)} \mathbb{C}.$$

Since we have a Cartesian square

$$\begin{array}{ccc} H_{dR}^0(X, \mathfrak{g} \otimes D_X) & \longrightarrow & H_{dR}^0(X-x, \mathfrak{g} \otimes D_X) \\ \downarrow & & \downarrow \\ H_{dR}^0(D_x, \mathfrak{g} \otimes D_X) = \mathfrak{g}(\mathcal{O}_x) & \longrightarrow & H_{dR}^0(D_x^\circ, \mathfrak{g} \otimes D_X) = \mathfrak{g}(\mathcal{K}_x) \end{array}$$

it follows that

$$\mathcal{A}(L)_x = \text{Ind}_{\mathfrak{g}(\mathcal{O}_x)}^{\mathfrak{g}(\mathcal{K}_x)} \mathbb{C} = \text{Vac}_x.$$

Thus in this case the fibers are given by the vacuum module for \mathfrak{g} . In fact, this chiral algebra agrees with the one we constructed before using the Beilinson-Drinfeld Grassmannian.

For Lie- $*$ algebras, we can consider two types of modules:

Definition 18. *Let L be a Lie- $*$ algebra.*

- *A Lie- $*$ L module is a D -module \mathcal{M} on X along with an action map*

$$\rho : L \boxtimes \mathcal{M} \rightarrow \Delta_!(\mathcal{M})$$

satisfying the Lie action identity.

- *A chiral L module is a D -module \mathcal{M} on X along with a chiral action map*

$$\rho : j_*j^*(L \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

satisfying the following condition. For $j' : U \rightarrow X^2 \times X$ the complement of the diagonals $\{x_1 = x\}$ and $\{x_2 = x\}$, we have (similarly to the case of chiral modules over a chiral algebra) the maps

$$\rho_{1(23)} : j'_*j'^*(L \boxtimes L \boxtimes \mathcal{M}) \xrightarrow{id \boxtimes \rho} \Delta_{23!}j'_*(L \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

with $\rho_{(12)3}$ and $\rho_{2(13)}$ defined similarly. We demand that

$$\rho_{(12)3} = \rho_{1(23)} - \rho_{2(13)}.$$

We have the following important fact about chiral modules over a Lie-* algebra.

Theorem 19. *Let L be a Lie-* algebra. Then there is an equivalence of categories*

$$\{\text{chiral } L\text{-modules}\} \simeq \{\text{chiral } \mathcal{A}(L)\text{-modules}\}.$$

Now, suppose \mathcal{M} is a chiral L -module. We have a forgetful functor to Lie-* modules given by the composition

$$L \boxtimes \mathcal{M} \rightarrow j_* j^*(L \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M}).$$

This functor has a left adjoint

$$\text{Ind} : \{\text{Lie-* } L\text{-modules}\} \rightarrow \{\text{chiral } L\text{-modules}\}$$

given as follows. Let M be a Lie-* L -module. Taking deRham cohomology of the action map

$$L \boxtimes M \rightarrow \Delta_!(M)$$

along the first component, we see that $M' := M \otimes \omega_X^{-1}$ is a Lie $\tilde{L}_0^{(1)}$ -module. Now, let

$$\text{Ind}'(M)^{(\tilde{I})} = \text{Ind}_{\tilde{L}_0^{(\tilde{I})}}^{\tilde{L}_0^{(\tilde{I})}} p^*(M')$$

for a finite set I where $p : X^{\tilde{I}} \rightarrow X$ is the projection to the last component. We have that

$$\text{Ind}'(M) := \{\text{Ind}'(M)^{\tilde{I}}\}$$

is a factorization module for the factorization algebra corresponding to $\mathcal{A}(L)$. We then define $\text{Ind}(M)$ to be the corresponding chiral L -module.

4 Commutative Chiral Algebras

Let \mathcal{A} be a chiral algebra. We say that \mathcal{A} is commutative if the composition

$$\mathcal{A} \boxtimes \mathcal{A} \rightarrow j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

vanishes.

We have the following characterization of commutative chiral algebras.

