

Weierstrass preparation theorem and singularities in the space of non-degenerate arcs

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1 Introduction

It has been long expected that there exists a deep connection between singularities of certain arc spaces and harmonic analysis over nonarchimedean fields. For instance, certain functions appearing naturally in harmonic analysis can be interpreted as the function attached to the trace of the Frobenius operators on what should be the stalks of the intersection complex of certain arc spaces, see [2]. However, a proper foundation of a theory of perverse sheaves on arc spaces is still missing even though a recent work of Bouthier and Kazhdan [1] outlines a strategy for setting it up.

As the theory of perverse sheaves is originally built for schemes of finite type, the basic difficulty in extending it to arc spaces is that those spaces are almost always infinite dimensional. The first inroad into this new territory is made by Grinberg and Kazhdan who prove that the formal completion of the arc spaces at a point representing a non-degenerate arc is isomorphic to the formal completion of a scheme of finite type, augmented by infinitely many free formal variables, under the assumption that the base field is the field of complex numbers. This result is later improved by Drinfeld who prove it over an arbitrary base field.

Let us fix the notations in order to state Grinberg-Kazhdan-Drinfeld's theorem. Let k be a field. Let X be an affine k -scheme of finite type. For every $n \in \mathbb{N}$, we consider the space of n th jets on X representing the functor $R \mapsto \mathcal{L}_n X(R) = X(R[t]/t^{n+1})$ on the categories of k -algebras. The arc space of X is the limit of $\mathcal{L}_n X$ as $n \rightarrow \infty$:

$$\mathcal{L}X(R) = \varprojlim X(R[t]/t^{n+1}) = X(R[[t]]). \quad (1.1)$$

If $X = \text{Spec}(A_0)$ is an affine k -scheme, for every $n \in \mathbb{N}$, $\mathcal{L}_n X$ is represented by an affine k -scheme $\mathcal{L}_n X = \text{Spec}(A_n)$ then $\mathcal{L}X = \text{Spec}(A)$ where $A = \varinjlim A_n$ is the filtered colimit of A_n as $n \rightarrow \infty$.

If X is a smooth, then the space $\mathcal{L}_n X$ of n -th jets on X is also smooth. For every $n \in \mathbb{N}$, the transition morphism $\mathcal{L}_{n+1} X \rightarrow \mathcal{L}_n X$ is smooth and surjective. More precisely, the transition morphism $\mathcal{L}_{n+1} X \rightarrow \mathcal{L}_n X$ is a torsor under certain vector bundles over $\mathcal{L}_n X$ of rank equal the

dimension of X . If X is not smooth, the situation is much more complicated: the transition morphism $\mathcal{L}_{n+1}X \rightarrow \mathcal{L}_nX$ is neither smooth nor surjective.

Let X' be a smooth open subscheme of X and Z a closed subscheme of X complement of X' . We are mainly interested in the open subscheme of non-degenerate arcs whose k -points form the set

$$\mathcal{L}^\bullet X(k) = \mathcal{L}X(k) - \mathcal{L}Z(k).$$

The definition of non-degenerate arc space depends thus on the choice of a smooth open subscheme X' of X . Although we may take X' to be the smooth locus of X , it is often more convenient to keep the possibility of choosing X' smaller than the smooth locus of X . For instance, the case of the affine line $X = \mathbb{A}^1$ and $X' = \mathbb{G}_m$ will be of special interest for the study of the Weierstrass preparation theorem. Some care is in order to enunciate the functorial description of $\mathcal{L}^\bullet X$.

Definition 1.1. *The non-degenerate arc space of X relative to the smooth subscheme X' is the functor on the category of k -algebras which attaches to each k -algebra R the set $\mathcal{L}^\bullet X(R)$ consisting of maps $x : \text{Spec}(R[[t]]) \rightarrow X$ such that the projection $x^{-1}(X') \rightarrow \text{Spec}(R)$ is surjective.*

We can now state the theorem of Grinberg-Kazhdan and Drinfeld, [4], [3].

Theorem 1.2. *Let $x \in \mathcal{L}X(k)$ be a point $x : \text{Spec}(k[[t]]) \rightarrow X$ such that the restriction to $\text{Spec}(k((t)))$ has image in X' . Then the completion of $\mathcal{L}X$ at x has a finite type formal model i.e. there exists a k -scheme Y of finite type with a point $y \in Y(k)$ such that there exists an isomorphism*

$$(\mathcal{L}X)_x^\wedge \simeq \hat{Y}_y \hat{\times} \hat{D}^\infty$$

where $\hat{D}^\infty = \text{Spf}(k[[x_1, x_2, \dots]])$.

