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Nearby Cycles for Local Models of Some Shimura Varieties

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Abstract. Kottwitz conjectured a formula for the (semi-simple) trace of Frobenius on the nearby cycles for the local model of a Shimura variety with Iwahori-type level structure. In this paper, we prove his conjecture in the linear and symplectic cases by adapting an argument of Gaitsgory, who proved an analogous theorem in the equal characteristic case.

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1. Introduction

For certain classical groups G and certain minuscule coweights μ of G, M. Rapoport and Th. Zink have constructed a projective scheme $M(G,\mu)$ over $\mathbb{Z}_p$ that is a local model for singularities at p of some Shimura variety with level structure of Iwahori type at p. Locally for the étale of topology, $M(G,\mu)$ is isomorphic to a natural $\mathbb{Z}_p$-model $\mathcal{M}(G,\mu)$ of the Shimura variety.

The semi-simple trace of the Frobenius endomorphism on the nearby cycles of $\mathcal{M}(G,\mu)$ plays an important role in the computation of the local factor at p of the semi-simple Hasse–Weil zeta function of the Shimura variety, see [17]. We can recover the semi-simple trace of Frobenius on the nearby cycles of $\mathcal{M}(G,\mu)$ from that of the local model $M(G,\mu)$, see loc. cit. Thus the problem to calculate the function.

$$x \in M(G,\mu)(\mathbb{F}_q) \mapsto \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi(\bar{\mathbb{Q}}_\ell)_x)$$

comes naturally. R. Kottwitz has conjectured an explicit formula for this function.

To state this conjecture, we note that the set of $\mathbb{F}_q$-points of $M(G,\mu)$ can be naturally embedded as a finite set of Iwahori-orbits in the affine flag variety of $G(\mathbb{F}_q((t)))$

$$M(G,\mu)(\mathbb{F}_q) \subset G(\mathbb{F}_q((t)))/I$$

where I is the standard Iwahori subgroup of $G(\mathbb{F}_q((t)))$.

CONJECTURE (Kottwitz). *For all* $x \in M(G, \mu)(\mathbb{F}_q)$,

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi(\bar{\mathbb{Q}}_\ell)_x) = q^{\langle\rho,\mu\rangle} z_\mu(x).$$

Here $q^{\langle\rho,\mu\rangle} z_\mu(x)$ is the unique function in the center of the Iwahori–Hecke algebra of I-bi-invariant functions with compact support in $G(\mathbb{F}_q((t)))$, characterized by

$$q^{\langle\rho,\mu\rangle} z_\mu(x) * \mathbb{I}_K = \mathbb{I}_{K\mu K}.$$

Here K denotes the maximal compact subgroup $G(\mathbb{F}_q[[t]])$ and $\mathbb{I}_{K\mu K}$ denotes the characteristic function of the double-coset corresponding to a coweight μ.

Kottwitz' conjecture was first proved for the local model of a special type of Shimura variety with Iwahori type reduction at p attached to the group GL(d) and minuscule coweight $(1, 0^{d-1})$ (the 'Drinfeld case') in [9]. The method of that paper was one of direct computation: Rapoport had computed the function $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi(\bar{\mathbb{Q}}_\ell)_x)$ for the Drinfeld case (see [17]), and so the result followed from a comparison with an explicit formula for the Bernstein function $z_{(1,0^{d-1})}$. More generally, the explicit formula for z_μ in [9] is valid for any minuscule coweight μ of any quasi-split p-adic group. Making use of this formula, U. Görtz verified Kottwitz' conjecture for a similar Iwahori-type Shimura variety attached to $G = \mathrm{GL}(4)$ and $\mu = (1, 1, 0, 0)$, by computing the function $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi(\bar{\mathbb{Q}}_\ell)_x)$ for x ranging over all 33 strata of the corresponding local model $M(G, \mu)$.

Shortly thereafter, A. Beilinson and D. Gaitsgory were motivated by Kottwitz' conjecture to attempt to produce all elements in the center of the Iwahori–Hecke algebra geometrically, via a nearby cycle construction. For this they used Beilinson's deformation of the affine Grassmannian: a space over a curve X whose fiber over a fixed point $x \in X$ is the affine flag variety of the group G, and whose fiber over every other point of X is the affine Grassmannian of G. In [5] Gaitsgory proved a key commutativity result (similar to our Proposition 21) which is valid for any split group G and any dominant coweight, in the function field setting. His result also implies that the semi-simple trace of Frobenius on nearby cycles (of a K-equivariant perverse sheaf on the affine Grassmannian) corresponds to a function in the center of the Iwahori–Hecke algebra of G.

The purpose of this article is to give a proof of Kottwitz' conjecture for the cases $G = \mathrm{GL}(d)$ and $G = \mathrm{GSp}(2d)$. In fact we prove a stronger result (Theorem 11) which applies to arbitrary coweights, and which was also conjectured by Kottwitz (although only the case of minuscule coweights seems to be directly related to Shimura varieties).

MAIN THEOREM. *Let G be either* GL(d) *or* GSp($2d$). *Then for any dominant coweight μ of G, we have*

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi^M(\mathcal{A}_{\mu,\eta})) = (-1)^{2\langle\rho,\mu\rangle} \sum_{\lambda \leqslant \mu} m_\mu(\lambda) z_\lambda.$$

Here M is a member of an increasing family of schemes $M_{n\pm}$ which contains the local models of Rapoport-Zink; the generic fiber of M can be embedded in the affine

Grassmannian of G, and $\mathcal{A}_{\mu,\eta}$ denotes the K-equivariant intersection complex corresponding to μ. The special fiber of M embeds in the affine flag variety of $G(\bar{\mathbb{F}}_q((t)))$ so we can think of the semi-simple trace of Frobenius on nearby cycles as a function in the Iwahori–Hecke algebra of G.

The crucial step in the proof of the theorem is to show that the function $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi^M(\mathcal{A}_{\mu,\eta}))$ is in the center of the Iwahori–Hecke algebra. The basic strategy to prove this is

(1) give a geometric construction of convolution of sheaves which corresponds to the usual product in the Hecke algebra,
(2) show that convolution commutes with the nearby cycle functor,
(3) show that on the generic fiber, convolution of appropriate sheaves is commutative.

While the strategy of proof is similar to that of Beilinson and Gaitsgory, in order to get a statement which is valid over all local non-Archimedean fields we use a somewhat different model, based on spaces of lattices, in the construction of the schemes $M_{n\pm}$ (we have not determined the precise relation between our model and that of Beilinson and Gaitsgory). This is necessary to compensate for the lack of an adequate notion of affine Grassmannian over p-adic fields. The union of the schemes $M_{n\pm}$ can be thought of as a p-adic analogue of Beilinson's deformation of the affine Grassmannian.

2. Rapoport-Zink Local Models

2.1. SOME DEFINITIONS IN THE LINEAR CASE

Let F be a local non-Archimedean field. Let $\mathcal{O}$ denote the ring of integers of F and let $k = \mathbb{F}_q$ denote the residue field of $\mathcal{O}$. We choose a uniformizer ϖ of $\mathcal{O}$. We denote by η the generic point of $S = \mathrm{Spec}(\mathcal{O})$ and by s its closed point.

For $G = \mathrm{GL}(d)$ and for μ the minuscule coweight

$$(\underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{d-r})$$

with $1 \leqslant r \leqslant d-1$, the local model M_μ represents the functor which associates to each $\mathcal{O}$-algebra R the set of $L_\bullet = (L_0, \ldots, L_{d-1})$ where $L_0, \ldots, L_{d-1}$ are R-submodules of R^d satisfying the following properties

- $L_0, \ldots, L_{d-1}$ are locally direct factors of corank r in R^d,
- $\alpha'(L_0) \subset L_1, \alpha'(L_1) \subset L_2, \ldots, \alpha'(L_{d-1}) \subset L_0$ where α is the matrix

$$\alpha' = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \varpi & & & 0 \end{pmatrix}$$

The projective S-scheme M_μ is a local model for singularities of p of some Shimura variety for unitary group with level structure of Iwahori type at p (see [17, 18]).

Following a suggestion of G. Laumon, we introduce a new variable t and rewrite the moduli problem of M_μ as follows. Let $M_\mu(R)$ be the set of $L_\bullet = (L_0, \ldots, L_{d-1})$ where $L_0, \ldots, L_{d-1}$ are $R[t]$-submodules of $R[t]^d/tR[t]^d$ satisfying the following properties

- as R-modules, $L_0, \ldots, L_{d-1}$ are locally direct factors of corank r in $R[t]^d/tR[t]^d$,
- $\alpha(L_0) \subset L_1, \alpha(L_1) \subset L_2, \ldots, \alpha(L_{d-1}) \subset L_0$ where α is the matrix

$$\alpha = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ t+\varpi & & & 0 \end{pmatrix}$$

Obviously, these two descriptions are equivalent because t acts as 0 on the quotient $R[t]^d/tR[t]^d$. Nonetheless, the latter description indicates how to construct larger S-schemes M_μ, where μ runs over a certain cofinal family of dominant (nonminuscule) coweights.

Let $n_- \leqslant 0 < n_+$ be two integers.

DEFINITION 1. Let $M_{r,n\pm}$ be the functor which associates each $\mathcal{O}$-algebra R the set of $L_\bullet = (L_o, \ldots, L_{d-1})$ where $L_0, \ldots, L_{d-1}$ are $R[t]$-submodules of

$$t^{n_-}R[t]^d/t^{n_+}R[t]^d$$

satisfying the following properties:

- as R-modules, $L_0, \ldots, L_{d-1}$ are locally direct factors rank $n_+d - r$ in $t^{n_+}R[t]^d/t^{n_+}R[t]^d$,
- $\alpha(L_0) \subset L_1, \alpha(L_1) \subset L_2, \ldots, \alpha(L_{d-1}) \subset L_0$.

This functor is obviously represented by a closed sub-scheme in a product of Grassmannians. In particular, $M_{r,n\pm}$ is projective over S.

In some cases, it is more convenient to adopt the following equivalent description of the functor $M_{r,n\pm}$. Let us consider α as an element of the group

$$\alpha \in \mathrm{GL}(d, \mathcal{O}[t, t^{-1}, (t+\varpi)^{-1}]).$$

Let $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_d$ be the fixed $\mathcal{O}[t]$-submodules of $\mathcal{O}[t, t^{-1}, (t+\varpi)^{-1}]^d$ defined by $\mathcal{V}_i = \alpha^{-i}\mathcal{O}[t]^d$. In particular, we have $\mathcal{V}_d = (t+\varpi)^{-1}\mathcal{V}_0$. Denote by $\mathcal{V}_{i,R}$ the tensor $\mathcal{V}_i \otimes_{\mathcal{O}} R$ for any $\mathcal{O}$-algebra R.

DEFINITION 2. Let $M_{r,n\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the set of

$$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d = (t+\varpi)^{-1}\mathcal{L}_0)$$

where $\mathcal{L}_0, \mathcal{L}_1, \dots$ are $R[t]$-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^d$ satisfying the following conditions

- for all $i = 0, \dots, d-1$, we have $t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^n - \mathcal{V}_{i,R}$,
- as R-modules, $\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor of $t^{n_-}\mathcal{V}_{i,R}/t^{n_+}\mathcal{V}_{i,R}$ with rank $n_+d - r$.

By using the isomorphism

$$\alpha^i : t^{n_-}\mathcal{V}_{i,R}/t^{n_+}\mathcal{V}_{i,R} \xrightarrow{\sim} t^{n_-}R[t]^d/t^{n_+}R[t]^d$$

we can associate to each sequence $L_\bullet = (L_i)$ as in Definition 1 of $M_{r,n\pm}$, the sequence $\mathcal{L}_\bullet = (\mathcal{L}_i)$ as in Definition 2, in such a way that $\alpha^i(\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}) = L_i$. This correspondence is clearly bijective. Therefore, the two definitions of the functor $M_{r,n\pm}$ are equivalent.

It will be more convenient to consider the disjoint union $M_{n\pm}$ of projective schemes $M_{r,n\pm}$ for all r for which $M_{r,n\pm}$ makes sense, namely $M_{n\pm} = \coprod_{dn \leqslant r \leqslant dn_+} M_{r,n\pm}$, instead of each connected component $M_{r,n\pm}$ individually.

2.2. GROUP ACTION

Definition 2 permits us to define a natural group action on $M_{n\pm}$. Every $R[t]$-module $\mathcal{L}_i$ as above is included in

$$t^{n_+}R[t]^d \subset \mathcal{L}_i \subset t^{n_-}(t+\varpi)^{-1}R[t]^d.$$

Let $\bar{\mathcal{L}}_i$ denote its image in the quotient

$$\bar{\mathcal{V}}_{n_\pm,R} = t^{n_-}(t+\varpi)^{-1}R[t]^d/t^n + R[t]^d.$$

Obviously, $\mathcal{L}_i$ is completely determined by $\bar{\mathcal{L}}_i$.

Let $\bar{\mathcal{V}}_i$ denote the image of $\mathcal{V}_i$ in $\bar{\mathcal{V}}_{n_\pm}$. We can view $\bar{\mathcal{V}}_{n_\pm}$ as the free R-module $R^{(n_+-n_-+1)d}$ equipped with the endomorphism t and with the filtration

$$\bar{\mathcal{V}}_\bullet = (\bar{\mathcal{V}}_0 \subset \bar{\mathcal{V}}_1 \cdots \subset \bar{\mathcal{V}}_d = (t+\varpi)^{-1}\bar{\mathcal{V}}_0)$$

which is stabilized by t.

We now consider the functor $J_{n_\pm}$ which associates to each $\mathcal{O}$-algebra R the group $J_{n_\pm}(R)$ of all $R[t]$-automorphisms of $\bar{\mathcal{V}}_{n_\pm}$ fixing the filtration $\bar{\mathcal{V}}_\bullet$. This functor is represented by a closed subgroup of $\mathrm{GL}((n_+ - n_- + 1)d)$ over S that acts in the obvious way of $M_{n_\pm}$.

LEMMA 3. *The group scheme $J_{n_\pm}$ is smooth over S.*

Proof. Consider the functor $\mathcal{J}_{n_\pm}$ which associates to each $\mathcal{O}$-algebra R the ring $\mathcal{J}_{n_\pm}(R)$ of all $R[t]$-endomorphisms of $\bar{\mathcal{V}}_{n_\pm}$ stabilizing the filtration $\bar{\mathcal{V}}_\bullet$. This functor is obviously represented by a closed sub-scheme of the S-scheme $\mathfrak{gl}((n_+ - n_- + 1)d)$ of square matrices with rank $(n_+ - n_- + 1)d$.

The natural morphism of functors $J_{n_\pm} \to \mathcal{J}_{n_\pm}$ is an open immersion. Thus it suffices to prove that $\mathcal{J}_{n_\pm}$ is smooth over S.

Giving an element of $\mathcal{J}_{n_\pm}$ is equivalent to giving d vectors $v_1, \ldots, v_d$ such that $v_i \in t^{n_-}\bar{\mathcal{V}}_i$. This implies that $\mathcal{J}_{n_\pm}$ is isomorphic to a trivial vector bundle over S of rank

$$\sum_{i=1}^{d} \mathrm{rk}_{\mathcal{O}}(t^{n_-}\mathcal{V}_i/t^{n_+}\mathcal{O}[t]^d) = d^2(n_+ - n_- + 1) - (d-1)d/2.$$

This finishes the proof of the lemma. □

2.3. DESCRIPTION OF THE GENERIC FIBER

For this purpose, we use Definition 1 of $M_{n_\pm}$. Let R be an F-algebra. The matrix α then is invertible as an element

$$\alpha \in \mathrm{GL}(d, R[t]/t^{n_+ - n_-}R[t]),$$

the group of automorphisms of $t^{n_-}R[t]^d/t^{n_+}R[t]^d$.