Theorem 20. *There is an equivalence of categories*

$$\{\text{commutative chiral algebras}\} \simeq \{\text{commutative left } D_X\text{-algebras}\}$$

given by

$$\mathcal{A} \mapsto \mathcal{A} \otimes \omega_X^{-1}.$$

Proof. Recall that we have a canonical exact sequence

$$0 \rightarrow \mathcal{A} \boxtimes \mathcal{A} \rightarrow j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A} \otimes^! \mathcal{A}) \rightarrow 0.$$

It follows that for a commutative chiral algebra, we have a map

$$m : \mathcal{A} \otimes^! \mathcal{A} \rightarrow \mathcal{A}.$$

In fact, as we'll see m makes $\mathcal{A}^l := \mathcal{A} \otimes \omega_X^{-1}$ a commutative algebra in the category of left D -modules. Commutativity of m (on \mathcal{A}^l) clearly follows from anti-commutativity of the chiral bracket. Furthermore, the unit gives the unit section

$$\eta : \mathcal{O}_X \rightarrow \mathcal{A}^l.$$

Note that we can factor the chiral bracket as

$$\begin{aligned} j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) &= (\mathcal{A}^l \boxtimes \mathcal{A}^l) \otimes j_* j^*(\omega_X \boxtimes \omega_X) \rightarrow (\mathcal{A}^l \boxtimes \mathcal{A}^l) \otimes \Delta_!(\omega_X) = \\ &= \Delta_!(\mathcal{A}^l \otimes \mathcal{A}^l) \otimes \Delta_!(\omega_X) \rightarrow \Delta_!(\mathcal{A}^l) \otimes \Delta_!(\omega_X) = \Delta_!(\mathcal{A}). \end{aligned}$$

Thus for a section

$$((a \boxtimes b) \otimes s_2) \in \mathcal{A}^l \boxtimes \mathcal{A}^l \otimes j_* j^*(\omega_X \boxtimes \omega_X)$$

we have

$$\mu((a \boxtimes b) \otimes s_2) = (a \cdot b) \otimes \mu^\omega(s_2)$$

where $(a \cdot b) := m(a, b)$ and μ^ω is the chiral bracket for ω . In these terms, the Jacobi identity for \mathcal{A} becomes

$$(a \cdot (b \cdot c)) \otimes \mu_{1(23)}^\omega(s_3) = ((a \cdot b) \cdot c) \otimes \mu_{(12)3}^\omega(s_3) + (b \cdot (a \cdot c)) \otimes \mu_{2(13)}^\omega(s_3)$$

for a section

$$(a \boxtimes b \boxtimes c) \otimes s_3 \in (\mathcal{A}^l \boxtimes \mathcal{A}^l \boxtimes \mathcal{A}^l) \otimes j_* j^*(\omega_X \boxtimes \omega_X \boxtimes \omega_X).$$

From the Jacobi identity for ω , we deduce that all three terms on the left side of the tensors are equal. Thus, the product m on \mathcal{A}^l is associative.

Now, suppose that \mathcal{B} is a commutative left D_X -algebra. We can define a chiral bracket on $\mathcal{A} = \mathcal{B} \otimes \omega_X$ by

$$\mu : j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A} \otimes^! \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

where the first map is the canonical map and the second is the one given by the multiplication map on \mathcal{B} . By a similar argument as above, this makes \mathcal{A} into a chiral algebra. \square

Now, suppose that \mathcal{B} is a commutative left D_X algebra. We will describe \mathcal{B} modules in terms of the corresponding chiral algebra.

Definition 21. *Let \mathcal{A} be a chiral algebra. A commutative \mathcal{A} -module is a chiral \mathcal{A} -module \mathcal{M} such that the composition*

$$\mathcal{A} \boxtimes \mathcal{M} \rightarrow j_*j^*(\mathcal{A} \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

vanishes.

Suppose \mathcal{B} is a commutative left D_X algebra and \mathcal{A} the corresponding commutative chiral algebra. We then have an equivalence of categories

$$\{(D_X\text{-}\mathcal{B})\text{-modules}\} \rightarrow \{\text{commutative } \mathcal{A}\text{-modules}\}$$

given by

$$M \mapsto \mathcal{M} := M \otimes \omega_X$$

with the chiral action map given by the composition

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{A} \otimes^! \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

where the last map comes from the action map $\mathcal{B} \otimes M \rightarrow M$.