Grinberg-Kazhdan-Drinfeld's theorem gives the hope that a reasonable theory of perverse sheaves on formal arc spaces may exist. In [2], it is proven that in a weak sense the formal finite dimensional model is independent of choices i.e if \hat{Y}_y and $\hat{Y}'_{y'}$ are finite type formal models of \hat{X}_x

$$(\mathcal{L}X)_x \simeq \hat{Y}_y \hat{\times} \hat{D}^\infty \simeq \hat{Y}'_{y'} \hat{\times} \hat{D}^\infty$$

then there exists integers $m, n \in \mathbb{N}$ such that there exists an isomorphism

$$\hat{Y}_y \hat{\times} D^m \simeq \hat{Y}'_{y'} \hat{\times} D^n. \quad (1.2)$$

As consequence in the case k is a finite field, using the ℓ -adic intersection complex on a scheme Y of finite type of which \hat{Y}_y is formal finite dimensional model of \hat{X}_x , we can define a canonical function $\text{IC}_{\mathcal{L}X} : \mathcal{L}^\bullet X(k) \rightarrow \mathbb{Q}_\ell$ which deserves the name of intersection complex

function although we don't know yet to define the intersection complex on the space of non-degenerate arcs. If $x \in \mathcal{L}X(k)$ is a non-degenerate arc on X , and \hat{Y}_y is a finite-dimensional formal as in 1.2, then we set

$$\mathrm{IC}_{\mathcal{L}X}(x) = \mathrm{IC}_Y(y).$$

The existence of the isomorphism (1.2) implies that $\mathrm{IC}_Y(y) = \mathrm{IC}_{Y'}(y')$ so that this number doesn't depend on the choice of the finite type formal model as argued in [2, Prop. 1.2].

In order to define perverse sheaves on the space of non-degenerate arcs, one may hope a stronger version of the Drinfeld-Grinberg-Kazhdan theorem in which formal completions are replaced by strict Henselizations for instance. In other words, instead of formal charts as in 1.2, one would like to construct Henselian charts. There are good reasons to believe that the analogue of Grinberg-Kazhdan theorem for henselizations doesn't hold.

In [1], Bouthier and Kazhdan attempt to construct certain type of coverings of the space of non-degenerate arcs which admit essentially smooth surjective map to schemes of finite type. The strategy of [1] consists in generalizing Drinfeld's construction in [3], which works over local Artinian test rings, to general test rings. There are unfortunately some gaps in the construction of [1].

Both constructions [3] and [1] rely in the Weierstrass preparation theorem and the Newton method of solving algebraic equation by approximation. We will study with some care the Weierstrass preparation and division theorem, and the Newton method over a general test rings. The study of the Weierstrass division theorem reveals that the coordinate ring of the arc space of the curve of equation $xy = 0$ contains non-zero functions which vanish at all points to the infinite order. We observe that these functions are annihilated in completed local rings but generally not in henselizations. We also observe that if X is a k -scheme of finite type, henselizations of the ring of coordinates $X \times \mathbb{A}^\infty$ don't contain non-zero elements which vanish at all points to the infinite order.

Nevertheless, following the method of [3] and [1], one can obtain a theorem on local structure of non-degenerate arc space. Although the description of the local structure is not very pleasant, one can derive from it the following statement.

Theorem 1.3. *Let X be a k -scheme locally of complete intersection. Let $x \in \mathcal{L}X(k)$ be a non-degenerate arc. There exist a k -scheme Y of finite type, a point $y \in Y(k)$, a morphism $\phi : Y \rightarrow \mathcal{L}X$ mapping y to x such that for every field k' containing k , for every point $y' \in Y(k')$ mapping to $x' \in \mathcal{L}X(k')$, there exists an isomorphism*

$$\mathcal{L}X_{x'}^\wedge \hat{\times} \hat{D}^\infty \simeq \hat{Y}_{y'} \hat{\times} \hat{D}^\infty.$$

Schemes of finite type as in the above statement should be thought of as good slices in non-degenerate arc spaces. One may hope to construct a reasonable theory of perverse sheaves on arc spaces using these slices. For instance, one may think of the hypothetical intersection complex $\mathrm{IC}_{\mathcal{L}X}$ of $\mathcal{L}X$ as the unique object whose restriction to every slice Y of finite type is the intersection complex IC_Y of Y up to a shift by the dimension.