Let $(L_0, \ldots, L_{d-1})$ be an element of $M_{n_\pm}(R)$. As R-modules, the L_i are locally direct factors of the same rank. For $i = 1, \ldots, d-1$, the inclusion $\alpha(L_{i-1}) \subset L_i$ implies the equality $\alpha(L_{i-1}) = L_i$. In this case, the last inclusion $\alpha(L_{d-1}) \subset L_0$ is automatically an equality, because the matrix

$$\alpha^d = \mathrm{diag}(t + \varpi, \ldots, t + \varpi)$$

satisfies the property: $\alpha^d(L_0) = L_0$. In others words, the whole sequence $(L_0, \ldots, L_{d-1})$ is completely determined by L_0.

Let us reformulate the above statement in a more precise way. Let $\mathrm{Grass}_{n_\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the set of $R[t]$-submodules L of $t^{n_-}R[t]^d/t^{n_+}R[t]^d$ which, as R-modules, are locally direct factors of $t^{n_-}R[t]^d/t^{n_+}R[t]^d$. Obviously, this functor is represented by a closed subscheme of a disjoint union of Grassmannians. In particular, it is proper over S.

Let π: $M_{n_\pm} \to \mathrm{Grass}_{n_\pm}$ be the morphism defined by

$$\pi(L_0, \ldots, L_{d-1}) = L_0.$$

The above discussion can be reformulated as follows.

LEMMA 4. *The morphism* π: $M_{n_\pm} \to \mathrm{Grass}_{n_\pm}$ *is an isomorphism over the generic point of* η *of* S. □

Let $K_{n\pm}$ the functor which associates to each $\mathcal{O}$-algebra R the group

$$K_{n\pm} = \mathrm{GL}(d, R[t]/t^{n_+ - n_-}R[t]).$$

Obviously, it is represented by a smooth group scheme over S and acts naturally on $\mathrm{Grass}_{n\pm}$. This action yields a decomposition into orbits that are smooth over S $\mathrm{Grass}_{n\pm} = \coprod_{\lambda \in \Lambda(n\pm)} O_\Lambda$, where $\Lambda(n_\pm)$ is the finite set of sequences of integers $\lambda = (\lambda_1, \ldots, \lambda_d)$ satisfying the following condition $n_+ \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_d \geqslant n_-$. This set $\Lambda(r, n_\pm)$ can be viewed as a finite subset of the cone of dominant coweights of

$G = \mathrm{GL}(d)$ and conversely, every dominant coweight of G occurs in some $\Lambda(n_\pm)$. For all $\lambda \in \Lambda(n_\pm)$, we have $O_\lambda(F) = K_F t^\lambda K_F/K_F$. Here $K_F = \mathrm{GL}(d, F[[t]])$ is the standard maximal 'compact' subgroup of $G_F = \mathrm{GL}(d, F((t)))$ and acts on $\mathrm{Grass}_{n\pm}(F)$ through the quotient $K_{n\pm}(F)$. The above equality holds if one replaces F by any field which is also an $\mathcal{O}$-algebra, since $K_{n\pm}$ is smooth; in particular it holds for the residue field k.

We derive from the above lemma the description

$$M_{n\pm}(F) = \coprod_{\lambda \in \Lambda(n\pm)} K_F t^\lambda K_F/K_F.$$

We will need to compare the action of $J_{n\pm}$ on $M_{n\pm}$ and the action of $K_{n\pm}$ on $\mathrm{Grass}_{n\pm}$. By definition, $J_{n\pm}(R)$ is a subgroup of

$$J_{n\pm}(R) \subset \mathrm{GL}(d, R[t]/t^{n_+-n_-}(t+\varpi)R[t])$$

for any $\mathcal{O}$-algebra R. By using the natural homomorphism

$$\mathrm{GL}(d, R[t]/t^{n_+-n_-}(t+\varpi)R[t]) \longrightarrow \mathrm{GL}(d, R[t]/t^{n_+-n_-}R[t])$$

we get a homomorphism $J_{n\pm}(R) \longrightarrow K_{n\pm}(R)$. This gives rises to a homomorphism of group schemes $\rho\colon J_{n\pm} \longrightarrow K_{n\pm}$, which is surjective over the generic point η of S.

The proof of the following lemma is straightforward.

LEMMA 5. *With respect to the homomorphism* $\rho\colon J_{n\pm} \longrightarrow K_{n\pm}$, *and to the morphism* $\pi\colon M_{n\pm} \longrightarrow \mathrm{Grass}_{n\pm}$, *the action of* $J_{n\pm}$ *on* $M_{n\pm}$ *and the action of* $K_{n\pm}$ *on* $\mathrm{Grass}_{n\pm}$ *are compatible.*

2.4. DESCRIPTION OF THE SPECIAL FIBER

For this purpose, we will use Definition 2 of $M_{n\pm}$. The functor $M_{r,n\pm}$ associates to each k-algebra R the set of

$$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d = t^{-1}\mathcal{L}_0)$$

where $L_0, L_1, \ldots$ are $R[t]$-submodules of $R[t, t^{-1}]^d$ satisfying the following conditions

- for all $i = 0, \ldots, d-1$, we have $t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^{n_-}\mathcal{V}_{i,R}$,
- as an R-module, each $\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor of $t^{n_-}\mathcal{V}_{i,R}/t^{n_+}\mathcal{V}_{i,R}$ with rank $n_+d - r$.

Let I_k denote the standard Iwahori subgroup of $G_k = \mathrm{GL}(d, k((t)))$, that is, the subgroup of integer matrices $\mathrm{GL}(d, k[[t]])$ whose reduction mod t lies in the subgroup of upper triangular matrices in $\mathrm{GL}(d, k)$. The set of k-points of $M_{n\pm}$ can be realized as a finite subset in the set of affine flags of $\mathrm{GL}(d)$, i.e., $M_{n\pm}(k) \subset G_k/I_k$. By definition, the k-points of $J_{n\pm}$ are the matrices in $\mathrm{GL}(d, k[t]/t^{n_+-n_-+1}k[t])$ whose reduction mod t is upper triangular. Thus, $J_{n\pm}(k)$ is a quotient of I_k. Obviously, the action of $J_{n\pm}$ on $M_{n\pm}(k)$ and the action of I_k on G_k/I_k are compatible. Therefore, for each r such that $dn_- \leqslant r \leqslant dn_+$ there exists a finite subset $\tilde{W}(r, n_\pm) \subset \tilde{W}$ of the affine Weyl group $\tilde{W}$ such that

$$M_{n\pm}(k) = \coprod_{w\in\tilde{W}(n_\pm)} I_k w I_K/I_k,$$

where $\tilde{W}(n_\pm) = \coprod_r \tilde{W}(r, n_\pm)$. One can see easily that any element $w \in \tilde{W}$ occurs in the finite subset $\tilde{W}(n_\pm)$ for some $n_\pm$. But the exact determination of the finite sets $\tilde{W}(r, n_\pm)$ is a difficult combinatorial problem; for the case of minuscule coweights of GL(d) (i.e., $n_+ = 1$ and $n_- = 0$) these sets have been described by Kottwitz and Rapoport [13].★

Let us recall that

$$\mathrm{Grass}_{n\pm}(k) = \coprod_{\lambda\in\Lambda(n\pm)} K_k t^\lambda K_k/K_k.$$

The proof of the text lemma is straightforward.

LEMMA 6. *The map* $\pi(k)$: $M_{n\pm}(k) \longrightarrow \mathrm{Grass}_{n\pm}(k)$ *is the restriction of the natural map* $G_k/I_k \longrightarrow G_k/K_k$.

2.5. SYMPLECTIC CASE

For the symplectic case, we will give only the definitions of the symplectic analogies of the objects which were considered in the linear case. The statements of Lemmas 3, 4, 5 and 6 remain unchanged.

In this section, the group G stands for GSp($2d$) associated to the symplectic form $\langle\, , \rangle$ represented by the matrix

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$$

where J is the anti-diagonal matrix with entries equal to 1. Let μ denote the minuscule coweight

$$\mu = (\underbrace{1, \ldots, 1}_{d}, \underbrace{0, \ldots, 0}_{d}).$$

Following Rapoport and Zink ([18]) the local model M_μ represents the functor which associates to each $\mathcal{O}$-algebra R the set of sequences $L_\bullet = (L_0, \ldots, L_d)$ where $L_0, \ldots, L_d$ are R-submodules of R^{2d} satisfying the following properties

- $L_0, \ldots, L_d$ are locally direct factors of R^{2d} of rank d,
- $\alpha^1(L_0) \subset L_1, \ldots \alpha'(L_{d-1}) \subset L_d$ where α' is the matrix of size $2d \times 2d$

★We refer to our subsequent work [10] for further progress in the description of the sets $W(r, n_\pm)$. In the terminology of Kottwitz and Rapoport [13], the set $\tilde{W}(r, n_\pm)$ is precisely the set of μ-permissible elements, for $\mu = (n_+^q, R + n_-, n_-^{d-q-1})$, where q and R are defined by $r - dn_- = q(n_+ - n_-) + R$, with $0 \leqslant R < n_+ - n_-$. By the main result of [10], it is also the set of μ-admissible elements. Similar remarks apply to the sets $\tilde{W}(n_\pm)$ occurring in the sympletic case, cf. end of Section 2.5.

$$\alpha' = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \varpi & & & 0 \end{pmatrix}$$

- L_0 and L_d are isotropic with respect to $\langle\, , \rangle$.

Just as in the linear case, let us introduce a new variable t and give the symplectic analogue of Definition 2. We consider the matrix of size $2d \times 2d$

$$\alpha' = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ t+\varpi & & & 0 \end{pmatrix}$$

viewed as an element of

$$\alpha \in \mathrm{GL}(2d, \mathcal{O}[t, t^{-1}, (t+\varpi)^{-1}]).$$

Denote by $\mathcal{V}, \ldots, \mathcal{V}_{2d-1}$ the fixed $\mathcal{O}[t]$-submodules of $\mathcal{O}[t, t^{-1}, (t+\varpi)^{-1}]^{2d}$ defined by $\mathcal{V}_i = \alpha^{-i}\mathcal{O}[t]^{2d}$. For an $\mathcal{O}$-algebra R, let $\mathcal{V}_{i,R}$ denote $\mathcal{V}_i \otimes_{\mathcal{O}} R$.

For any $R[t]$-submodule $\mathcal{L}$ of $R[t, t^{-1}, (t+\varpi)^{-1}]^{2d}$, the R[t]-module

$$\mathcal{L}^{\perp'} = \left\{ x \in R[t, t^{-1}, (t+\varpi)^{-1}]^{2d} \;|\forall y \in \mathcal{L}, t^n(t+\varpi)^{n'}\langle x, y\rangle \in R[t] \right\}$$

is called the *dual* of $\mathcal{L}$ with respect to the form $\langle\, , \rangle' = t^n(t+\varpi)^{n'}\langle\, , \rangle$. Thus $\mathcal{V}_0$ is autodual with respect to the form $\langle\, , \rangle$ and $\mathcal{V}_d$ is autodual with respect to the form $(t+\varpi)\langle\, , \rangle$.

Here is the symplectic analogue of Definition 2 of the model $M_{n\pm}$. For $n_- = 0$ and $n_+ = 1$, $M_{n\pm}$ will coincide with M_μ, for $\mu = (1^d, 0^d)$:

DEFINITION 7. For any $n_- \leqslant 0 < n_+$, let $M_{n\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the set of sequences

$$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d)$$

where $\mathcal{L}_0, \ldots, \mathcal{L}_d$ are R[t]-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^{2d}$ satisfying the following properties:

- for all $i = 0, \ldots, d$, we have $t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^{n_-}\mathcal{V}_{i,R,}$
- as R-modules, $\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor of $t^{n_-}\mathcal{V}_{i,R}/t^{n_+}\mathcal{V}_{i,R}$ of rank $(n_+ - n_-)d$,
- L_0 is autodual with respect to the form $t^{-n_- - n_+}\langle\, , \rangle$, and $\mathcal{L}_d$ is autodual with respect to the form $t^{-n_- - n_+}(t+\varpi)\langle\, , \rangle$.

Let us now define the natural group action on $M_{n\pm}$. The functor $J_{n\pm}$ associates to each $\mathcal{O}$-algebra R the group $J_{n\pm}(R)$ of $R[t]$-linear automorphisms of

$$\bar{\mathcal{V}}_{n\pm,R} = t^{n_-}(t+\varpi)^{-1}R[t]^{2d}/t^{n_+}R[t]^{2d}$$

which fix the filtration $\bar{\mathcal{V}}_{\bullet,R} = (\bar{\mathcal{V}}_{0,R} \subset \cdots \subset \bar{\mathcal{V}}_{d,R})$ (the image of $\mathcal{V}_{\bullet,R}$ in $\bar{\mathcal{V}}_{n_\pm,R}$ and which fix, up to a unit R, the symplectic form $t^{-n_- - n_+}(t+\varpi)\langle\, ,\rangle$. This functor is represented by an S-group scheme $J_{n\pm}$ which acts on $M_{n\pm}$. Lemma 3 remains true in the symplectic case: $J_{n\pm}$ is a *smooth* group scheme over S. The proof is completely similar to the linear case.

Let us now describe the generic fiber of $M_{n\pm}$. Let $\mathrm{Grass}_{n\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the set of R[t]-submodules L of $t^{n_-}R[t]^{2d}/t^{n_+}R[t]^{2d}$ which, as R-modules, are locally direct factors of rank $(n_+ - n_-)d$ and which are isotropic with respect to $t^{-n_- - n_+}\langle\, ,\rangle$. Then the morphism π: $M_{n\pm} \longrightarrow \mathrm{Grass}_{n\pm}$ defined by $\pi(L_\bullet) = L_0$ is an isomorphism over the generic point η of S. Let $K_{n\pm}$ denote the functor which associates to each $\mathcal{O}$-algebra R the group of $R[t]$-automorphisms of $t^{n_-}R[t]^{2d}/t^{n_+}R[t]^{2d}$ which fix the symplectic form $t^{-n_- - n_+}\langle\, ,\rangle$ up to a unit in R. Then $K_{n\pm}$ is represented by a smooth group scheme over S, and it acts in the obvious way on $\mathrm{Grass}_{n\pm}$. Consequently, we have a stratification in orbits of the generic fiber $M_{n\pm,\eta}$, i.e., $M_{n\pm,\eta} = \coprod_{\lambda\in\Lambda(n_\pm)} O_{\lambda,\eta}$. Here $\Lambda(n_\pm)$ is the set of sequences $\lambda = (\lambda_1, \ldots, \lambda_d)$ satisfying

$$n_+ \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_d \geqslant \frac{n_+ + n_-}{2},$$

and can be viewed as finite subset of the cone of dominant, coweights of $G = \mathrm{GSp}(2d)$. One can easily check that each dominant coweight of $\mathrm{GSp}(2d)$ occurs in some $\Lambda(n_\pm)$. For any $\lambda \in \Lambda(n_\pm)$, we have also $O_{\lambda,\eta}(F) = K_F t^\lambda K_F/K_F$, where $K_F = G(F[[t]])$ is the 'maximal compact' subgroup of $G_F = G(F((t)))$.