Now, for a commutative D_X algebra \mathcal{B} , we can describe the corresponding chiral algebra in factorization terms as follows. Let $Z = \text{Spec}(\mathcal{B}^l)$. Then Z is a D_X -scheme. In this situation, we can construct a counital factorization space $Z_I \rightarrow X^I$ by considering multijets. An S point of Z_I is given by a map $\phi : S \rightarrow X^I$ along with a horizontal section $\hat{X}_S \rightarrow Z$, where \hat{X}_S is the

completion of $X \times S$ along the subscheme given by the union of the graphs of ϕ . When Z is an affine D_X -scheme, each Z_I is an affine D_{X_I} -scheme.

Let $(\mathcal{B})^I$ be the left D_{X_I} -module of global sections of Z_I . As quasi-coherent sheaves, these form a factorization algebra. Let \mathcal{A} be the corresponding chiral algebra.

Claim 22. *The chiral algebra \mathcal{A} is commutative and the chiral bracket is given by the multiplication map on \mathcal{B} .*

To prove the claim, we will make use of the Eckmann-Hilton argument. We will write down a formal algebraic proof, but here's the basic idea of the proof, which is extremely simple. Suppose you wanted to explain to someone that addition was commutative. For instance, say you wanted to show that $7 + 15 = 15 + 7$. Well, $7 + 15$ is the number of marbles you have if you have two piles of marbles - 7 marbles in the left pile and 15 marbles in the right pile. Well, $15 + 7$ is what you would get by moving one pile of marbles past the other. This is essentially the argument. A slightly more sophisticated version of this argument says that a monoid in the category of monoids is a commutative monoid. In this case, one can denote one multiplication as vertical composition and the other as horizontal and essentially carry out the same argument as with the marbles, moving one multiplication past the other. This is the Eckmann-Hilton argument and it shows for instance that higher homotopy groups of a topological space are commutative. In our context, we have the following theorem.

Theorem 23. *Let \mathcal{A} be a chiral algebra with a compatible unital binary operation*

$$m : \mathcal{A} \otimes^! \mathcal{A} \rightarrow \mathcal{A}$$

Then \mathcal{A} is a commutative chiral algebra, m makes \mathcal{A}^l into a commutative algebra and the chiral bracket on \mathcal{A} factors through m .

Proof. Clearly, it suffices to show that the chiral bracket factors through m . Compatibility of chiral bracket with the binary operation means that the following diagram is commutative (i.e. m is a map of chiral algebras):

$$\begin{array}{ccc} j_*j^*((\mathcal{A} \otimes^! \mathcal{A}) \boxtimes (\mathcal{A} \otimes^! \mathcal{A})) & \longrightarrow & \Delta_!(\mathcal{A} \otimes^! \mathcal{A}) \\ \downarrow & & \downarrow \\ j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) & \longrightarrow & \Delta_!(\mathcal{A}) \end{array}$$

We have two unit maps

$$\alpha, \beta : \omega_X \rightarrow \mathcal{A}$$

for the chiral bracket and the binary operation respectively. Let's show that these agree. From the commutativity of the above diagram and the unit axioms, we have the following commutative diagram

$$\begin{array}{ccc} j_*j^*((\omega_\alpha \otimes^! \omega_\beta) \boxtimes (\omega_\beta \otimes^! \omega_\alpha)) & \twoheadrightarrow & \Delta_!(\omega_\beta \otimes^! \omega_\beta) = \Delta_!(\omega_\beta) \\ \downarrow & & \searrow \\ j_*j^*(\omega_\alpha \boxtimes \omega_\alpha) & \twoheadrightarrow & \Delta_!(\omega_\alpha) \end{array} \quad \begin{array}{c} \nearrow \\ \nearrow \end{array} \Delta_!(\mathcal{A})$$

It follows that the two units agree. Now, we can construct a section

$$s : j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow j_*j^*((\mathcal{A} \otimes^! \mathcal{A}) \boxtimes (\mathcal{A} \otimes^! \mathcal{A}))$$

given by

$$s : j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) = j_*j^*((\mathcal{A} \otimes^! \omega) \boxtimes (\omega \otimes^! \mathcal{A})) \rightarrow j_*j^*((\mathcal{A} \otimes^! \mathcal{A}) \boxtimes (\mathcal{A} \otimes^! \mathcal{A}))$$