Although it may be possible to drop the complete intersection assumption, in this paper, I want, as much as possible, to stick with the computational approach of [3]. One should also note that in many moduli problems, slices are given to us by replacing maps from the formal disc to a given target scheme, or algebraic stack, by maps from a given curve. This idea is in fact behind the calculations in the complete intersection case as in [3]. Drinfeld explains how one can approach the general case by means of the Newton groupoid in the Geometric Langlands Seminar in Spring 2017.

2 Formal completion and tensor product

There is a potential danger in dealing with formal series with coefficients in a varying commutative ring R for the operation of tensor product doesn't commute with the operation of completion. The purpose of this section is to provide warnings by means of some examples where what we observe isn't quite what we would naively expect. Our examples are all related to the problem of Weierstrass division.

For a commutative ring R , an ideal I of R , and an R -module M , we define the I -completion of M to be the limit

$$\hat{M}_I = \varprojlim M/I^n M.$$

For instance, the p -completion of the ring \mathbb{Z} of integers is the ring \mathbb{Z}_p of p -adic integers. At this place, it may be necessary to clear some confusion in terminology: although the field \mathbb{Q}_p of p -adic numbers, defined as the quotient field of \mathbb{Z}_p , is p -adically complete in the sense that all Cauchy sequences in \mathbb{Q}_p for the p -adic distance converge, the p -completion of \mathbb{Q}_p is zero.

For every commutative ring R , the ring of formal series $R[[t]]$ consists in set of all series of the form $x = x_0 + x_1 t + x_2 t^2 + \dots$ where x_0, x_1, \dots are arbitrary elements subjected to usual rules of addition and multiplication. It contains the ring $R[t]$ of polynomials consisting of $x = x_0 + x_1 t + x_2 t^2 + \dots$ with coefficients x_0, x_1, \dots equal to 0 except for finitely many of them. We may also define $R[[t]]$ as the t -completion of $R[t]$:

$$R[[t]] = \varprojlim R[t]/(t^n).$$

As unproblematic as this definition may appear, some good cares are in order. If R is a commutative ring, I is an ideal of R , M is a R -module, for an R -algebra R' , the canonical morphism

$$\hat{M}_I \otimes_R R' \rightarrow (M \otimes_R R')_{IR'} \quad (2.1)$$

may not be an isomorphism.

For instance, $\mathbb{Q}[[t]] \otimes_{\mathbb{Q}} \mathbb{C}$ consists of finite complex linear combinations of formal series with rational coefficients while $\mathbb{C}[[t]]$ consists of all formal series with complex coefficients. In this example the map (2.1) is injective but not surjective.

Now we consider the ring $R = \mathbb{Z}[t]$ of polynomial with integers coefficients, the ideal I of R generated by t and the module

$$M = \mathbb{Z}[t]/(t-p). \quad (2.2)$$

As an abelian group, R is canonically isomorphic with \mathbb{Z} . In fact, we may see R as the $\mathbb{Z}[t]$ -module \mathbb{Z} on which t acts as the multiplication by p . The t -completion of M is thus $\hat{M}_I = \mathbb{Z}_p$. Now if $R' = \mathbb{Q}[t]$, then on the one hand we have $\hat{M}_I \otimes_R R' = \mathbb{Q}_p$, but on the other hand the t -completion of $\mathbb{Q}[t]/(t-p)$ is zero as $p \in \mathbb{Q}^\times$. In particular, in this example the map (2.1) is surjective but not injective.

The following lemma won't be used in the sequel except for a side observation, but it may serve as yet another warning.

Lemma 2.1. *Let R is a commutative ring R , I a finitely generated ideal of R and $\bar{R} = R/I$. Then the natural map $\phi : R[[t]] \otimes_R \bar{R} \rightarrow \bar{R}[[t]]$ is an isomorphism. The assertion doesn't hold without the assumption I finitely generated*