Next we turn to the special fiber of $M_{n_\pm}$. For this it is most convenient to give a slight reformulation of Definition 7 above. Let R be any $\mathcal{O}$-algebra. It is easy to see that specifying a sequence $\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \cdots \mathcal{L}_d)$ as in Definition 7 is the same as specifying a periodic "lattice chain"

$$\cdots \subset \mathcal{L}_{-1} \subset \mathcal{L}_0 \subset \cdots \subset \mathcal{L}_{2d} = (t+\varpi)^{-1}\mathcal{L}_0 \subset \cdots$$

consisting of $R[t]$-submodules of $R[t,\ t^{-1},\ (t+\varpi)^{-1}]^{2d}$ with the following properties:

- $t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^{n_-}\mathcal{V}_{i,R}$, where $\mathcal{V}_{i,R} = \alpha^{-i}\mathcal{V}_{0,R}$, for every $i \in \mathbb{Z}$,
- $\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor of rank $(n_+ - n_-)d$, for every $i \in \mathbb{Z}$,
- $\mathcal{L}_i^\perp = t^{-n_- - n_+}\mathcal{L}_{-i}$, for every $i \in \mathbb{Z}$,

where $\perp$ is defined using the original symplectic form $\langle\, ,\rangle$ on $R[t,\ t^{-1},\ (t+\varpi)^{-1}]^{2d}$. We denote by I_k the standard Iwahori subgroup of $\mathrm{GSp}(2d,\ k[[t]])$, namely, the stabilizer in this group of the periodic lattice chain $\mathcal{V}_{\bullet,k[[t]]}$. There is a canonical surjection $I_k \to J_{n_\pm}(k)$ and so the Iwahori subgroup I_k acts via its quotient $J_{n_\pm}(k)$ on the set $M_{n_\pm}(k)$. Moreover, the I_k-orbits in $M_{n_\pm}(k)$ are parametrized by a certain finite set $\tilde{W}(n_\pm)$ of the affine Weyl group $\tilde{W}(\mathrm{GSp}(2d))$, i.e., $M_{n_\pm}(k) = \coprod_{w\in\tilde{W}(n_\pm)} I_k\, w\, I_k/I_k$. The precise description of the sets $\tilde{W}(n_\pm)$ is a difficult combinatorial problem (see [13] for the case $n_+ = 1, n_- = 0$), but one can easily see that any $w \in \tilde{W}(\mathrm{GSp}(2d))$ is contained in some $\tilde{W}(n_\pm)$.

The definitions of the group scheme action of $K_{n_\pm}$ on $\mathrm{Grass}_{n_\pm}$, of the homomorphism ρ: $J_{n_\pm} \to K_{n_\pm}$ and the compatibility properties (Lemmas 5, 6) are obvious and will be left to the reader.

3. Semi-Simple Trace on Nearby Cycles

3.1. SEMI-SIMPLE TRACE

The notion of semi-simple trace was introduced by Rapoport in [17] and its good properties were mentioned there. The purpose of this section is only to give a more systematic presentation in insisting on the important fact that the semi-simple trace furnish a kind of sheaf-function dictionary à la Grothendieck. In writing this section, we have benefited from very helpful explanations of Laumon.

Let $\bar{F}$ be a separable closure of the local field F. Let Γ be the Galois group $\mathrm{Gal}(\bar{F}/F)$ of F and let Γ_0 be the inertia subgroup of Γ defined by the exact sequence

$$1 \to \Gamma_0 \to \Gamma \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

For any prime $\ell \neq p$, there exists a canonical surjective homomorphism $t_\ell\colon \Gamma_0 \to \mathbb{Z}_\ell(1)$.

Let $\mathcal{R}$ denote the Abelian category of continuous, finite dimensional ℓ-adic representations of Γ. Let (ρ, V) be an object of $\mathcal{R}$, i.e., ρ: $\Gamma \to \mathrm{GL}(V)$.
According to a theorem of Grothendieck, the restricted representation $\rho(\Gamma_0)$ is *quasi-unipotent*, i.e. there exists a finite-index subgroup Γ_1 of Γ_0 which acts unipotently on V (the residue field k is supposed finite). There exists then an unique nilpotent morphism, the *logarithm* of ρ, N: $V(1) \to V$ characterized by the following property: for all $g \in \Gamma_1$, we have $\rho(g) = \exp(Nt_\ell(g))$.

Following Rapoport, an increasing filtration $\mathcal{F}$ of V will be called *admissible* if it is stable under the action of Γ and such that Γ_0 operates on the associated graded $\mathrm{gr}^{\mathcal{F}}_\bullet(V)$ through a finite quotient. Admissible filtrations always exist: we can take for instance the filtration defined by the kernels of the powers of N.

We define the semi-simple trace of Frobenius on V as

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, V) = \sum_k \mathrm{Tr}(\mathrm{Fr}_q, \mathrm{gr}^{\mathcal{F}}_k(V)^{\Gamma_0}).$$

LEMMA 8. *The semi-simple trace* $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, V)$ *does not depend on the choice of the admissible filtration* $\mathcal{F}$.

Proof. Let us first consider the case where Γ_0 acts on V through a finite quotient. Since taking invariants under a finite group acting on a $\bar{\mathbb{Q}}_\ell$-vector space is an exact functor, the graded associated to the filtration $\mathcal{F}'$ of V^{Γ_0} induced by $\mathcal{F}$ is equal to $\mathrm{gr}^{\mathcal{F}}_\bullet(V)^{\Gamma_0}$, i.e., $\mathrm{gr}^{\mathcal{F}'}_k(V^{\Gamma_0}) = \mathrm{gr}^{\mathcal{F}}_k(V)^{\Gamma_0}$. Consequently

$$\mathrm{Tr}(\mathrm{Fr}_q, V^{\Gamma_0}) = \sum_k \mathrm{Tr}(\mathrm{Fr}_q, \mathrm{gr}^{\mathcal{F}}_k(V)^{\Gamma_0}).$$

In the general case, any two admissible filtrations admit a third finer admissible filtration. By using the above case, one sees the semi-simple trace associated to each of the two first admissible filtrations is equal to the semi-simple trace associated to the third one and the lemma follows. □

COROLLARY 9. *The function defined by* $V \mapsto \mathrm{Tr}^{ss}(\mathrm{Fr}_q, V)$ *on the set of isomorphism classes* V *of* $\mathcal{R}$*, factors through the Grothendieck group of* $\mathcal{R}$.

For any object C of the derived category associated to $\mathcal{R}$, we put

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, C) = \sum_i (-1)^i \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{H}^i(C)) .$$

By the above corollary, for any distinguished triangle $C \to C' \to C'' \to C[1]$ the equality

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, C) + \mathrm{Tr}^{ss}(\mathrm{Fr}_q, C'') = \mathrm{Tr}^{ss}(\mathrm{Fr}_q, C')$$

holds.

Let X be a k-scheme of finite type, $X_{\bar{s}} = X \otimes_k \bar{k}$. Let $D_c^b(X \times_k \eta)$ denote the derived category associated to the abelian category of constructible ℓ-adic sheaves on $X_{\bar{s}}$ equipped with an action of Γ compatible with the action of Γ on $X_{\bar{s}}$ through $\mathrm{Gal}(\bar{k}/k)$, see [3].★ Let $\mathcal{C}$ be an object of $D_c^b(X \times_k \eta)$. For any $x \in X(k)$, the fiber $\mathcal{C}_x$ is an object of the derived category of $\mathcal{R}$. Thus we can define the function semi-simple trace

$$\tau_{\mathcal{C}}^{ss} X(k) \to \mathbb{Q}_\ell \text{ by } \tau_{\mathcal{C}}^{ss}(x) = \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathcal{C}_x).$$

This association $\mathcal{C} \mapsto \tau_{\mathcal{C}}^{ss}$ furnishes an analogue of the usual sheaf-function dictionary of Grothendieck (see [7]):

PROPOSITION 10. *Let* $f\colon X \to Y$ *be a morphism between k-schemes of finite type*

(1) *Let* $\mathcal{C}$ *be an object of* $D_c^b(Y \times_k \eta)$. *For all* $x \in X(k)$, *we have* $\tau_{f^*\mathcal{C}}^{ss}(x) = \tau_{\mathcal{C}}^{ss}(f(x))$.
(2) *Let* $\mathcal{C}$ *be an object of* $D_c^b(X \times_k \eta)$. *For all* $y \in Y(k)$, *we have*

$$\tau_{\mathrm{R}f_!\mathcal{C}}^{ss}(y) = \sum_{\substack{x \in X(k)\\ f(x)=y}} \tau_{\mathcal{C}}^{ss}(x). .$$

Proof. The first statement is obvious because $f^*\mathcal{C}_x$ and $\mathcal{C}_{f(x)}$ are canonically isomorphic as objects of the derived category of $\mathcal{R}$.

It suffices to prove the second statement in the case $Y = s$. By Corollary 9 and 'shifting', it suffices to consider the case where $\mathcal{C}$ is concentrated in only one degree, say in the degree zero. Denote $C = \mathcal{H}^0(\mathcal{C})$ and choose an admissible filtration of $\mathcal{C}$

$$0 = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_n = C.$$

The associated spectral sequence

$$\mathrm{E}_1^{i,j-i} = \mathrm{H}_c^j(X_{\bar{s}}, C_i/C_{i-1}) \Longrightarrow \mathrm{H}_c^j(X_{\bar{s}}, C)$$

★The category $D_c^b(X \times_k \eta)$ is defined, following [4], to be $\bar{\mathbb{Q}}_\ell \otimes$ the projective 2-limit of the categories $D_{ctf}^b(X \times_k \eta, \mathbb{Z}/\ell^n\mathbb{Z})$, so it is not strictly speaking the derived category of the abelian category of constructible ℓ-adic sheaves.

yields an abutment filtration on $\mathrm{H}^j_c(X_{\bar{s}}, C)$ with associated graded $\mathrm{E}^{i,j-i}_\infty$. Since the inertia group acts on $\mathrm{E}^{i,j-i}_1$ through a finite quotient, the same property holds for $\mathrm{E}^{i,j-i}_\infty$ because $\mathrm{E}^{i,j-i}_\infty$ is a subquotient of $\mathrm{E}^{i,j-i}_\infty$. Consequently, the abutment filtration on $\mathrm{H}^j_c(X_{\bar{s}}, C)$ is an admissible filtration and by definition, we have

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, Rf_!C) = \sum_{i,j}(-1)^j \mathrm{Tr}(\mathrm{Fr}_q, (\mathrm{E}^{i,j-i}_\infty)^{\Gamma_0}).$$

Now, the identity in the Grothendieck group

$$\sum_{i,j}(-1)^j \mathrm{E}^{i,j-i}_1 = \sum_{i,j}(-1)^j \mathrm{E}^{i,j-i}_\infty$$

implies

$$\sum_{i,j}(-1)^j (\mathrm{E}^{i,j-i}_1)^{\Gamma_0} = \sum_{i,j}(-1)^j (\mathrm{E}^{i,j-i}_\infty)^{\Gamma_0}$$

because taking the invariants under a finite group is an exact functor.

The same exactness implies

$$(\mathrm{E}^{i,j-i}_1)^{\Gamma_0} = \mathrm{H}^j_c(X_{\bar{s}}, C_i/C_{i-1})^{\Gamma_0} = \mathrm{H}^j_c(X_{\bar{s}}, (C_i/C_{i-1})^{\Gamma_0}).$$

By putting the above equalities together, we obtain

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, Rf_!C) = \sum_{i,j}(-1)^j \mathrm{Tr}(\mathrm{Fr}_q, \mathrm{H}^j_c(X_{\bar{s}}, (C_i/C_{i-1})^{\Gamma_0})).$$

By using now the Grothendieck–Lefschetz formula, we have

$$\sum_{x \in X(k)} \mathrm{Tr}(\mathrm{Fr}_q, (C_i/C_{i-1})^{\Gamma_0}_x) = \sum_j (-1)^j \mathrm{Tr}(\mathrm{Fr}_q, \mathrm{H}^j_c(X_{\bar{s}}, (C_i/C_{i-1})^{\Gamma_0})).$$

Consequently,

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, Rf_!C) = \sum_{x \in X(k)} \mathrm{Tr}^{ss}(\mathrm{Fr}_q, C_x). \qquad \square$$

3.2. NEARBY CYCLES

Let $\bar{\eta} = \mathrm{Spec}(\bar{F})$ denote the geometric generic point of S, $\bar{S}$ be the normalization of S in $\bar{\eta}$ and $\bar{s}$ be the closed point of $\bar{S}$. For an S-scheme X of finite type, let us denote by $\bar{j}^X\colon X_{\bar{\eta}} \to X_{\bar{S}}$ the morphism deduced from $\bar{j}\colon \bar{\eta} \to \bar{S}$ and denote by $\bar{\imath}^X\colon X_{\bar{s}} \to X_{\bar{S}}$ the morphism deduced from $\bar{\imath}\colon \bar{s} \to \bar{S}$.

The nearby cycles of an ℓ-adic complex C_η on X_η, is the complex of ℓ-adic sheaves defined by

$$\mathrm{R}\Psi^X(C_\eta) = \bar{\imath}^{X,*}\mathrm{R}\bar{j}^X_* \bar{j}^{X,*} C_\eta.$$

The complex $\mathrm{R}\Psi^X(C_\eta)$ is equipped with an action of Γ compatible with the action of Γ on $X_{\bar{s}}$ through the quotient $\mathrm{Gal}(\bar{k}/k)$.

For X a proper S-scheme, we have a canonical isomorphism

$$\mathrm{R}\Gamma(X_{\bar{s}}, \mathrm{R}\Psi(C_\eta)) = \mathrm{R}\Gamma(X_{\bar{\eta}}, C_\eta)$$

compatible with the natural actions of Γ on the two sides.

Let us suppose moreover the generic fiber X_η is smooth. In the order the compute the local factor of the Hasse-Weil zeta function, one should calculate the trace

$$\sum_j (-1)^j \mathrm{Tr}(\mathrm{Fr}_q, \mathrm{H}^j(X_{\bar{\eta}}, \bar{\mathbb{Q}}_\ell)^{\Gamma_0}).$$

Assuming that the graded pieces in the monodromy filtration of $\mathrm{H}^j(X_{\bar{\eta}}, \bar{\mathbb{Q}}_\ell)$ are pure (Deligne's conjecture), Rapoport proved that the true local factor is completely determined by the semi-simple local factor, see [17]. Now by the above discussion the semi-simple trace can be computed by the formula

$$\sum_j (-1)^j \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{H}^j(X_{\bar{\eta}}, \bar{\mathbb{Q}}_\ell)) = \sum_{x \in X(k)} \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi(\bar{\mathbb{Q}}_\ell)_x).$$

4. Statement of the Main Result

4.1. NEARBY CYCLES ON LOCAL MODELS

We have been in Subsection 2.3 (resp. 2.5 for sympletic case) that the generic fiber of $M_{n_\pm}$ admits a stratification with smooth strata $M_{n_\pm,\eta} = \coprod_{\lambda \in \Lambda(n_\pm)} O_{\lambda,\eta}$.