It follows that we the chiral bracket factors as

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A} \otimes^! \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

where the last map is given by $\Delta_!(m)$. □

Remark 24. *We can actually strengthen the above theorem slightly by not imposing the condition that \mathcal{A} is a chiral algebra. All that is necessary is that there is a unital chiral operation*

$$\mu : j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A})$$

and a unital binary operation

$$m : \mathcal{A} \otimes^! \mathcal{A} \rightarrow \mathcal{A}$$

which are compatible. In this case, μ makes \mathcal{A} into a commutative chiral algebra, m makes \mathcal{A}^l into a commutative left D_X -algebra and one determines the other. We leave the proof as an exercise for the reader.

Let us now return to considering multijets. In this case, \mathcal{B} is a commutative left D_X algebra, and \mathcal{A} is the chiral algebra corresponding to the factorization space given by multijets of $\text{Spec}(\mathcal{B})$. As a right D_X -module, we clearly have $\mathcal{A} = \mathcal{B} \otimes \omega_X$, and the multiplication on \mathcal{B} is compatible with the chiral bracket. It follows by Eckmann-Hilton that \mathcal{A} is a commutative chiral algebra and the chiral bracket is given by the multiplication map.

5 Factorization Modules on Higher Powers

Let \mathcal{A} be a chiral algebra. Thus far, we have considered chiral modules supported at a point and chiral modules on the curve. We can also define modules on higher powers of X . We do so in factorization terms. Let \mathcal{B} be the corresponding factorization algebra.

Definition 25. *Let I_0 be a finite set. A factorization \mathcal{B} module \mathcal{M} on X^{I_0} is a collection $D_{X^{\tilde{I}}}$ -modules $\{\mathcal{M}^{\tilde{I}}\}$ for finite sets I with $\tilde{I} := I \sqcup I_0$. For every surjection $\pi : \tilde{J} \rightarrow \tilde{I}$ such that $\pi|_{I_0} = id$, we have the corresponding diagonal $X^{\tilde{I}} \hookrightarrow X^{\tilde{J}}$ are we are given isomorphisms*

$$\mathcal{M}^{(\tilde{J})}|_{X^{\tilde{I}}} \simeq \mathcal{M}^{(\tilde{I})}$$

and for every partition $I = I_1 \sqcup \dots \sqcup I_n$, we are given factorization isomorphisms

$$\mathcal{M}^{\tilde{I}}|_U \simeq \left(\bigotimes_{0 < i < n} \mathcal{B}^{(I_i)} \otimes \mathcal{M}^{\tilde{I}_n} \right)|_U$$

where $U \subset X^{\tilde{I}}$ is the open set corresponding to the partition $\tilde{I} = I_1 \sqcup \dots \sqcup \tilde{I}_n$. We require that the isomorphisms be mutually compatible.

One can also give a definition of factorization modules on X^{I_0} in terms of chiral operations similarly to the definition of chiral modules on X . In fact as in the case of chiral modules on X , we have that the forgetful functor

$$\{\text{factorization } \mathcal{B}\text{-modules on } X^{I_0}\} \rightarrow \{D_{X^{I_0}}\text{-modules}\}$$

is faithful. Furthermore, the category of factorization modules on X^{I_0} is an abelian category.

Example 26. 1. For any I_0 , $\mathcal{B}^{(I_0)}$ is a factorization module on X^{I_0} .

2. Suppose \mathcal{M} is a module on X^{I_0} , and for a finite set $J_0 = J \sqcup I_0$, we have that $\mathcal{N} = \mathcal{M}^{(J_0)}$ is a factorization module on X^{J_0} .

3. Suppose \mathcal{M} is a factorization module on X^{I_0} then for any surjection $\pi : J_0 \rightarrow I_0$, we have the corresponding diagonal $i : X^{I_0} \hookrightarrow X^{J_0}$. We then have $i_*(\mathcal{M})$ is a factorization module on X^{J_0} . Similarly, if \mathcal{N} is a factorization module on X^{J_0} , $i^*(\mathcal{N})$ is a factorization module on X^{I_0} .