Proof. Since $R \rightarrow \bar{R}$ is surjective, the map $R[[t]] \rightarrow \bar{R}[[t]]$ is surjective and so is $\phi : R[[t]] \otimes_R \bar{R} \rightarrow \bar{R}[[t]]$. We only need to prove that ϕ is injective. The injectivity of ϕ is equivalent to that the kernel of $R[[t]] \rightarrow \bar{R}[[t]]$ is the ideal of $R[[t]]$ generated by I . Let us denote $I[[t]]$ the kernel of $R[[t]] \rightarrow \bar{R}[[t]]$. This kernel consists in formal series $x = x_0 + x_1 t + \dots$ such that $x_n \in I$ for all n . If a_1, \dots, a_m is a system of generators of I , then for every $n \in \mathbb{N}$, the coefficient x_n of x can be written possibly non uniquely in the form $x_n = a_1 y_{n,1} + \dots + a_m y_{n,m}$. If we choose such an expression for each $n \in \mathbb{N}$ then we have $x = a_1 y_1 + \dots + a_m y_m$ where y_i is the formal series $y_i = y_{i,0} + y_{i,1} t + \dots$. It follows that x belongs to the ideal of $R[[t]]$ generated by I .

The assertion doesn't hold without the assumption that I is finitely generated. Let $R = k[x_0, x_1, x_2, \dots]$ be the ring of polynomials in infinitely many variables x_0, x_1, x_2, \dots with coefficients in a field k , I the ideal of R generated by x_0, x_1, x_2, \dots , $\bar{R} = R/I = k$. Then the formal series $x_0 + x_1 t + x_2 t^2 + \dots$, lying in the kernel of $R[[t]] \rightarrow k[[t]]$, doesn't belong to the ideal of $R[[t]]$ generated by I but its t -completion. \square

3 Variations on the Weierstrass preparation theorem

If f is a nonzero germ of a holomorphic function at the origin in \mathbb{C}^{n+1} , given with coordinates t, x_1, \dots, x_n , the classical Weierstrass preparation theorem asserts the existence of a germ of holomorphic function u at the origin in \mathbb{C}^{n+1} non vanishing at the origin and a polynomial

$$q = q_0 + q_1 t + \dots + q_{d-1} t^{d-1} + t^d \quad (3.1)$$

where q_0, \dots, q_{d-1} are germs of holomorphic functions at the origin \mathbb{C}^n , and vanishing at the origin, such that $f = uq$.

Let $R_n = \mathbb{C}[[x_1, \dots, x_n]]$ be the ring of formal series of variables x_1, \dots, x_n . Let $f \in R_n[[t]]$ be a formal series of variables t, x_1, \dots, x_n . We denote $\bar{f} \in \mathbb{C}[[t]]$ the reduction of f modulo x_1, \dots, x_n and write $\bar{f} = t^d \bar{u}$ where $u \in \mathbb{C}[[t]]^\times$. By usual manipulations with formal series, we can prove that there exist a unique formal series $u \in R_n[[t]]^\times$ with reduction $\bar{u} \in \mathbb{C}[[t]]^\times$ and a unique polynomial $q \in R_n[t]$ of the form (3.1) whose coefficients $q_0, \dots, q_{d-1} \in R_n$ are formal series vanishing constant coefficients such that $f = uq$. The holomorphic content of the Weierstrass preparation theorem consists in showing that if f is absolutely convergent in a neighborhood of 0 in \mathbb{C}^{n+1} then q_0, \dots, q_{d-1} are also absolutely convergent in a neighborhood of 0 in \mathbb{C}^n .

Let A_n denote the ring of germs of holomorphic functions defined in a neighborhood of 0 in \mathbb{C}^n . A_n is a Henselian local ring whose completion with respect to the maximal ideal is $R_n = \mathbb{C}[[x_1, \dots, x_n]]$. Every $f \in A_n[[t]]$ can be factored uniquely in the form $f = uq$ with $u \in R_n[[t]]^\times$ and $q \in R_n[t]$ a Weierstrass polynomial as above but there is no guarantee that the coefficients q_0, \dots, q_{d-1} belong to A_n unless f is assumed to be a convergent series. We draw the conclusion that the existence of Weierstrass factorization for $f \in A[[t]]$ as $f = uq$ where $u \in A[[t]]^\times$ and $q \in A[t]$ is a Weierstrass polynomial doesn't hold generally for Henselian local ring A as opposed to complete local rings. The aim of this section is to study the existence of the Weierstrass factorization over a general test ring and its connection with the geometry of the arc space of the affine line.