Denote by $\bar{O}_{\lambda,\eta}$ the Zariski closure $\bar{O}_{\lambda,\eta}$ in $M_{n_\pm,\eta}$; in general $\bar{O}_{\lambda,\eta}$ is no longer smooth. It is natural to consider $\mathcal{A}_{\lambda,\eta} = \mathrm{IC}(O_{\lambda,\eta})$, its ℓ-adic intersection complex.

We want to calculate the function

$$\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})}(x) = \mathrm{Tr}^{ss}(\mathrm{Fr}_q, \mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})_x)$$

of semi-simple trace of the Frobenius endomorphism on the nearby cycle complex $\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})$ defined in the last section. We are denoting the scheme $M_{n_\pm}$ simply by M here.

As $O_{\lambda,\eta}$ is an orbit of $J_{n_\pm,\eta}$, the intersection complex $\mathcal{A}_{\lambda,\eta}$ is naturally $J_{n_\pm,\eta}$-equivariant. As we know that $J_{n_\pm}$ is smooth over S by Lemma 3, its nearby cycle complex $\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})$ is $J_{n_\pm,\bar{s}}$-equivariant. In particular, the function $\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}\lambda,\eta)}\colon M_{n_\pm}(k) \to \bar{\mathbb{Q}}_\ell$ is $J_{n_\pm}(k)$-invariant.

Now following the group theoretic description of the action of $J_{n_\pm}(k)$ on $M_{n_\pm}(k)$ in Subsection 2.4 (resp. 2.5), we can consider the function $\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})}$ as a function on G_k with compact support which is invariant on the left and on the right by the Iwahori subgroup I_k, i.e., $\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})} \in \mathcal{H}(G_k//I_k)$.

The following statement was conjectured by R. Kottwitz, and is the main result of this paper.

THEOREM 11. *Let G be either* $\mathrm{GL}(d)$ *or* $\mathrm{GSp}(2d)$. *Let $M = M_{n_\pm}$ be the scheme associated to the group G and the pair of integers $n_\pm$, as above. Then we have the formula*

$$\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})} = (-1)^{2\langle\rho,\lambda\rangle} \sum_{\lambda' \leqslant \lambda} m_\lambda(\lambda') z_{\lambda'},$$

where $z_{\lambda'}$ *is the function of Bernstein associated to the dominant coweight* λ'*, which lies in the center* $Z(\mathcal{H}(G_k//I_k))$ *of* $\mathcal{H}(G_k//I_k)$.

Here, ρ is half the sum of positive roots for G and thence $2\langle\rho, \lambda\rangle$ is the dimension of $O_{\lambda,\eta}$. The integer $m_\lambda(\lambda')$ is the multiplicity of weight λ' occurring in the representation of highest weight λ. The partial ordering $\lambda' \leqslant \lambda$ is defined to mean that $\lambda - \lambda'$ is a sum of positive coroots of G.

Comparing with the formula for minuscule μ given in Kottwitz' conjecture (cf. Introduction), one notices the absence of the factor $q^{\langle\rho,\mu\rangle}$ and the appearance of the sign $(-1)^{2\langle\rho,\mu\rangle}$. This difference is explained by the normalization of the intersection complex $\mathcal{A}_{\mu,\eta}$. For minuscule coweights μ, the orbit O_μ is closed. Consequently, the intersection complex $\mathcal{A}_{\mu,\eta}$ differs from the constant sheaf only by a normalization factor

$$\mathcal{A}_{\mu,\eta} = \bar{\mathbb{Q}}_\ell[2\langle\rho, \mu\rangle](\langle\rho, \mu\rangle).$$

We refer to Lusztig's article [14] for the definition of Bernstein's functions. In fact, what we need is rather the properties that characterize these functions. We will recall these properties in the next subsection.

4.2. THE SATAKE AND BERNSTEIN ISOMORPHISMS

Denote by K_k the standard maximal compact subgroup $G(k[[t]])$ of G_k, where G is either GL(d) or GSp($2d$). The $\bar{\mathbb{Q}}_\ell$-valued functions with compact support in G_k invariant on the left and on the right by K_k form a commutative algebra $\mathcal{H}(G_k//K_k)$ with respect to the convolution product. Here the convolution is defined using the Haar measure on G_k which gives K_k measure 1. Denote by $\mathbb{I}_K$ the characteristic function of K_k. This element is the unit of the algebra $\mathcal{H}(G_k//K_k)$. Similarly we define the convolution on $\mathcal{H}(G_k//I_k)$ using the Haar measure on G_k which gives I_k measure 1.

We consider the following triangle

$$\begin{array}{ccc} & \bar{\mathbb{Q}}_\ell[X_*]^W & \\ {}^{\text{Bern.}}\swarrow & & \nwarrow^{\text{Sat.}} \\ Z(\mathcal{H}(G_k//P_k)) & \xrightarrow[-*\mathbb{I}_K]{} & \mathcal{H}(G_k//K_k) \end{array}$$

Here $\bar{\mathbb{Q}}_\ell[X_*]^W$ is the W-invariant sub-algebra of the $\bar{\mathbb{Q}}_\ell$-algebra associated to the group of cocharacters of the standard (diagonal) torus T in G and W is the Weyl group associated to T. For the case $G = \mathrm{GL}(d)$, this algebra is isomorphic to the algebra of symmetric polynomials with d variables and their inverses: $\bar{\mathbb{Q}}_\ell[X_1^\pm, \ldots, X_d^\pm]^{S_d}$.

The above maps

$$\text{Sat: } \mathcal{H}(G_k//K_k) \to \bar{\mathbb{Q}}_\ell[X_*]^W$$

and

$$\text{Bern}: \ \bar{\mathbb{Q}}_\ell[X_*]^W \to Z(\mathcal{H}(G_k//I_k))$$

are the isomorphisms of algebras constructed by Satake, see [19] and by Bernstein, see [14]. It follows immediately from its definition that the Bernstein isomorphism sends the irreducible character χ_λ of highest weight λ to

$$\text{Bern}(\chi_\lambda) = \sum_{\lambda' \leqslant \lambda} m_\lambda(\lambda') z_{\lambda'}.$$

The horizontal map

$$Z(\mathcal{H}(G_k//I_k)) \to \mathcal{H}(G_k//K_k)$$

is defined by $f \mapsto f * \mathbb{I}_K$ where

$$f * \mathbb{I}_K(g) = \int_{G_k} f(gh^{-1})\mathbb{I}_K(h)\mathrm{d}h.$$

The next statement seems to be known to the experts. It can be deduced easily see [8], from results of Lusztig [14] and Kato [12]. Another proof can be found in an article of Dat [2].

LEMMA 12. *The above triangle is commutative.*

It follows that the horizontal map is an isomorphism, and that $(-1)^{2\langle\rho,\lambda\rangle}\sum_{\lambda'\leqslant\lambda} m_\lambda(\lambda')z_{\lambda'}$ is the unique element in $Z(\mathcal{H}(G_k//I_k))$ whose image in $\mathcal{H}(G_k//K_k)$ has Satake transform $(-1)^{2\langle\rho,\lambda\rangle}\chi_\lambda$.

Thus in order to prove the Theorem 11, it suffices now to prove the two following statements.

PROPOSITION 13. *The function $\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})}$ lies in the center $Z(\mathcal{H}(G_k//I_k))$ of the algebra $\mathcal{H}(G_k//I_k)$.*

PROPOSITION 14. *The Satake transform of $\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})} * \mathbb{I}_K$ is equal to $(-1)^{2\langle\rho,\lambda\rangle}\chi_\lambda$, where χ_λ is the irreducible character of highest weight λ.*

In fact we can reformulate Proposition 14 in such a way that it becomes independent of Proposition 13. We will prove Proposition 14 in the next section.

In order to prove Proposition 13, we have to adapt Lusztig's construction of geometric convolution to our context. This will be done in the Section 7. The proof of Proposition 13 itself will be given in Section 8.

5. Proof of Proposition 14

5.1. AVERAGING BY K

The map

$$Z(\mathcal{H}(G_k//I_k)) \to \mathcal{H}(G_k//K_k)$$

defined by $f \mapsto f * \mathbb{I}_K$ can be obviously extended to a map

$$C_c(G_k/I_k) \to C_c(G_k/K_K)$$

where $C_c(G_k/I_k)$ (resp. $C_c(G_k/K_k)$) is the space of functions with compact support in G_k invariant on the right by I_k (resp. K_k). This map can be rewritten as follows

$$f * \mathbb{I}_K(g) = \sum_{h \in K_k/I_k} f(gh).$$

Therefore, this operation corresponds to summing along the fibers of the map $G_k/I_k \to G_k/K_k$. For the particular function $\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})}$, it amounts to summing along the fibers of the map

$$\pi(k) : M_{n_\pm}(k) \to \mathrm{Grass}_{n_\pm}(k),$$

(see Lemma 6).

By using now the sheaf-function dictionary for semi-simple trace, we get

$$\tau^{ss}_{\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})} * \mathbb{I}_K = \tau^{ss}_{\mathrm{R}\pi_{\bar{s},*}\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})}.$$

The nearby cycle functor commutes with direct image by a proper morphism, so that

$$\mathrm{R}\pi_{\bar{s},*}\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) = \mathrm{R}\Psi^{\mathrm{Grass}}\mathrm{R}\pi_{\eta,*}(\mathcal{A}_{\lambda,\eta}).$$

By Lemma 4, π_η is an isomorphism. Consequently, $\mathrm{R}\pi_{\eta,*}(\mathcal{A}_{\lambda,\eta}) = \mathcal{A}_{\lambda,\eta}$.

According to the description of $\mathrm{Grass} = \mathrm{Grass}_{n_\pm}$ (see Subsections 2.3 and 2.5), we can prove that $\mathrm{R}\Psi^{\mathrm{Grass}}\mathcal{A}_{\lambda,\eta} = \mathcal{A}_{\lambda,\bar{s}}$ (note that the complex $\mathcal{A}_{\lambda,\eta}$ over Grass_η can be extended in a canonical fashion to a complex $\mathcal{A}_\lambda$ over the S-scheme Grass, thus $\mathcal{A}_{\lambda,\bar{s}}$ makes sense). In particular, the inertia subgroup Γ_0 acts trivially on $\mathrm{R}\Psi^{\mathrm{Grass}}\mathcal{A}_{\lambda,\eta}$ and the semi-simple trace is just the ordinary trace. The proof of a more general statement will be given in the following appendix.

By putting together the above equalities, we obtain $\mathrm{R}\pi_{\bar{s},*}\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) = \mathcal{A}_{\lambda,s}$.

To conclude the proof of Proposition 14, we quote an important theorem of Lusztig and Kato, see [14] and [12]. We remark that Ginzburg and also Mirkovic and Vilonen have put this result in its natural framework: a Tannakian equivalence, see [6, 16].

THEOREM 15 (Lusztig, Kato). *The Satake transform of the function* $\tau_{\mathcal{A}_{\lambda,s}}$ *is equal to* $\mathrm{Sat}(\tau_{\mathcal{A}_{\lambda,s}}) = (-1)^{2\langle\rho,\lambda\rangle}\chi_\lambda$, *where* χ_λ *is the irreducible character of highest weight* λ.

5.2. APPENDIX

This appendix seems to be well known to the experts. We thank G. Laumon who has kindly explained it to us.

Let us consider the following situation.

Let X be a proper scheme over S equipped with an action of a group scheme J smooth over S. We suppose there is a stratification $X = \coprod_{\alpha \in \Delta} X_\alpha$ with each stratum

X_α smooth over S. We assume that the group scheme J acts transitively on all fibers of X_α. Moreover, we suppose there exists, for each α, a J-equivariant resolution of singularities $\tilde{X}_\alpha$, π_α: $\tilde{X}_\alpha \to \bar{X}_\alpha$ of the closure $\bar{X}_\alpha$ of X_α, such that this resolution $\tilde{X}_\alpha$, smooth over S, contains X_α as a Zariski open; the complement $\tilde{X}_\alpha - X_\alpha$ is also supposed to be a union of normal crossing divisors.

If X is an invariant subscheme of the affine Grassmannian or of the affine flag variety, we can use the Demazure resolution.

Let i_α denote the inclusion map $X_\alpha \to X$ and let $\mathcal{F}_\alpha$ denote $i_{\alpha,!}\bar{\mathbb{Q}}_\ell$. A bounded complex of sheaves $\mathcal{F}$ with constructible cohomology sheaves (more precisely an object of $D_c^b(X, \mathbb{Q}_\ell)$ – cf. the second footnote), is said to be Δ-*constant* if the cohomology sheaves of $\mathcal{F}$ are successive extensions of $\mathcal{F}_\alpha$ with $\alpha \in \Delta$. The intersection complex of $\bar{X}_\alpha$ is Δ-constant.

For an ℓ-adic complex $\mathcal{F}$ of sheaves on X, there exists a canonical morphism $\mathcal{F}_{\bar{s}} \to \mathrm{R}\Psi^X(\mathcal{F}_\eta)$ whose mapping cylinder is the vanishing cycle $\mathrm{R}\Phi^X(\mathcal{F})$.

LEMMA 16. *If $\mathcal{F}$ is a Δ-constant complex, then* $\mathrm{R}\Phi^X(\mathcal{F}) = 0$.

Proof. Clearly, it suffices to prove $\mathrm{R}\Phi^X(\mathcal{F}_\alpha) = 0$. Consider the equivariant resolution $\pi_\alpha : \tilde{X}_\alpha \to \bar{X}_\alpha$. We have a canonical isomorphism

$$\mathrm{R}\pi_{\alpha,*}\mathrm{R}\Phi^{\tilde{X}_\alpha}(\mathcal{F}_\alpha) \xrightarrow{\sim} \mathrm{R}\Phi^{\bar{X}_\alpha}(\mathcal{F}_\alpha).$$

It suffices then to prove $\mathrm{R}\Phi^{\tilde{X}_\alpha}(\mathcal{F}_\alpha) = 0$. This is known because $\tilde{X}_\alpha$ is smooth over S and $\tilde{X}_\alpha - X_\alpha$ is a union of normal crossing divisors. □

COROLLARY 17. *If $\mathcal{F}$ is Δ-constant and bounded, the inertia group Γ_0 acts trivially on the nearby cycle* $\mathrm{R}\Psi^X(\mathcal{F}_\eta)$.

Proof. The morphism $\mathcal{F}_{\bar{s}} \to \mathrm{R}\Psi^X(\mathcal{F}_\eta)$ is an isomorphism compatible with the actions of Γ. The inertia subgroup Γ_0 acts trivially on $\mathcal{F}_{\bar{s}}$, thus it acts trivially on $\mathrm{R}\Psi^X(\mathcal{F}_\eta)$, too. □

6. Invariant Subschemes of G/I

We recall here the well known ind-scheme structure of G_k/I_k where G denotes the group $\mathrm{GL}(d, k((t+\varpi)))$ or the group $\mathrm{GSp}(2d, k((t+\varpi)))$ and where I is its standard Iwahori subgroup. The variable $t+\varpi$ is used instead of t in order to be compatible with the definitions of local models given in Section 2.