4. Suppose \mathcal{M} is a factorization module on X^{I_0} and \mathcal{N} is a factorization module on X^{J_0} , then $j_*j^*(\mathcal{M} \boxtimes \mathcal{N})$ is a factorization module on $X^{I_0 \sqcup J_0}$, where

$$j : U = \{(x_i), (y_j) \mid x_i \neq y_j\} \hookrightarrow X^{I_0} \times X^{J_0}.$$

The last example above allows us to define a pseudo-tensor structure on the category of chiral \mathcal{A} -modules on X by setting

$$\text{Hom}(\{M_1, M_2\}, N) = \text{Hom}(j_*j^*(M_1 \boxtimes M_2), \Delta_*(N))$$

for M_1, M_2, N factorization \mathcal{A} modules on X , where the Hom on the right-hand side is in the category of factorization \mathcal{A} -modules on X^2 .

6 Chiral Algebra of Endomorphisms

Let \mathcal{A} be a chiral algebra and \mathcal{B} the corresponding factorization algebra. We have seen that each $\mathcal{B}^{(I)}$ is a factorization \mathcal{B} -module on X^I . Let

$$R\text{End}(\mathcal{B})^{(I)} = R\text{Hom}(\mathcal{B}^{(I)}, \mathcal{B}^{(I)})$$

where $R\text{Hom}$ is in the derived category of factorization \mathcal{B} -modules on X^I . Let

$$R\text{End}(\mathcal{A})^{(I)} = R\text{End}(\mathcal{B})^{(I)} \otimes \omega_{X^I}$$

be the corresponding right D -modules.

We have that $R\text{End}(\mathcal{B})$ is a dg-factorization algebra. Namely, for a diagonal $\Delta : X^I \hookrightarrow X^J$, we have

$$L\Delta^*(R\text{End}(\mathcal{B})^{(J)}) \simeq R\text{End}(\mathcal{B})^{(I)}$$

and for a partition $I = I_1 \sqcup \dots \sqcup I_n$ and $j : U \hookrightarrow X^I$ the corresponding open set

$$j^*(R\text{End}(\mathcal{B})^{(I)}) \simeq j^*(\boxtimes R\text{End}(\mathcal{B})^{(I_i)}).$$

Suppose that we are given a decomposition $I = I_0 \sqcup J$. For the factorization \mathcal{B} module on X^{I_0} $\mathcal{M} = \mathcal{B}^{I_0}$, we have that $\mathcal{M}^I = \mathcal{B}^I$. It follows that we have maps

$$p^*(R\text{End}(\mathcal{B})^{I_0}) = R\text{End}(\mathcal{B})^{(I_0)} \boxtimes \mathcal{O}_{X^J} \rightarrow R\text{End}(\mathcal{B})^{(I)}$$

which makes $R\text{End}(\mathcal{B})$ into a unital dg-factorization algebra. It is coconnective, i.e. it has no cohomology in negative degrees.

Lemma 27. *Let \mathcal{B} be a coconnective unital dg-factorization algebra. Then $(\mathcal{B}_0)^{(I)} = H^0(\mathcal{B}^{(I)})$ is a unital factorization algebra.*

Proof. Let $\mathcal{A}^{(I)} = \mathcal{B}^{(I)} \otimes \omega_{X^I}$ and $\mathcal{A}_0^{(I)} = \mathcal{B}_0^{(I)} \otimes \omega_{X^I}$ be the corresponding right D -module. The Cousin complex for $\mathcal{A}^{(2)}$ gives the triangle

$$\mathcal{A}^{(2)} \rightarrow j_*j^*(\mathcal{A}^{(2)}) \rightarrow \Delta_!(\mathcal{A}^{(1)}).$$

The long exact sequence in cohomology gives the exact sequence

$$0 \rightarrow \mathcal{A}_0^{(2)} \rightarrow j_*j^*(\mathcal{A}_0^{(2)}) \rightarrow \Delta_!(\mathcal{A}_0^{(1)}).$$