The arc space $\mathcal{L}\mathbb{A}^1$ of the affine line \mathbb{A}^1 is spectrum of the ring of polynomials with countably infinite number of variables:

$$\mathcal{L}\mathbb{A}^1 = \text{Spec}(k[x_0, x_1, x_2, \dots]) \quad (3.2)$$

for a R -point $x \in \mathcal{L}\mathbb{A}^1(R)$ is uniquely represented by a formal series $x = x_0 + x_1 t + \dots$ for every k -algebra R . We consider the space $\mathcal{L}^\bullet \mathbb{A}^1$ of non-degenerate arcs with respect to the open subset \mathbb{G}_m of \mathbb{A}^1 . On the level of k -points we have

$$\mathcal{L}^\bullet \mathbb{A}^1(k) = k[[t]] - \{0\} \quad (3.3)$$

The description of set of R -points of $\mathcal{L}^\bullet \mathbb{A}^1$ is as follows:

Definition 3.1. For every commutative ring R , a formal series $x \in R[[t]]$ is said to be non-degenerate, and denoted $x \in \mathcal{L}^\bullet \mathbb{A}^1(R)$, if for every ring homomorphism $\nu_K : R \rightarrow K$ from R to a field K the formal series $\nu_K(x)$ is a nonzero element of $K[[t]]$, or in other words:

$$\nu_K(x) \in K[[t]] \cap K((t))^\times.$$

A strictly stronger condition is $x \in R[[t]] \cap R((t))^\times$ where $R((t))$ is the ring of Laurent formal series with coefficients in R . In other words, we have an inclusion

$$R[[t]] \cap R((t))^\times \subset \mathcal{L}^\bullet \mathbb{A}^1(R). \quad (3.4)$$

The functor $R \mapsto R[[t]] \cap R((t))^\times$ is closely related to the affine Grassmannian of \mathbb{G}_m defined as the sheafification of the functor $R \mapsto R((t))^\times / R[[t]]^\times$. Although our purpose is not to review the theory of affine Grassmannian, we will borrow some familiar arguments from that theory helping to clarify the relation between the properties of being invertible in $R((t))^\times$ and the existence of a strict Weierstrass factorization.

Definition 3.2. *Let R be a commutative ring.*

1. *We will call a Weierstrass factorization of a formal series $x \in R[[t]]$ an expression of the form $x = uq$ where $u \in R[[t]]^\times$ and q is a monic polynomial of some degree d .*
2. *We will call the expression $x = uq$ as above a strict Weierstrass factorization of x if q divides t^n for some $n \in \mathbb{N}$ i.e. $t^n \in (q)$ where (q) is the ideal of $R[[t]]$ generated by q .*

If we denote $\mathcal{L}^\beta \mathbb{A}^1(R)$ and $\mathcal{L}^{\beta+} \mathbb{A}^1(R)$ the set of formal series $x \in R[[t]]$ having a Weierstrass factorization and strict Weierstrass factorization respectively:

$$\mathcal{L}^{\beta+} \mathbb{A}^1(R) \subset \mathcal{L}^\beta \mathbb{A}^1(R) \subset \mathcal{L}^\bullet \mathbb{A}^1(R). \quad (3.5)$$

We will prove that the equality

$$R[[t]] \cap R((t))^\times = \mathcal{L}^{\beta+} \mathbb{A}^1(R) \quad (3.6)$$

holds for all commutative rings R .

Proposition 3.3. *For every formal series $x \in R[[t]]$, the following conditions are equivalent:*

1. $x \in R[[t]] \cap R((t))^\times$;
2. $R[[t]]/(x)$ is a projective R -module of finite type, (x) being the ideal of $R[[t]]$ generated by x , and $t^n \in (x)$ for some integer $n \in \mathbb{N}$;
3. x admits a strict Weierstrass factorization: $x = uq$ where $u \in R[[t]]^\times$ and q is a monic polynomial dividing t^n for some $n \in \mathbb{N}$.

If these conditions are satisfied, then the strict Weierstrass factorization of x is unique.

Proof. First we prove that if $x \in R[[t]] \cap R((t))^\times$ then, for some $n \in \mathbb{N}$, t^n belongs to the ideal (x) of $R[[t]]$ generated by x . Let y be the inverse of x in $R((t))^\times$, we have $xy = 1$. Since $y \in R((t))$ there exists $n \in \mathbb{N}$ such that $t^n y \in R[[t]]$. It follows that $t^n = x(yt^n) \in (x)$. It will also follow that $R[[t]]/xR[[t]]$ is a R -module of finite type as a quotient of the R -module of finite type $R[[t]]/t^n R[[t]]$.