6.1. LINEAR CASE

Let $N_{n_\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the set of

$$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d = (t+\varpi)^{-1}\mathcal{L}_0)$$

where $\mathcal{L}_0, \mathcal{L}_1, \ldots$ are $R[t]$-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^d$ such that for $i = 0, 1, \ldots, d-1$

$$(t+\varpi)^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset (t+\varpi)^{n_-}\mathcal{V}_{i,R}$$

and $\mathcal{L}_i/(t+\varpi^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor, of fixed rank independent of i, of the free R-module $(t+\varpi)^{n_-}\mathcal{V}_{i,R}/(t+\varpi)^{n_+}\mathcal{V}_{i,R}$. Obviously, this functor is represented by a closed subscheme in a disjoint union of products of Grassmannians. In particular, $N_{n_\pm}$ is proper.

Let $I_{n_\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the group $R[t]$-linear automorphisms of

$$(t+\varpi)^{n_- -1}R[t]^d/(t+\varpi)^{n_+}R[t]^d$$

fixing the image in this quotient of the filtration

$$\mathcal{V}_{0,R}\subset\mathcal{V}_{1,R}\subset\cdots\subset\mathcal{V}_{d,R}=(t+\varpi)^{-1}\mathcal{V}_{0,R}.$$

This functor is represented by a smooth group scheme over S which acts on $N_{n_\pm}$.

6.2. SYMPLECTIC CASE

Let $N_{n_\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the set of sequences

$$\mathcal{L}_\bullet=(\mathcal{L}_0\subset\mathcal{L}_1\subset\cdots\subset\mathcal{L}_d).$$

where $\mathcal{L}_0,\mathcal{L}_1,\ldots$ are $R[t]$-submodules of $R[t,t^{-1},(t+\varpi)^{-1}]^{2d}$ satisfying

$$(t+\varpi)^{n_+}\mathcal{V}_{i,R}\subset\mathcal{L}_i\subset(t+\varpi)^{n_-}\mathcal{V}_{i,R}$$

and such that $\mathcal{L}_i/(t+\varpi)^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor of $(t+\varpi)^{n_-}\mathcal{V}_{i,R}/(t+\varpi)^{n_+}\mathcal{V}_{i,R}$ of rank $(n_+-n_-)d$ for all $i=0,1,\ldots,d$, and $\mathcal{L}_0$ (resp. $\mathcal{L}_d$) is autodual with respect to the symplectic form $(t+\varpi)^{-n_--n_+}\langle\,,\rangle$ (resp. $(t+\varpi)^{-n_--n_++1}\langle\,,\rangle$).

Let $I_{n\pm}$ be the functor which associates to each $\mathcal{O}$-algebra R the group $R[t]$-linear automorphisms of

$$(t+\varpi)^{n_- -1}R[t]^{2d}/(t+\varpi)^{n_+}R[t]^{2d}$$

fixing the image in this quotient, of the filtration

$$\mathcal{V}_{0,R}\subset\mathcal{V}_{1,R}\subset\cdots\subset\mathcal{V}_{2d,R}=(t+\varpi)^{-1}\mathcal{V}_{0,R},$$

and fixing the symplectic form $(t+\varpi)^{-n_--n_+1}+\langle\,,\rangle$ up to a unit in R. This functor is represented by a smooth group scheme over S which acts on $N_{n\pm}$.

6.3. THERE IS NO VANISHING CYCLE ON N

For any algebraically closed field k over $\mathcal{O}$, each $N_{n\pm}(k)$ is an $I_{n\pm}$-invariant subset of the direct limit

$$\lim_{\substack{\longrightarrow\\ n_\pm\to\pm\infty}} N_{n_\pm}(k)=G(k((t+\varpi)))/I_{n_\pm}$$

where G denotes either the linear group or the group of symplectic similitudes. It follows from the Bruhat-Tits decomposition that $N_{n_\pm}$ admits a stratification by $I_{n_\pm}$-orbits $N_{n_\pm}=\coprod_{w\in\tilde{W}'(n_\pm)}O_w$, where $\tilde{W}'(n_\pm)$ is a finite subset of the affine Weyl group $\tilde{W}$ of GL(d) (resp. GSp($2d$).) Moreover, for all $w\in\tilde{W}'(n_\pm)$, O_w is isomorphic to the affine space $\mathbb{A}_S^{\ell(w)}$ of dimension $\ell(w)$ over S, in particular it is smooth over S.

By construction, $I_{n_\pm}$ acts transitively on each O_w. All this remains true if we replace S by any other base scheme.

Let $\bar{O}_w$ denote the closure of O_w. Let $\mathcal{I}_{w,\eta}$ (resp. $\mathcal{I}_{w,s}$) denote the intersection complex of $\bar{O}_{w,\eta}$ (resp. $\bar{O}_{w,s}$). We have $\mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}) = \mathcal{I}_{w,\bar{s}}$ (see Appendix 5.2 for a proof). In particular, the inertia subgroup Γ_0 acts trivially on $\mathrm{R}\Psi^N(\mathcal{I}_{w,\eta})$.

Let $\tilde{W}$ be the affine Weyl group of GL(d), respectively GSp($2d$). It can be easily checked that $\tilde{W} = \cup_{n_\pm} \tilde{W}'(n_\pm)$ for the linear case as well as for the symplectic case.

7. Convolution Product of $\mathcal{A}_\lambda$ with $\mathcal{I}_w$

7.1. CONVOLUTION DIAGRAM

In this section, we will adapt a construction due to Lusztig in order to define the convolution product of an equivariant perverse sheaf $\mathcal{A}_\lambda$ over $M_{n_\pm}$ with an equivariant perverse sheaf $\mathcal{I}_w$ over $N_{n'_\pm}$. See Lusztig's article [15] for a quite general construction.

For any dominant coweight λ and any $w \in \tilde{W}$, we can choose $n_\pm$ and $n'_\pm$ so that $\lambda \in \Lambda(n_\pm)$ and $w \in \tilde{W}'(n'_\pm)$. From now on, since λ and w as well as $n_\pm$ and $n'_\pm$ are fixed, we will often write M for $M_{n_\pm}$ and N for $N_{n'_\pm}$. This should not cause any confusion.

The aim of this subsection is to construct the convolution diagram à la Lusztig

$$\begin{array}{ccccc} & \tilde{M}\times\tilde{N} & & & \\ {}^{p_1}\swarrow & & \searrow^{p_2} & & \\ M\times N & & M\tilde{\times}N & \xrightarrow{m} & P \end{array}$$

with the usual properties that will be made precise later.

7.2. LINEAR CASE

- The functor $M \tilde{\times} N$ associates to each $\mathcal{O}$-algebra R the set of pairs $(\mathcal{L}_\bullet, \mathcal{L}'_\bullet)$

  $$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d = (t+\varpi)^{-1}\mathcal{L}_0),$$

  $$\mathcal{L}'_\bullet = (\mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_d = (t+\varpi)^{-1}\mathcal{L}'_0),$$

  where $\mathcal{L}_I$, $\mathcal{L}'_I$ are $R[t]$-submodules of $R[t, t^{-1}, t(+\varpi)^{-1}]^d$ satisfying the following conditions

  $$t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^{n_-}\mathcal{V}_{i,R},$$

  $$(t+\varpi)^{n'_+}\mathcal{L}_i \subset \mathcal{L}'_i \subset (t+\varpi)^{n'_-}\mathcal{L}_i.$$

  As usual, $\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}$ is supposed to be locally a direct factor of $t^{n_-}\mathcal{V}_{i,R}/t^{n_+}\mathcal{V}_{i,R}$, and $\mathcal{L}'_i/(t+\varpi)^{n'_+}\mathcal{L}_i$ locally a direct factor of $(t+\varpi)^{n'_-}\mathcal{L}_i/(t+\varpi)^{n'_+}\mathcal{L}_i$ as R-modules. The ranks of the projective R-modules $\mathcal{L}_i/t^{n_+}$ and $\mathcal{L}_{i,R}$ and $\mathcal{L}'_i/(t+\varpi)^{n'_+}\mathcal{L}_i$ are each also supposed to be independent of i. It follows from the above conditions that

  $$t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R} \subset \mathcal{L}'_i \subset t^{n_-}(t+\varpi)^{n'_-}\mathcal{V}_{i,R}$$

and $\mathcal{L}'_i/t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R}$ is locally a direct factor of $t^{n_-}(t+\varpi)^{n'_-}\mathcal{V}_{i,R}/t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R}$ as an R-module. Thus defined the functor $M \tilde{\times} N$ is represented by a projective scheme over S.

- The functor P associates to each $\mathcal{O}$-algebra R the set of chains $\mathcal{L}'_\bullet$

  $$\mathcal{L}'_\bullet = (\mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_d = (t+\varpi)^{-1}\mathcal{L}'_0),$$

  where $\mathcal{L}'_i$ are R[t]-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^d$ satisfying

  $$t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R} \subset \mathcal{L}'_i \subset t^{n_-}(t+\varpi)^{n_-}\mathcal{V}_{i,R}$$

  and the usual conditions 'locally a direct factor as R-modules'. As above, $(\mathrm{k}_R(\mathcal{L}'_i/t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R})$ is supposed to be independent of i. Obviously, this functor is represented by a projective scheme over S.
- The forgetting map $m(\mathcal{L}_\bullet, \mathcal{L}'_\bullet) = \mathcal{L}'_\bullet$ yields a morphism m: $M \tilde{\times} N \to P$. This map is defined: it suffices to note that $t^{n_-}(t+\varpi)^{n'_-}\mathcal{V}_{i,R}/\mathcal{L}'_i$ is locally free as an R-module, being an extension of $t^{n_-}\mathcal{V}_{i,R}/\mathcal{L}_i$ by $(t+\varpi)^{n'_-}\mathcal{L}_i/\mathcal{L}'_i$ each of which is locally free. Clearly, this morphism is a proper morphism because it source and its target are proper schemes over S.

Now before we can construct the schemes $\tilde{M}$, $\tilde{N}$, and the remaining morphisms in the convolution diagram, we need the following simple remark.

LEMMA 18. *The function which associates to each $\mathcal{O}$-algebra R the set of matrices $g \in \mathfrak{gl}_s(R)$ such that the image of g: $R^s \to R^s$ is locally a direct factor of rank r of R^s is representable by a locally closed subscheme of $\mathfrak{gl}_s$.*

Proof. For $1 \leqslant i \leqslant s$, denote by St_i the closed subscheme of $\mathfrak{gl}_s$ defined by the equations: all minors of order at least $i+1$ vanish. By using Nakayama's lemma, one can see easily that the above functor is represented by the quasi-affine, locally closed subscheme $\mathrm{St}_r - \mathrm{St}_{r\text{-}1}$ of $\mathfrak{gl}_s$. □

Now let $\bar{\mathcal{V}}_0 \subset \bar{\mathcal{V}}_1 \subset \cdots$ be the image of $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots$ in the quotient

$$\bar{\mathcal{V}} = t^{n_-}(t+\varpi)^{n'_- - 1}\mathcal{O}[t]^d/t^{n_+}(t+\varpi)^{n'_+}\mathcal{O}[t]^d.$$

Let $\bar{\mathcal{L}}_0 \subset \bar{\mathcal{L}}_1 \subset \cdots$ be the image of $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots$ in the quotient $\bar{\mathcal{V}}_R = \bar{\mathcal{V}} \otimes_{\mathcal{O}} R$. Because $\mathcal{L}_i$ is completely determined by $\bar{\mathcal{L}}_i$, we can write $\bar{\mathcal{L}}_\bullet \in M(R)$ for $\mathcal{L}_\bullet \in M(R)$ and so on.

- We consider the functor $\tilde{M}$ which associates to each $\mathcal{O}$-algebra R the set of R[t]-endomorphisms $g \in \mathrm{End}\,(\bar{\mathcal{V}}_R)$ such that if $\bar{\mathcal{L}}_i = g(t^{n_-}\bar{\mathcal{V}}_i)$ then

  $$t^{n_+}\bar{\mathcal{V}}_{i,R} \subset \bar{\mathcal{L}}_i \subset t^{n_-}\bar{\mathcal{V}}_{i,R}$$

  and $\bar{\mathcal{L}}_i/t^{n_+}\bar{\mathcal{V}}_{i,R}$ is locally a direct factor of $t^{n_-}\bar{\mathcal{V}}_{i,R}/t^{n_+}\bar{\mathcal{V}}_{i,R}$, of the same rank, for all $i = 0, \ldots, d-1$. Using Lemma 18 one sees this functor is representable and comes naturally with a morphism $p : \tilde{M} \to M$.

- In a totally analogous way, we consider the functor $\tilde{N}$ which associates to each $\mathcal{O}$-algebra R the set of R[t]-endomorphisms $g \in$ End $(\bar{\mathcal{V}}_R)$ such that if $\bar{\mathcal{L}}_i = g((t+\varpi)^{n'_-}\bar{\mathcal{V}}_{i,R})$ then

  $$(t+\varpi)^{n'_+}\bar{\mathcal{V}}_{i,R} \subset \bar{\mathcal{L}}_i \subset (t+\varpi)^{n'_-}\bar{\mathcal{V}}_{i,R}$$

  and $\bar{\mathcal{L}}_i/(t+\varpi)^{n'_+}\bar{\mathcal{V}}_{i,R}$ is locally a direct factor of $(t+\varpi)^{n'_-}\bar{\mathcal{V}}_{i,R}/(t+\varpi)^{n'_+}\bar{\mathcal{V}}_{i,R}$, of the same rank for all $i = 0, \ldots, d-1$. As above, the representability follows from Lemma 18. This functor comes naturally with a morphism p: $\tilde{N} \to N$.
- Now we define the morphism p_1: $\tilde{M} \times \tilde{N} \to M \times N$ by $p_1 = p \times p'$.
- We define the morphism p_2: $\tilde{M} \times \tilde{N} \to M \tilde{\times} N$ by $p_2(g, g') = (\mathcal{L}_\bullet, \mathcal{L}'_\bullet)$ with

  $$(\mathcal{L}_\bullet, \mathcal{L}'_\bullet) = (g(t^{n_-}\mathcal{V}_\bullet), gg'(t^{n_-}(t+\varpi)^{n'_-}\mathcal{V}_\bullet)).$$

We have now achieved the construction of the convolution diagram. We need to prove some facts related to this diagram.

LEMMA 19. *The morphisms p_1 and p_2 are smooth and surjective. Their restrictions to connected components of $\tilde{M} \times \tilde{N}$ with image in the corresponding connected components of $M \times N$ and of $M \tilde{\times} N$, have the same relative dimensions.*

Proof. The proof is very similar to that of Lemma 3. Let us note that the morphism p: $\tilde{M} \to M$ can be factored as $p = f \circ j$ where j: $\tilde{M} \to U$ is an open immersion and f: $U \to M$ is the vector bundle defined as follows. For any $\mathcal{O}$-algebra R and any $\mathcal{L}_\bullet \in M(R)$, the fiber of U over $\mathcal{L}_\bullet$ is the R-module

$$U(\mathcal{L}_\bullet) = \bigoplus_{i=0}^{d-1}(t+\varpi)^{n'_-}\mathcal{L}_i/t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R}.$$

The morphisms p', p_1 and p_2 can be described in the same manner. The equality of relative dimensions of p_1 and p_2 follows from Lemma 24 (proved in Section 8) and the fact that they are each smooth. □

Just as in Subsection 2.2, we can consider the group valued functor $\tilde{J}$ which associates to each $\mathcal{O}$-algebra R the group of $R[t]$-linear automorphisms of $\bar{\mathcal{V}}_R$ which fix the filtration $\bar{\mathcal{V}}_0 \subset \bar{\mathcal{V}}_1 \subset \cdots \subset \bar{\mathcal{V}}_d$. Obviously, this functor is represented by a connected affine algebraic group scheme over S. The same proof as that of Lemma 3 proves that $\tilde{J}$ is smooth over S. Moreover, there are canonical morphisms of S-group schemes $\tilde{J} \to J$ and $\tilde{J} \to I$, where $J = J_{n\pm}$ (resp. $I = I_{n'_\pm}$) is the group scheme defined in Subsection 2.2 (resp. 6.1).