The unit gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A}_0^{(2)} & \longrightarrow & j_*j^*(\mathcal{A}_0^{(2)}) & \longrightarrow & \Delta_!(\mathcal{A}_0^{(1)}) \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & \mathcal{A}_0^{(1)} \boxtimes \omega_X & \longrightarrow & j_*j^*(\mathcal{A}_0^{(1)} \boxtimes \omega_X) & \longrightarrow & \Delta_!(\mathcal{A}_0^{(1)}) \longrightarrow 0 \end{array}$$

It follows that $j_*j^*(\mathcal{A}_0^{(2)}) = j_*j^*(\mathcal{A}_0^{(1)} \boxtimes \mathcal{A}_0^{(1)}) \rightarrow \Delta_!(\mathcal{A}_0^{(1)})$ is surjective. Similar considerations on higher powers of X shows that $\mathcal{A}_0^{(1)}$ is a chiral algebra with \mathcal{B}_0 the corresponding factorization algebra. \square

For a chiral algebra \mathcal{A} with corresponding factorization algebra \mathcal{B} , let $End(\mathcal{A})$ be the chiral algebra corresponding to the factorization algebra given by $H^0(REnd(\mathcal{B}))$. Composition of morphisms gives us an algebra map

$$End(\mathcal{A}) \otimes^! End(\mathcal{A}) \rightarrow End(\mathcal{A})$$

which is compatible with the chiral algebra structure. It follows that $End(\mathcal{A})$ is a commutative chiral algebra.

Let $\mathcal{A}_{\mathfrak{g}}$ be the chiral envelope of the Lie-* algebra $D_X \otimes \mathfrak{g}$ for a Lie algebra \mathfrak{g} . We want to show that the algebra of endomorphisms of the vacuum module of $\mathcal{A}_{\mathfrak{g}}$ supported at a point is commutative. Even though the chiral algebra $End(\mathcal{A})$ is commutative for any chiral algebra \mathcal{A} , the corresponding statement for the vacuum module supported at a point is not necessarily true. However, it is true in the case of vertex operator algebras.

Definition 28. A vertex operator algebra V is an assignment $X \mapsto V_X$ of a chiral algebra V_X on every smooth curve X , along with compatible isomorphisms

$$\phi^*(V_Y) \simeq V_X$$

for etale maps $\phi : X \rightarrow Y$.

Example 29. The chiral algebra \mathcal{A}_g is defined for any smooth curve X and these form a vertex operator algebra.

Now, let's show that in the case of a vertex operator algebra V , the algebra of endomorphisms of the vacuum module is commutative. Since the statement is etale local, we can restrict ourselves without loss of generality to the case that $X = \mathbb{A}^1$. Since V is a vertex operator algebra, we have that $\mathcal{A} := V_{\mathbb{A}^1}$ is a translation equivariant D -module on \mathbb{A}^1 . Let $i : \{0\} \rightarrow \mathbb{A}^1$ be the inclusion map and

$$Vac_{\mathcal{A}} := i_! i^!(\mathcal{A})[1]$$

be the vacuum module supported at $\{0\}$. We have that the algebra of endomorphisms of $Vac_{\mathcal{A}}$ is given by

$$End(Vac_{\mathcal{A}}) = End(H^0(Ri^!(\mathcal{A})[1])) = H^0(REnd(Ri^!(\mathcal{A})[1])) = H^1(REnd(Ri^!(\mathcal{A}))).$$

The Grothendieck spectral sequence gives the exact sequence

$$0 \rightarrow H^1(Ri^!(REnd(\mathcal{A}))) \rightarrow H^1(REnd(i^!(\mathcal{A}))) \rightarrow i^!(H^1(REnd(\mathcal{A}))).$$

Since \mathcal{A} is translation equivariant so is $H^1(REnd(\mathcal{A}))$. In particular, $H^1(REnd(\mathcal{A}))$ is flat. It follows that $i^!(H^1(REnd(\mathcal{A}))) = 0$ and therefore

$$H^1(Ri^!(REnd(\mathcal{A}))) \simeq H^1(REnd(i^!(\mathcal{A}))) \simeq End(Vac_{\mathcal{A}}).$$

By commutativity of $End(\mathcal{A})$, the algebra $H^1(Ri^!(REnd(\mathcal{A})))$ is commutative and therefore so is $End(Vac_{\mathcal{A}})$.