Next we prove that if $x \in R[[t]] \cap R((t))^\times$ then $R[[t]]/xR[[t]]$ is a flat R -module. We consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R[[t]] & \longrightarrow & R((t)) & \longrightarrow & R((t))/R[[t]] \longrightarrow 0 \\
 & & \downarrow x & & \downarrow x & & \downarrow x \\
 0 & \longrightarrow & R[[t]] & \longrightarrow & R((t)) & \longrightarrow & R((t))/R[[t]] \longrightarrow 0
 \end{array} \tag{3.7}$$

with exact horizontal lines. Since $x \in R((t))^\times$, the middle vertical map is an isomorphism. The snake lemma provides us with an exact sequence of R -modules

$$0 \longrightarrow R[[t]]/xR[[t]] \longrightarrow R((t))/R[[t]] \xrightarrow{x} R((t))/R[[t]] \longrightarrow 0 \tag{3.8}$$

For $R((t))$ decomposes as a direct sum of R -modules

$$R((t)) = t^{-1}R[t^{-1}] \oplus R[[t]]$$

we can identify $R((t))/R[[t]]$ with $t^{-1}R[t^{-1}]$. It follows that $R((t))/R[[t]]$ is flat as R -module and thus $R[[t]]/xR[[t]]$ is also flat as R -module.

Now we prove that if $R[[t]]/xR[[t]]$ is a projective R -module of finite rank d and if $t^n \in (x)$ for some $n \in \mathbb{N}$, then x has a strict Weierstrass factorization. We consider the exact sequence of R -modules

$$0 \longrightarrow (x)/(t^n) \longrightarrow R[[t]]/(t^n) \longrightarrow R[[t]]/(x) \longrightarrow 0 \tag{3.9}$$

where (x) and (t^n) are ideals of $R[[t]]$ generated by x and t^n respectively. We know that $R((t))/t^n R[[t]]$ and $R((t))/xR[[t]]$ are flat R -modules of finite type. It follows that $(x)/(t^n)$ is a flat R -module. Being generated by x, xt, \dots, xt^{n-1} , it is a R -module of finite type, and therefore $(x)/(t^n)$ is a projective module of finite type.

Let $q \in R[t]$ denote the characteristic polynomial of the R -linear operator t acting on $R[[t]]/xR[[t]]$. By the Cayley-Hamilton theorem, the multiplication by q in $R[[t]]/(x)$ is null. This implies that $q \in (x)$ or in other words x divides q .

We claim that q divides t^n . Indeed if q' is the characteristic polynomial of t acting on the projective R -module of finite type $(x)/(t^n)$, then we have $t^n = qq'$ by the exact sequence (3.9).

Now we consider morphisms of R -modules:

$$R[t]/(q) \rightarrow R[[t]]/(q) \rightarrow R[[t]]/(x) \tag{3.10}$$

where $R[t]/(q)$ is a free R -module of rank d since q is a monic polynomial and $R[[t]]/(x)$ is also known to be projective R -modules of rank d by assumption. Since q divides t^n , t acts nilpotently on $R[t]/(q)$. It follows that the map $R[t]/(q) \rightarrow R[[t]]/(q)$ is an

isomorphism. It follows that $R[[t]]/q$ is also a free R -module of rank d . The surjective map $R[[t]]/(q) \rightarrow R[[t]]/(x)$ between projective R -modules of rank d has to be an isomorphism. It follows that $x = qu$ where $u \in R[[t]]^\times$ and q is a monic polynomial dividing t^n thus x has a strict Weierstrass factorization.

Now we assume that $x \in R[[t]]$ has a strict Weierstrass factorization $x = uq$ with $u \in R[[t]]^\times$ and q is a monic polynomial dividing t^n for some $n \in \mathbb{N}$ and we prove that $x \in R[[t]] \cap R((t))^\times$. In this case $q \in R((t))^\times$ and therefore $x \in R[[t]] \cap R((t))^\times$.