- We consider the action α_1 of $\tilde{J} \times \tilde{J}$ on $\tilde{M} \times \tilde{N}$ defined by

  $$\alpha_1(h, h'; g, g') = (gh^{-1}, g'h'^{-1}).$$

  Clearly, this action leaves stable the fibers of $p_1 : \tilde{M} \times \tilde{N} \to M \times N$.

- We also consider the action α_2 of $\tilde{J} \times \tilde{J}$ on the same $\tilde{M} \times \tilde{N}$ defined by

  $\alpha_2(h, h'; g, g') = (gh^{-1}, hg'h'^{-1}).$

  Clearly, this action leaves stable the fibers of $p_2 : \tilde{M} \times \tilde{N} \to M \tilde{\times} N$.

LEMMA 20. (i) *The action α_1, respectively α_2, is transitive on all geometric fibers of p_1, respectively p_2. The geometric fibres of p_1, respectively p_2, are therefore connected.*

(ii) *Moreover, the stabilizer under the action α_1, respectively α_2, of any geometric point is a smooth connected subgroup of $\tilde{J} \times \tilde{J}$.*

Proof. Let E be a (separably closed) field containing the fraction field F of $\mathcal{O}$ or its residue field k. Let g, g' be elements of $\tilde{M}(E)$ such that

$\mathcal{L}_\bullet = p(g) = p(g') \in M(E).$

For all $i = 0, \ldots, d-1$, denote by $\hat{\mathcal{V}}_i$ and $\hat{\mathcal{L}}_i$ the tensors

$$\hat{\mathcal{V}}_i = \mathcal{V}_i \otimes_{\mathcal{O}[t]} E[t]_{(t(t+\varpi))}, \qquad \hat{\mathcal{L}}_i = \mathcal{L}_i \otimes_{E[t]} E[t]_{(t(t+\varpi))}$$

where $E[t]_{(t(t+\varpi))}$ is the localized ring of $E[t]$ at the ideal $(t(t+\varpi))$, i.e., the ring $\mathcal{S}^{-1}E[t]$ where $\mathcal{S} = E[t] - \{(t) \cup (t+\varpi)\}$; this is a semi-local ring. Of course, we can consider the modules $\hat{\mathcal{V}}_i$ and $\hat{\mathcal{L}}_i$ as $E[t]_{(t(t+\varpi))}$-submodules of $E(t)^d$.

Clearly, we have an isomorphism

$$\hat{\mathcal{V}}_E = t^{n_-}(t+\varpi)^{n'_- -1}\hat{\mathcal{V}}_0 / t^{n_+}(t+\varpi)^{n'_+}\hat{\mathcal{V}}_0$$

so that $E[t]$-endomorphisms of $\bar{\mathcal{V}}_E$ are the same as $E[t]_{(t(t+\varpi))}$-endomorphisms of $\hat{\mathcal{V}}_0$ taken modulo $t^{n_+ - n_-}(t+\varpi)^{n'_+ - n'_- +1}$.

By using the Nakayama lemma, g and g' can be lifted to $\hat{g}, \hat{g}' \in \mathrm{GL}(d, E(t))$ such that

$$\hat{\mathcal{L}}_i = \hat{g}t^{n_-}\hat{\mathcal{V}}_i; \qquad \hat{\mathcal{L}}_i = \hat{g}'t^{n_-}\hat{\mathcal{V}}_i.$$

This of course induces $\hat{h}\bar{\mathcal{V}}_i = \bar{\mathcal{V}}_i$ with $\bar{h} = \hat{g}^{-1}\hat{g}'$ and for all $i = 0, \ldots, d-1$.

Let h be the reduction modulo $t^{n_+ - n_-}(t+\varpi)^{n'_+ - n'_- +1}$ of $\hat{h}$. It is clear that $g' = gh$ and h lies in $\tilde{J}(E)$.

We have proved that $\tilde{J}$ acts transitively on the geometric fibres of $\tilde{M} \to M$. We can prove in a completely similar way that $\tilde{J}$ acts transitively on the geometric fibers of $\tilde{N} \to N$. Consequently, the action α_1 is transitive on the geometric fibers of p_1.

The proof of the statement for α_2 and p_2 is similar. This completes the proof of (i).

For (ii), let $\mathcal{L}_\bullet \in M(E)$ and take $g \in \tilde{M}(E)$ over $\mathcal{L}_\bullet$. We have to look at the points $h \in \tilde{J}(E)$ such that $gh = g$ as endomorphisms of $\bar{\mathcal{V}}_E$. Let $e_1, \ldots, e_d$ be the standard generators of $\bar{\mathcal{V}}_E$ so that the image in $\bar{\mathcal{V}}_E$ of $t^{n_-}(t+\varpi)^{n'_-}\mathcal{V}_i$ is generated by $e_1, \ldots, e_i, (t+\varpi)e_{i+1}, \ldots, (t+\varpi)e_n$. Let us denote that image by $t^{n_-}(t+\varpi)^{n'_-}\bar{\mathcal{V}}_i$. Now $gh = g$ if and only if $h(e_i) - e_i$ belongs to $\mathrm{Ker}(g)$ for all $i = 1, \ldots, d$. The condition $h \in \tilde{J}(E)$ says that $h(e_i)$ lies in the submodule generated by $t^{n_-}(t+\varpi)^{n'_-}\bar{\mathcal{V}}_i$ and h is invertible. The dimension of the E-vector space $\mathrm{Ker}(g) \cap t^{n_-}(t+\varpi)^{n'_-}\bar{\mathcal{V}}_i$ depends only on $\mathcal{L}_\bullet$ and it is constant along each connected of M as the dimension of $g(t^{n_-}(t+\varpi)^{n'_-} - \bar{\mathcal{V}}_i) = (t+\varpi)^{n'_-}\bar{\mathcal{L}}_i$ is. This proves that the stabilizer group scheme of $\tilde{J}$ acting on M (i.e., the subscheme of $\tilde{J} \times \tilde{M}$ on which the action and projection morphisms a, $\mathrm{pr}_2 : \tilde{J} \times \tilde{M} \to \tilde{M}$ agree) is an open subscheme of a vector bundle over $\tilde{M}$. This shows that the stabilizer of a single point $g \in \tilde{M}(E)$ is a connected smooth subgroup.

The same proof works for the actions α_1 and α_2. □

We remark that Lemmas 19 and 20 are essential for the construction of the convolution of perverse sheaves, discussed in Section 7.4.

The symmetric construction yields the following diagram

$$\begin{array}{ccccc} & \tilde{N} \times \tilde{M} & & & \\ {}^{p'_1}\swarrow & & \searrow^{p'_2} & & \\ N \times M & & N \tilde{\times} M & \xrightarrow{m'} & P \end{array}$$

enjoying the same structures and properties. More precisely, we define $N \tilde{\times} M$ as follows: for each $\mathcal{O}$-algebra R, let $(N \tilde{\times} M)(R)$ be the set of pairs $(\mathcal{L}'_\bullet, \mathcal{L}_\bullet)$

$$\mathcal{L}'_\bullet = (\mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_d = (t+\varpi)^{-1}\mathcal{L}'_0),$$
$$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d = (t+\varpi)^{-1}\mathcal{L}_0),$$

where $\mathcal{L}'_i$, $\mathcal{L}_i$ are $R[t]$-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^d$ satisfying the following conditions

$$(t+\varpi)^{n'_+}\mathcal{V}_{i,R} \subset \mathcal{L}'_i \subset (t+\varpi)^{n'_-}\mathcal{V}_{i,R}$$
$$t^{n_+}\mathcal{L}'_i \subset \mathcal{L}_i \subset t^{n_-}\mathcal{L}'_i$$

such that for each $i = 0, \ldots, d-1$, the R-module $\mathcal{L}'_i/(t+\varpi)^{n'_+}\mathcal{V}_{i,R}$ is locally a direct factor of $(t+\varpi)^{n'_-}\mathcal{V}_{i,R}/(t+\varpi)^{n'_+}\mathcal{V}_{i,R}$, and the R-module $\mathcal{L}_i/t^{n_+}\mathcal{L}'_i$ is locally a direct factor of $t^{n_-}\mathcal{L}'_i/t^{n_+}\mathcal{L}'_i$. It is also supposed that $\mathrm{rk}_R(\mathcal{L}'_i/(t+\varpi)^{n'_+}\mathcal{V}_{i,R})$ and $\mathrm{rk}_R(\mathcal{L}_i/t^{n_+}\mathcal{L}'_i)$ are independent of i.

The morphisms p'_1, p'_2, and m' are defined in the obvious way: $p'_1 = p' \times p$, $m'(\mathcal{L}'_\bullet, \mathcal{L}_\bullet) = \mathcal{L}_\bullet$, and $p'_2(g', g) = (g'(t+\varpi)^{n'_-}\mathcal{V}_{i,R}, g'g(t^{n_-}(t+\varpi)^{n'_-})\mathcal{V}_{i,R})$.

7.3. SYMPLECTIC CASE

In this section we construct the symplectic analogue of the convolution diagram just discussed. In particular we need to define the schemes $M \tilde{\times} N$, $\tilde{M}$, $\tilde{N}$, P, and the morphisms p_1, p_2, and m. Moreover, we need to construct the smooth group scheme $\tilde{J}$ which acts on the whole convolution diagram. Once this is done, defining the symplectic analogues of the actions α_1 and α_2, proving the symplectic analogues of Lemmas 19 and 20, and defining the symmetric construction are all straightforward tasks and will be left to the reader.

- The functor $M\tilde{\times}N$ associates to each $\mathcal{O}$-algebra R the set of pairs $(\mathcal{L}_\bullet, \mathcal{L}'_\bullet)$

  $$\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_d),$$
  $$\mathcal{L}'_\bullet = (\mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_d),$$

  where $\mathcal{L}_i, \mathcal{L}'_i$ are $R[t]$-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^{2d}$ satisfying the following conditions

  $$t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^{n_-}\mathcal{V}_{i,R},$$
  $$(t+\varpi)^{n'_+}\mathcal{L}_i \subset \mathcal{L}'_i \subset (t+\varpi)^{n'_-}\mathcal{L}_i,$$

  satisfying the usual 'locally direct factors as R-modules' conditions: $\mathcal{L}_i/t^{n_+}\mathcal{V}_{i,R}$ is locally a direct factor of $t^{n_-}\mathcal{V}_{i,R}/t^{n_+}\mathcal{V}_{i,R}$ of rank $(n_+ - n_-)d$ and $\mathcal{L}'_i/(t+\varpi)^{n'_+}\mathcal{L}_i$ is locally a direct factor of $(t+\varpi)^{n'_-}\mathcal{L}_i/(t+\varpi)^{n'_+}\mathcal{L}_i$ of rank $(n'_+ - n'_-)d$. Moreover we suppose $\mathcal{L}_0, \mathcal{L}_d, \mathcal{L}'_0$ and $\mathcal{L}'_d$ are autodual with respect to $t^{-n_- - n_+}\langle\, ,\rangle$, $t^{-n_- - n_+}(t+\varpi)\langle\, ,\rangle$, $t^{-n_- - n_+}(t+\varpi)^{-n'_- - n'_+}\langle\, ,\rangle$ and $t^{-n_- - n_+}(t+\varpi)^{-n'_- - n'_+ + 1}\langle\, ,\rangle$ respectively.
- The functor P associates to each $\mathcal{O}$-algebra R the set of chains $\mathcal{L}'_\bullet$

  $$\mathcal{L}'_\bullet = (\mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_d)$$

  where $\mathcal{L}'_i$ are $R[t]$-submodules of $R[t, t^{-1}, (t+\varpi)^{-1}]^{2d}$ satisfying

  $$t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{i,R} \subset \mathcal{L}'_i \subset t^{n_-}(t+\varpi)^{n'_-}\mathcal{V}_{i,R},$$

  such that the usual 'locally a direct factor as R-modules of rank $(n_+ - n_- + n'_+ - n'_-)d$' condition holds, and such that $\mathcal{L}'_0$ and $\mathcal{L}'_d$ are autodual with respect to $t^{-n_- - n_+}(t+\varpi)^{-n'_- - n'_+}\langle\, ,\rangle$ and $t^{-n_- - n_+}(t+\varpi)^{-n'_- - n'_+ + 1}\langle\, ,\rangle$ respectively.
- The forgetting map $m(\mathcal{L}_\bullet, \mathcal{L}'_\bullet) = \mathcal{L}'_\bullet$ yields a morphism m: $M\tilde{\times}N \to P$. Clearly, m is a proper morphism between proper S-schemes.
- We consider the functor $\tilde{M}$ which associates to each $\mathcal{O}$-algebra R the set of $R[t]$-endomorphisms g of

  $$\bar{\mathcal{V}}_R = t^{n_-}(t+\varpi)^{n'_- - 1}\mathcal{V}_{0,R}/t^{n_+}(t+\varpi)^{n'_+}\mathcal{V}_{0,R}$$

  satisfying

  $$\langle gx, gy\rangle = c_g t^{n_+ - n_-}\langle x, y\rangle$$

  for some $c_g \in R^\times$, and such that if $\bar{\mathcal{L}}_i = g(t^{n_-}\bar{\mathcal{V}}_i)$ for $i = 0, \ldots, d$, then we have $t^{n_+}\bar{\mathcal{V}}_{i,R} \subset \bar{\mathcal{L}}_i \subset t^{n_-}\bar{\mathcal{V}}_{i,R}$, and $\bar{\mathcal{L}}_i/t^{n_+}\bar{\mathcal{V}}_{i,R}$ is locally a direct factor of $t^{n_-}\bar{\mathcal{V}}_{i,R}/t^{n_+}\bar{\mathcal{V}}_{i,R}$ of rank $(n_+ - n_-)d$. If $g \in \tilde{M}(R)$ then one sees using the definitions that automatically, $\bar{\mathcal{L}}_\bullet = gt^{n_-}\mathcal{V}_{\bullet,R} \in M(R)$. The functor $\tilde{M}$ is representable and comes naturally with a morphism p: $\tilde{M} \to M$.
- Next consider the functor $\tilde{N}$ which associates to each $\mathcal{O}$-algebra R the set of $R[t]$-endomorphisms g of $\bar{\mathcal{V}}_R$ satisfying $\langle gx, gy\rangle = c_g(t+\varpi)^{n'_+ - n'_-}\langle x, y\rangle$ for some $c_g \in R^\times$ and such that if $\bar{\mathcal{L}}'_i = g(t+\varpi)^{n'_-}\bar{\mathcal{V}}_{i,R}$ for $i = 0, \ldots, d$ then we have

  $$(t+\varpi)^{n'_+}\bar{\mathcal{V}}_{i,R} \subset \bar{\mathcal{L}}'_i \subset (t+\varpi)^{n'_-}\bar{\mathcal{V}}_{i,R},$$

and $\bar{\mathcal{L}}_i'/(t+\varpi)^{n'_+}\bar{\mathcal{V}}_{i,R}$ is locally a direct factor of $(t+\varpi)^{n'_-}\bar{\mathcal{V}}_{i,R}/(t+\varpi)^{n'_+}\bar{\mathcal{V}}_{i,R}$ of rank $(n'_+ - n'_-)d$. From the definitions one sees that $\bar{\mathcal{L}}_\bullet' \in N(R)$. The functor $\tilde{N}$ is representable and comes with a morphism p: $N \to N$.

- We define $p_1 = p \times p'$. We define p_2: $\tilde{M} \times \tilde{N} \to M \tilde{\times} N$ exactly as in the linear case.
- We let $\tilde{J}$ denote the functor which associates to any $\mathcal{O}$-algebra R the group of $R[t]$-linear automorphisms of $\bar{\mathcal{V}}_R$ which fix the form $t^{-n_- - n_+}(t + \varpi^{-n'_- - n'_+ + 1}\langle\, ,\rangle$ up to an element in $R^\times$ and which fix the filtration $\bar{\mathcal{V}}_{i,R}$. As in Lemma 3, the group scheme $\tilde{J}$ is smooth over S. There are canonical S-group scheme morphisms $\tilde{J} \to J$ and $\tilde{J} \to I$, where $J = J_{n_\pm}$ (resp. $I = I_{n'_\pm}$) was defined in Subsection 2.5 (resp. 6.2).

7.4. DEFINITION OF THE CONVOLUTION PRODUCT

Let us recall the standard definition of convolution product due to Lusztig [15] (see also [6] and [16]).

Let E be a field containing the fraction field F of $\mathcal{O}$ or its residue field k and let $\epsilon = \mathrm{Spec}(E) \to S$ be the corresponding morphisms. For an S-scheme X, let X_ϵ denote the base change $X \times_S \epsilon$.

Let $\mathcal{A}$ be a perverse sheaf over M_ϵ that is J_ϵ-equivariant. Let $\mathcal{I}$ be a perverse sheaf over N_ϵ that is I_ϵ-equivariant. Both I_ϵ and J_ϵ are quotients of $\tilde{J}_\epsilon$, so we can say that $\mathcal{A}$ and $\mathcal{I}$ are $\tilde{J}_\epsilon$-equivariant.

Since p_1 is a smooth morphism, the pull-back $p_1^*(\mathcal{A} \boxtimes_\epsilon \mathcal{I})$ is also perverse up to the shift by the relative dimension of p_1. A priori, this pull-back is only α_1-equivariant. As $\mathcal{A}$ and $\mathcal{I}$ are $\tilde{J}_\epsilon$-equivariant, $p_1^*(\mathcal{A} \boxtimes_\epsilon \mathcal{I})$ is also α_2-equivariant. Since p_2 is smooth and the action α_2 is transitive on its geometric fibers, the perverse sheaf $\mathcal{F} = p_1^*(\mathcal{A} \boxtimes_\epsilon \mathcal{I})$ is constant along the fibers of p_2. Moreover, the stabilizers for α_2 of geometric points are smooth and connected. Under these hypotheses there exists a perverse sheaf $\mathcal{A} \tilde{\boxtimes}_\epsilon \mathcal{I}$, unique up to unique isomorphism, such that $p_1^*(\mathcal{A} \boxtimes_\epsilon \mathcal{I}) = p_2^*(\mathcal{A} \tilde{\boxtimes}_\epsilon \mathcal{I})$. The uniqueness follows from Proposition 4.2.5 of [1] which only requires that p_2 is smooth and its geometric fibers are connected.

To prove the existence of the perverse sheaf $\mathcal{A} \tilde{\boxtimes}_\epsilon \mathcal{I}$ we need the transitive group action on the fiber of p_2, the fact that p_2 is smooth and surjective, and the fact that the action α_2 has smooth connected stabilizers. We make use of the following general lemma.

LEMMA 21. *Suppose π: $X \to Y$ is a smooth surjective morphism of S-schemes, and suppose G_Y is a smooth connected Y-group scheme which acts trivially on Y and on X such that the action on each geometric fiber of π is transitive. Assume further that the stabilizer in G_Y of any geometric point of X is a smooth connected subgroup. Then a G_Y-equivariant perverse sheaf $\mathcal{F}$ on X descends along π.*

Proof. Assume temporarily that π possesses a section s. Then the action map and the section s give rise to a morphism a: $G_Y \to X$, which is smooth and surjective with geometrically connected fibers. Using the equivariance it follows that $a^*\pi^*s^*\mathcal{F} = a^*\mathcal{F}$. Since a and $\pi \circ a$ are both smooth with geometrically connected fibers, this implies that $s^*\mathcal{F}$ is perverse up to the shift by the relative dimension of π. Indeed, since the other perverse cohomologies of $s^*\mathcal{F}$ are killed by $a^*\pi^*$, they must be zero since this functor is fully faithful, by [1] 4.2.5. By applying the same proposition for a^* now, we obtain an isomorphism between perverse sheaves $\pi^*s^*\mathcal{F} = \mathcal{F}$.

In the general case there is an étale covering $U_i \to Y$ such that each π_i: $X \times_Y U_i \to U_i$ has a section. Using the group action of $G_Y \times_Y U_i$, the previous discussion shows that étale locally $\mathcal{F}$ descends along π. Using 4.2.5 of loc.cit. again to descend the gluing data and using Theorem 3.2.4 of loc.cit. to glue perverse sheaves, we see that $\mathcal{F}$ descends along π globally. □

By Lemmas 19 and 20, the morphism p_2 satisfies the hypotheses on π in Lemma 21, with $G_Y = (\tilde{J}_\epsilon \times \tilde{J}_\epsilon) \times (M \tilde{\times} N)$ acting via α_2. We have thus proved the existence of $\mathcal{A} \tilde{\boxtimes}_\epsilon \mathcal{I}$. Note that no shift is needed because on each connected component, p_1 and p_2 have the same relative dimension by Lemma 19.

Now set $\mathcal{A} *_\epsilon \mathcal{I} = \mathrm{R}m_*(\mathcal{A} \tilde{\boxtimes}_\epsilon \mathcal{I})$. By the symmetric construction, we can define the convolution product $\mathcal{I} *_\epsilon \mathcal{A}$.

Let E be now the algebraic closure $\bar{k}$ of the residual field k. We suppose that the peverse sheaves $\mathcal{A}$ and $\mathcal{I}$ are equipped with an action of $\mathrm{Gal}(\bar{F}/F)$ compatible with the action of $\mathrm{Gal}(\bar{F}/F)$ on the geometric special fiber through $\mathrm{Gal}(\bar{k}/k)$. In practice, the inertia subgroup Γ_0 acts trivially on $\mathcal{I}$ and nontrivially on $\mathcal{A}$. As the semi-simple trace provides a sheaf-function dictionary, we have:

$$\tau^{ss}_{\mathcal{A}} * \tau^{ss}_{\mathcal{I}} = \tau^{ss}_{\mathcal{A} *_{\bar{s}} \mathcal{I}}, \qquad \tau^{ss}_{\mathcal{I}} * \tau^{ss}_{\mathcal{A}} = \tau^{ss}_{\mathcal{I} *_{\bar{s}} \mathcal{A}}$$

where the convolution on the left hand is the ordinary convolution in the Hecke algebra $\mathcal{H}(G_k//I_k)$. For the convenience of the reader let us recall the argument for this rather standard statement.

Recall that the convolution product $f * f'$ of two functions in $\mathcal{H}(G_k//I_k)$ can be defined by

$$(f * f')(x) = \sum_{y \in G_k/I_k} f(y) f'(y^{-1}x)$$

for all $x \in G_k/I_k$. The right hand side is well defined since f' is bi-I_k-invariant. If $f = \tau^{ss}_{\mathcal{A}}$ and $f' = \tau^{ss}_{\mathcal{I}}$, one can check the above sum is exactly the summation along the fiber of m of the semi-simple trace function associated to the perverse sheaf $\mathcal{A} \tilde{\boxtimes}_{\mathrm{Spec}(k)} \mathcal{I}$. The compatibility between the ordinary convolution and the geometric convolution now follows from this, since the semi-simple trace behaves well with respect to a proper push-forward, as was made explicit in Section 3.

8. Proof of Proposition 13

8.1. COHOMOLOGICAL PART

According to the sheaf-function dictionary for semi-simple traces, it suffices to prove the following statement. Beilinson and Gaitsgory have proved a related result in the equal characteristic case, using a deformation of the affine Grassmannian of G, see [5].

PROPOSITION 22. *We have an isomorphism*

$$\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) *_{\bar{s}} \mathcal{I}_{w,\bar{s}} \xrightarrow{\sim} \mathcal{I}_{w,\bar{s}} *_{\bar{s}} \mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}).$$

Proof. The above statement makes sense because the functor $\mathrm{R}\Psi$ sends perverse sheaves to perverse sheaves, by a theorem of Gabber, see Corollary 4.5 in [11]. In particular, $\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta})$ is a perverse sheaf.

Let us recall that $\mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathcal{I}_{w,s}$ so that we have to prove

$$\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) *_{\bar{s}} \mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^M(\mathcal{I}_{w,\eta}) *_{\bar{s}} \mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}).$$

First, let us prove that nearby cycle commutes with convolution product.

LEMMA 23. *We have the isomorphisms*

$$\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) *_{\bar{s}} \mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^P(\mathcal{A}_{\lambda,\eta} *_{\eta} \mathcal{I}_{w,\eta})$$

$$\mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}) *_{\bar{s}} \mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^P(\mathcal{I}_{w,\eta} *_{\eta} \mathcal{A}_{\lambda,\eta})$$

Proof. According to a theorem of Beilinson-Bernstein (see Theorem 4.7 in [11]) we have an isomorphism of perverse sheaves

$$\mathrm{R}\Psi^{M\times N}(\mathcal{A}_{\lambda,\eta} \boxtimes_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) \boxtimes_{\bar{s}} \mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}).$$

This induces an isomorphism between the pull-backs

$$p_1^*\mathrm{R}\Psi^{M\times N}(\mathcal{A}_{\lambda,\eta} \boxtimes_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} p_1^*(\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) \boxtimes_{\bar{s}} \mathrm{R}\Psi^N(\mathcal{I}_{w,\eta}))$$

which are up to the shift by the relative dimension of p_1, perverse too. By definition, we have

$$p_1^*(\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) \boxtimes_{\bar{s}} \mathrm{R}\Psi^N(\mathcal{I}_{w,\eta})) \xrightarrow{\sim} p_2^*(\mathrm{R}\Psi^M(\mathcal{A}_{\lambda,\eta}) \tilde{\boxtimes}_{\bar{s}} \mathrm{R}\Psi^N(\mathcal{I}_{w,\eta})).$$

As p_1, p_2 are smooth, p_1^* and p_2^* commute with nearby cycle, so applying $\mathrm{R}\Psi^{\tilde{M}\times\tilde{N}}$ to

$$p_1^*(\mathcal{A}_{\lambda,\eta} \boxtimes_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} p_2^*(\mathcal{A}_{\lambda,\eta} \tilde{\boxtimes}_{\eta} \mathcal{I}_{w,\eta})$$

gives an isomorphism

$$p_1^*\mathrm{R}\Psi^{M\times N}(\mathcal{A}_{\lambda,\eta} \boxtimes_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} p_2^*\mathrm{R}\Psi^{M\tilde{\times}N}(\mathcal{A}_{\lambda,\eta} \tilde{\boxtimes}_{\eta} \mathcal{I}_{w,\eta}).$$

Since p_2 is smooth with connected geometric fibers, Proposition 4.2.5 of Beilinson-Bernstein-Deligne [1] implies that we have an isomorphism

$$\mathrm{R}\Psi^{M \tilde{\times} N}(\mathcal{A}_{\lambda,\eta} \tilde{\boxtimes}_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^{M}(\mathcal{A}_{\lambda,\eta}) \tilde{\boxtimes}_{\bar{s}} \mathrm{R}\Psi^{N}(\mathcal{I}_{w,\eta}).$$

By applying now the functor $\mathrm{R}m_*$, we have an isomorphism

$$\mathrm{R}m_*\mathrm{R}\Psi^{M \tilde{\times} N}(\mathcal{A}_{\lambda,\eta} \tilde{\boxtimes}_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^{M}(\mathcal{A}_{\lambda,\eta}) *_{\bar{s}} \mathrm{R}\Psi^{N}(\mathcal{I}_{w,\eta}).$$

Since the functor $\mathrm{R}\Psi$ commutes with the direct image of a proper morphism, we have

$$\mathrm{R}\Psi^{P}(\mathcal{A}_{\lambda,\eta} *_{\eta} \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}m_*\mathrm{R}\Psi^{M \tilde{\times} N}(\mathcal{A}_{\lambda,\eta} \tilde{\boxtimes}_{\eta} \mathcal{I}_{w,\eta}).$$

By composing the above isomorphisms, we get

$$\mathrm{R}\Psi^{M}(\mathcal{A}_{\lambda,\eta}) *_{\bar{s}} \mathrm{R}\Psi^{N}(\mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^{P}(\mathcal{A}_{\lambda,\eta} *_{\eta} \mathcal{I}_{w,\eta}).$$

By the same argument, we prove

$$\mathrm{R}\Psi^{N}(\mathcal{I}_{w,\eta}) *_{\bar{s}} \mathrm{R}\Psi^{M}(\mathcal{A}_{\lambda,\eta}) \xrightarrow{\sim} \mathrm{R}\Psi^{P}(\mathcal{I}_{w,\eta} *_{\eta} \mathcal{A}_{\lambda,\eta}).$$

This finishes the proof of the lemma. □

Now it clearly suffices to prove $\mathcal{A}_{\lambda,\eta} *_{\eta} \mathcal{I}_{w,\eta} \xrightarrow{\sim} \mathcal{I}_{w,\eta} *_{\eta} \mathcal{A}_{\lambda,\eta}$ which is an easy consequence of the following lemma.

LEMMA 24. (1) *Over the generic point η, we have two commutative triangles*

$$\begin{array}{ccc} & M_\eta \tilde{\times} N_\eta & \\ {}^{i}\swarrow & & \searrow^{m} \\ M_\eta \times N_\eta & \xrightarrow{\ \ j\ \ } & P_\eta \\ {}^{i'}\nwarrow & & \nearrow_{m'} \\ & N_\eta \tilde{\times} M_\eta & \end{array}$$

where all arrows are isomorphisms.

(2) *Moreover, we have the following isomorphisms*

$$i^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathcal{A}_{\lambda,\eta} \tilde{\boxtimes}\, \mathcal{I}_{w,\eta}, \qquad i'^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \mathcal{I}_{w,\eta} \tilde{\boxtimes}\, \mathcal{A}_{\lambda,\eta}$$

8.2. PROOF OF LEMMA 24

Let us prove the above lemma in the linear case.