Finally if $x = uq$ is a strict Weierstrass factorization then the linear maps (3.10) are isomorphisms of free R -modules. It follows that q is the characteristic polynomial of t acting on $R[[t]]/(x)$. Thus the strict Weierstrass factorization of x is unique if it exists. \square

Corollary 3.4. *Let $x \in \mathcal{L}^{\beta+} \mathbb{A}^1(R)$ be a formal series with strict Weierstrass factorization. For every ring homomorphism $R \rightarrow R'$, the canonical map $(R[[t]]/(x)) \otimes_R R' \rightarrow R'[[t]]/(x)$ of (2.1) is an isomorphism.*

Proof. In the course of the proof of Prop.3.3, we have showed that if $x = qu$ is a strict Weierstrass factorization of x , the R -linear map $R[t]/(q) \rightarrow R[[t]]/(x)$ is an isomorphism. The assertion can now be reduced to the obvious equality $R[t]/(q) \otimes_R R' = R'[t]/(q)$. \square

For every prime ideal \mathfrak{p} of R , we will denote $\nu_{\mathfrak{p}} : R \rightarrow k_{\mathfrak{p}}$ the canonical homomorphism from R to the fraction field $k_{\mathfrak{p}}$ of R/\mathfrak{p} . A formal series $x \in R[[t]]$ is non-degenerate if and only if for every $\mathfrak{p} \in \text{Spec}(R)$, $\nu_{\mathfrak{p}}(x) \in k_{\mathfrak{p}}[[t]]$ is nonzero. In this case, we will denote $\text{ord}_{\mathfrak{p}}(x)$ the vanishing order of the formal series $\nu_{\mathfrak{p}}(x) \in k_{\mathfrak{p}}[[t]]$.

Proposition 3.5. *Let $x \in R[[t]]$ be a non-degenerate formal series with coefficients in R . We consider the two assertions:*

- (1) x has a strict Weierstrass factorization;
- (2) the function $\mathfrak{p} \rightarrow \text{ord}_{\mathfrak{p}}(x)$ is a locally constant function on $\text{Spec}(R)$.

For an arbitrary k -algebra R , (1) implies (2). The converse statement that (2) implies (1) holds under the assumption that $\text{Nil}(R)^n = 0$ for some $n \in \mathbb{N}$, $\text{Nil}(R)$ being the nilradical of R . In particular if R is either reduced or Noetherian then (1) and (2) are equivalent.

Proof. Assume that x has a strict Weierstrass factorization i.e. $x \in \mathcal{L}^{\beta+} \mathbb{A}^1(R)$, then by Cor. 3.4, $R[[t]]/(x)$ is a projective R -module of finite type compatible with base change. In particular, for every prime ideal \mathfrak{p} we have

$$k_{\mathfrak{p}}[[t]]/(x) = R[[t]]/(x) \otimes_R k_{\mathfrak{p}} \tag{3.11}$$

where $\dim_{k_{\mathfrak{p}}}(k_{\mathfrak{p}}[[t]]/(x)) = \text{ord}_{\mathfrak{p}}(x)$. For $R[[t]]/(x)$ is a projective R -module of finite type, $\mathfrak{p} \mapsto \text{ord}_{\mathfrak{p}}(x)$ is a locally constant function of \mathfrak{p} .

Now we assume that $\text{Nil}(R)^n = 0$ for some $n \in \mathbb{N}$. Let $x \in R[[t]]$ be a formal series with coefficients in R such that the function $\mathfrak{p} \rightarrow \text{ord}_{\mathfrak{p}}(x)$ is locally constant on $\text{Spec}(R)$. We will prove that x has a strict Weierstrass factorization. We may assume that $\text{Spec}(R)$ is connected and $\text{ord}_{\mathfrak{p}}(x) = d$ for all $\mathfrak{p} \in \text{Spec}(R)$. Let us consider the coefficients x_0, x_1, \dots of the formal series $x = x_0 + x_1 t + \dots$. For $0 \leq i \leq d-1$, we have $\nu_{\mathfrak{p}}(x_i) = 0$ for all $\mathfrak{p} \in \text{Spec}(R)$. It follows that x_i belongs to $\text{Nil}(R)$ the intersection of all prime ideals of R . We recall that $\text{Nil}(R)$ can also be characterized as the ideal of all nilpotent elements of R . Let us denote $\bar{R} = R/\text{Nil}(R)$, \bar{x}_i the image of x_i in \bar{R} , and \bar{x} the image of x in $\bar{R}[[t]]$. We have $\bar{x}_i = 0$ for all i in the range $0 \leq i \leq d-1$. On the other hand, for every $\mathfrak{p} \in \text{Spec}(R)$ we have $\nu_{\mathfrak{p}}(x_d) \neq 0$. Since x_d doesn't lie in any maximal ideal of R , we have $x_d \in R^\times$. Now $\bar{x} = \bar{x}_d t^d + \bar{x}_{d+1} t^{d+1} + \dots \in \bar{R}[[t]]$ with $\bar{x}_d \in \bar{R}^\times$ obviously has a strict Weierstrass factorization. Under the assumption $\text{Nil}(R)^n = 0$ for some $n \in \mathbb{N}$, we deduce that x has a strict Weierstrass factorization in virtue of Prop. 3.6 that follows. \square