Over the generic point η, we have the canonical decomposition of

$$\bar{\mathcal{V}}_F = t^{n_-}(t+\varpi)^{n'_- -1}F[t]^d / t^{n_+}(t+\varpi)^{n'_+}F[t]^d$$

into the direct sum $\bar{\mathcal{V}}_F = \bar{\mathcal{V}}_F^{(t)} \oplus \bar{\mathcal{V}}_F^{(t+\varpi)}$ where

$$\bar{\mathcal{V}}_F^{(t)} = t^{n_-}F[t]^d/t^{n_+}F[t]^d$$

$$\bar{\mathcal{V}}_F^{(t+\varpi)} = (t+\varpi)^{n'_- -1}F[t]^d/(t+\varpi)^{n'_+}F[t]^d.$$

With respect to this decomposition, all the terms of the filtration

$$\bar{\mathcal{V}}_0 \subset \bar{\mathcal{V}}_1 \subset \cdots \subset \bar{\mathcal{V}}_{d-1}$$

decompose to $\bar{\mathcal{V}}_i = \bar{\mathcal{V}}_i^{(t)} \oplus \bar{\mathcal{V}}_i^{(t+\varpi)}$ for all $i = 0, \ldots, d-1$. Here, we have

$$\bar{\mathcal{V}}_0^{(t)} = \cdots = \bar{\mathcal{V}}_{d-1}^{(t)} = F[t]^d/t^{n_+}F[t]^d.$$

Let R be an F-algebra and let $(\mathcal{L}_\bullet, \mathcal{L}'_\bullet)$ be an element of $(M \tilde{\times} N)(R)$. These chains of $R[t]$-modules verify

$$t^{n_+}\mathcal{V}_{i,R} \subset \mathcal{L}_i \subset t^{n_-}\mathcal{V}_{i,R}, \qquad (t+\varpi)^{n'_+}\mathcal{L}_i \subset \mathcal{L}'_i \subset (t+\varpi)^{n'_-}\mathcal{L}_i.$$

As usual, let $\bar{\mathcal{L}}_i, \bar{\mathcal{L}}'_i$ denote the image of $\mathcal{L}_i, \mathcal{L}'_i$ in $\bar{\mathcal{V}}_R$. As $R[t]$-modules, they decompose to $\bar{\mathcal{L}}_i = \bar{\mathcal{L}}_i^{(t)} \oplus \bar{\mathcal{L}}_i^{(t+\varpi)}$ and $\bar{\mathcal{L}}'_i = \bar{\mathcal{L}}_i^{(t)} \oplus \bar{\mathcal{L}}_i'^{(t+\varpi)}$. The above inclusion conditions imply indeed

$$\bar{\mathcal{L}}_i^{(t)} = \bar{\mathcal{L}}_i'^{(t)}; \quad \bar{\mathcal{L}}_i^{(t+\varpi)} = \bar{\mathcal{V}}_{i,R}^{(t+\varpi)}.$$

Consequently, $\mathcal{L}_\bullet$ is completely determined by $\mathcal{L}'_\bullet$. In other terms, the map $m(\bar{\mathcal{L}}_\bullet, \bar{\mathcal{L}}'_\bullet) = \bar{\mathcal{L}}'_\bullet$ is an isomorphism of functors over η. In the same way, the map

$$i(\bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}, \bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{L}}_\bullet'^{(t+\varpi)}) = (\bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}, \bar{\mathcal{V}}_{\bullet,R}^{(t)} \oplus \bar{\mathcal{L}}_\bullet'^{(t+\varpi)})$$

yields an isomorphism i: $M_\eta \tilde{\times} N_\eta \xrightarrow{\sim} M_\eta \times N_\eta$. The composed isomorphism $j = m \circ i^{-1}$ is given by

$$j(\bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}, \bar{\mathcal{V}}_{\bullet,R}^{(t)} \oplus \bar{\mathcal{L}}_\bullet'^{(t+\varpi)}) = \bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{L}}_\bullet'^{(t+\varpi)}.$$

The analogous statement for the lower triangle in the diagram can be proved in the same way and the first part of the lemma is proved.

By the very definition of $\mathcal{A}_{\lambda,\eta} \tilde{\boxtimes} \mathcal{I}_{w,\eta}$, in order to prove the second part of the lemma, it suffices to construct an isomorphism

$$p_1^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}) \xrightarrow{\sim} p_2^* i^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}).$$

In fact, the triangle

$$\begin{array}{ccc} & \tilde{M}_\eta \times \tilde{N}_\eta & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ M_\eta \times N_\eta & \xleftarrow{\;i\;} & M_\eta \tilde{\times} N_\eta \end{array}$$

does not commute. Nevertheless this lack of commutativity can be corrected by equivariance properties. We consider the diagram

$$\begin{array}{ccccc} & & \tilde{M}_\eta \times \tilde{N}_\eta & & \\ & {}^{q_1}\swarrow & & \searrow^{q_2} & \\ & \tilde{J}_\eta \times M_\eta \times N_\eta & \xleftarrow{\mathrm{Id}\times i} & \tilde{J}_\eta \times M_\eta \tilde{\times} N_\eta & \\ {}^{\mathrm{pr}_1}\swarrow & & \swarrow_{\mathrm{pr}_2} & & \searrow^{\alpha} \\ M_\eta \times N_\eta & \xleftarrow{\;\;i\;\;} & M_\eta \tilde{\times} N_\eta & & M_\eta \tilde{\times} N_\eta \end{array}$$

defined as follows.

For any F-algebra R, an element $g \in \tilde{M}(R)$ is an $R[t]$-endomorphism of $\bar{\mathcal{V}}_R$ such that $\bar{\mathcal{L}}_\bullet = g(t^{n_-}\bar{\mathcal{V}}_{\bullet,R}) \in M(R)$. As $\bar{\mathcal{V}}_R$ decomposes to $\bar{\mathcal{V}}_R = \bar{\mathcal{V}}_R^{(t)} \oplus \bar{\mathcal{V}}_R^{(t+\varpi)}$, its $R[t]$-endomorphism g can be identified to a pair $g = (g^{(t)}, g^{(t+\varpi)})$ where $g^{(t)}$, respectively $g^{(t+\varpi)}$, is an endomorphism of $\bar{\mathcal{V}}_R^{(t)}$, respectively of $\bar{\mathcal{V}}_R^{(t+\varpi)}$.

As we have seen above, for $\bar{\mathcal{L}}_\bullet \in M(R)$, we have $\bar{\mathcal{L}}_i = \bar{\mathcal{L}}_i^{(t)} \oplus \bar{\mathcal{L}}_i^{(t+\varpi)}$ with $\bar{\mathcal{L}}_i^{(t+\varpi)} = \bar{\mathcal{V}}_{i,R}^{(t+\varpi)}$. Consequently, $g^{(t+\varpi)}$ is an automorphism of $\bar{\mathcal{V}}_R^{(t+\varpi)}$ fixing the filtration $\bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}$. In a similar way, an element $g' \in \tilde{N}(R)$ can be identified with a pair $(g'^{(t)}, g'^{(t+\varpi)})$ where $g'^{(t)}$ is an automorphism of $\bar{\mathcal{V}}_R^{(t)}$ fixing the filtration $\bar{\mathcal{V}}_{\bullet,R}^{(t)}$.

- The morphism q_1 is defined by

$$q_1(g, g') = ((g'^{(t)}, g^{(t+\varpi)}), g^{(t)}t^{n_-}\bar{\mathcal{V}}_{\bullet,R}^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)},\\ \bar{\mathcal{V}}_{\bullet,R}^{(t)} \oplus g'^{(t+\varpi)}(t+\varpi)^{n'_-}\bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}).$$

- The morphism q_2 is defined by

$$q_2(g, g') = ((g'^{(t)}, g^{(t+\varpi)}), g^{(t)}t^{n_-}\bar{\mathcal{V}}_{\bullet,R}^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)},\\ g^{(t)}t^{n_-}\bar{\mathcal{V}}_{\bullet,R}^{(t)} \oplus g'^{(t+\varpi)}(t+\varpi)^{n'_-}\bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}).$$

- The morphism α is defined by

$$\alpha((g'^{(t)}, g^{(t+\varpi)}), \bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}, \bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{L}}_\bullet'^{(t+\varpi)})\\ = (\bar{\mathcal{L}}_\bullet^{(t)} \oplus \bar{\mathcal{V}}_{\bullet,R}^{(t+\varpi)}, \bar{\mathcal{L}}_\bullet^{(t)} \oplus g^{(t+\varpi)}\bar{\mathcal{L}}_\bullet'^{(t+\varpi)}).$$

- pr_1 and pr_2 are the obvious projections

We can easily check that this diagram commutes and that

$$\mathrm{pr}_1 \circ q_1 = p_1; \quad \alpha \circ q_2 = p_2.$$

Now it is clear that

$$p_1^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}) \xrightarrow{\sim} q_2^*\mathrm{pr}_2^* i^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}).$$

Moreover, by equivariant properties of $\mathcal{A}_\lambda$ and $\mathcal{I}_w$, we have

$$\mathrm{pr}_2^* i^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}) \xrightarrow{\sim} \alpha^* i^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}).$$

(Note that the group I_η acts on $M_\eta \tilde{\times} N_\eta$ by acting on the second factor of $M_\eta \times N_\eta \cong M_\eta \tilde{\times} N_\eta$ and α gives the corresponding action of $\tilde{J}_\eta$ via the projection $\tilde{J}_\eta \longrightarrow I_\eta$.) In putting these things together, we get the required isomorphism

$$p_1^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}) \xrightarrow{\sim} p_2^* i^*(\mathcal{A}_{\lambda,\eta} \boxtimes \mathcal{I}_{w,\eta}).$$

This finishes the proof of the lemma in the linear case.

In the symplectic case, let us mention that the F-vector space

$$t^{n_-}(t+\varpi)^{n'_- -1}F[t]^{2d}/t^{n_+}(t+\varpi)^{n'_+}F[t]^{2d}$$

equipped with the symplectic form $t^{-n_- -n_+}(t+\varpi)^{-n'_- -n'_+ +1}\langle\,,\rangle$ splits into the direct sum of two vector spaces

$$t^{n_-}F[t]^{2d}/t^{n_+}F[t]^{2d} \oplus (t+\varpi)^{n'_- -1}F[t]^{2d}/(t+\varpi)^{n'_+}F[t]^{2d}$$

equipped with symplectic forms $t^{-n_- -n_+}\langle\,,\rangle$ and $(t+\varpi)^{-n'_- -n'_+ +1}\langle\,,\rangle$ respectively. Further, note that $g \in \tilde{M}(R)$ decomposes as $g = (g^{(t)}, g^{(t+\varpi)})$ where $g^{(t)} \in \mathrm{Aut}_{R[t]}(t^{n_-}R[t]^{2d}/t^{n_+}R[t]^{2d})$ is such that $\langle g^{(t)}x, g^{(t)}y\rangle = c_{g^{(t)}}t^{-n_- +n_+}\langle x, y\rangle$ (for some $c_{g^{(t)}} \in R^\times$), and $g^{(t+\varpi)} \in \mathrm{Aut}_{R[t]}((t+\varpi)^{n'_- -1}R[t]^{2d}/(t+\varpi)^{n'_+}R[t]^{2d})$ is such that $\langle g^{(t+\varpi)}x, g^{(t+\varpi)}y\rangle = c_{g^{(t+\varpi)}}\langle x, y\rangle$ (for some $c_{g^{(t+\varpi)}} \in R^\times$). A similar decomposition $g' = (g'^{(t)}, g'^{(t+\varpi)})$ holds, and thus ones sees $(g'^{(t)}, g^{(t+\varpi)}) \in \tilde{J}(R)$. Thus, the maps q_1 and q_2 as defined above make sense in the symplectic case as well. The rest of the argument goes through without change as in the linear case.

This finishes the proof of Lemma 24. We have therefore finished the proof of Proposition 22, and thus Proposition 13 and Theorem 11 as well. □

9. The Parahoric Case

Similar results in the parahoric cases follow easily from the Iwahori case treated above.

Let G denote a split connected reductive over F, let K denote a special good maximal compact subgroup of $G(F)$, let I denote an Iwahori subgroup contained in K, and let P denote a parahoric subgroup with $I \subset P \subset K$. We have the corresponding Hecke algebras of $\bar{\mathbb{Q}}_\ell$-valued compactly supported bi-invariant functions $\mathcal{H}_1, \mathcal{H}_P$, and $\mathcal{H}_K$. Let us define the convolution on the Hecke algebras using the Haar measures such that I (resp. P, K) has measure 1. There is map between their centers $Z(\mathcal{H}_I) \longrightarrow Z(\mathcal{H}_P)$ given by $z \mapsto z^P = z * \mathbb{I}_P$. (This is an algebra isomorphism; see the Remark following Theorem 25.)

Let G be one of the groups $\mathrm{GL}(d)$ or $\mathrm{GSp}(2d)$, and return to the notation G_k, I_k, $\mathcal{H}(G_k//I_k)$, etc. of Section 4.2.

Assume we are in the linear case (the symplectic case is similar and will be omitted from this discussion). To a standard parahoric subgroup P_k with $I_k \subset P_k \subset K_k$ we can associate a set of integers $\{0 = i_0 < i_1 < \cdots < i_{p-1} < i_p = d\}$ such that P_k is the stabilizer in $G(k[[t]])$ of the 'standard' partial lattice chain $\mathcal{V}_\bullet^P = (\mathcal{V}_0 \subset \mathcal{V}_{i_1} \subset \cdots \subset \mathcal{V}_{i_p} = (t+\varpi)^{-1}\mathcal{V}_0)$.

It is easy to define the analogue $M^P_{r,n_\pm}$ of the local model $M_{r,n_\pm}$ (cf. Definitions 1 and 2) as a scheme whose points are lattice chains

$$\mathcal{L}_\bullet^P = (\mathcal{L}_0 \subset \mathcal{L}_{i_1} \subset \cdots \subset \mathcal{L}_{i_p} = (t+\varpi)^{-1}\mathcal{L}_0)$$

such that for every j, the lattice $\mathcal{L}_{i_j}$ is in a specified position relative to the lattices $t^{n_\pm}\mathcal{V}_{i_j}$.

The obvious forgetful functor defines a proper map $M_{r,n_\pm} \longrightarrow M^P_{r,n_\pm}$ which is an isomorphism over the generic fibers. By using the same argument as in Section 5.1 we obtain the following result.

THEOREM 25. *Let λ be a dominant coweight of $G = \mathrm{GL}(d)$ or $\mathrm{GSp}(2d)$, and let P be a standard parahoric subgroup of G. Then*

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\Psi^{M^P}(\mathcal{A}_{\lambda,\eta})) = (-1)^{2\langle\rho,\lambda\rangle} \sum_{\lambda' \leqslant \lambda} m_\lambda(\lambda') z^P_{\lambda'}.$$

Remark. Let W_P denote the parabolic subgroup of the Weyl group corresponding to P, and let W^P denote the set of *minimal* representatives for the cosets in $W_P\backslash W$. One can show using the theory of the Bernstein center that we have the following commutative diagram of algebra isomorphisms

$$\begin{array}{ccc} & \bar{\mathbb{Q}}_\ell[X_*]^W & \\ {\scriptstyle (-*\mathbb{I}_P)\circ\mathrm{Bern.}}\swarrow & & \nwarrow{\scriptstyle \mathrm{Sat.}} \\ Z(\mathcal{H}(G_k//P_k)) & \xrightarrow[-*\mathbb{I}_{I_k W^P I_k}]{} & \mathcal{H}(G_k//K_k). \end{array}$$

Therefore the right-hand side in Theorem 25 can be characterized as follows: it is the unique element in $Z(\mathcal{H}(G_k//P_k))$ such that the Satake transform of its image under $-*\mathbb{I}_{I_k W^P I_k}$ is equal to $(-1)^{2\langle\rho,\lambda\rangle}\chi_\lambda$. □

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