Prop. 3.5 provides a handy way to check whether a formal series in $x \in R[[t]]$ has a strict Weierstrass factorization by calculating the function $\mathfrak{p} \mapsto \text{ord}_{\mathfrak{p}}(x)$. For instant, for $R = k[a]$ the formal series $x = -a + t \in R[[t]]$ does not have strict Weierstrass factorization for $\text{ord}_{\mathfrak{p}}(x) = 0$ if $\mathfrak{p} \neq (a)$ and $\text{ord}_{\mathfrak{p}}(x) = 1$ if $\mathfrak{p} = (a)$.

We also observe that both statement and proof of the above proposition can be simplified if R is a k -algebra of finite type. For a k -algebras of finite type, we may replace in the statement of Prop. 3.5 $\text{Spec}(R)$ by the space $\text{Spm}(R)$ of maximal ideals. For maximal ideals, the base change formula (3.11) holds in virtue of Lemma 2.1 so that we don't need to appeal to Prop. 3.3. This assures that (1) implies (2). The converse statement holds because for a k -algebra of finite type R , the intersection of all maximal ideals, its Jacobson radical $\text{Jac}(R)$, is equal to the intersection of all prime ideals, its nilradical.

Proposition 3.6. *The morphism $\mathcal{L}^{\beta+\mathbb{A}^1} \rightarrow \mathcal{L}^\bullet \mathbb{A}^1$ is surjective and formally étale: for every k -algebra R , every nilpotent ideal I of R i.e. $I^n = 0$ for some $n \in \mathbb{N}$, and $\bar{R} = R/I$, if $x \in R[[t]]$ is a formal series whose reduction modulo I satisfies $\bar{x} \in \bar{R}((t))^\times \cap \bar{R}[[t]]$ then $x \in R((t))^\times \cap R[[t]]$. In particular, if R is an Artinian local k -algebra then*

$$R[[t]] \cap R((t))^\times = \mathcal{L}^\bullet \mathbb{A}^1(R).$$

In other words, every non-degenerate formal series $x \in \mathcal{L}^\bullet \mathbb{A}^1(R)$ has a unique strict Weierstrass factorization under the assumption that R is an Artinian local ring.

Proof. Let $I((t))$ denote the ideal of $R((t))$ consisting of all Laurent formal series whose coefficients belong to I . We have an exact sequence

$$0 \rightarrow I((t)) \rightarrow R((t)) \rightarrow \bar{R}((t)) \rightarrow 0.$$

Since $I^n = 0$, we have $I((t))^n = 0$. It follows that the homomorphism $R((t)) \rightarrow \bar{R}((t))$ induces a bijection between their spectra. For an element in a ring is invertible if it doesn't

belong to any maximal ideal, we deduce that $x \in R((t))$ is invertible if and only its reduction $\bar{x} \in \bar{R}((t))$ is invertible. \square

Proposition 3.7. *The functor $R \mapsto \mathcal{L}^{\beta+} \mathbb{A}^1(R)$ is representable by a strict ind-scheme. The projection on the degree of q defines a bijection from the set of connected components of $\mathcal{L}^{\beta+} \mathbb{A}^1$ on \mathbb{N} . For every $d \in \mathbb{N}$, the component $\mathcal{L}^{\beta+,d} \mathbb{A}^1$ is isomorphic to $\hat{Q}_d \times \mathcal{L} \mathbb{G}_m$ where \hat{Q}_d is the completion of the space Q_d of all monic polynomials of degree d at the point $q = t^d$.*

Proof. The uniqueness of the strict Weierstrass factorization implies that

$$\mathcal{L}^{\beta+,d} \mathbb{A}^1(R) = Q_d^0(R) \times R[[t]]^\times$$

where $Q_d^0(R)$ is the set of monic polynomial of degree d dividing t^n for some $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we denote $Q_d^{0,n}(R)$ the set of monic polynomials of degree d dividing t^n . By definition $Q_d^{0,n}(R)$ is the subset of $Q_d(R)$ consisting of monic polynomials of degree d such that there exists $q' \in Q_{n-d}(R)$ satisfying $qq' = t^n$. This is equivalent to say that $(t^{-d}q)(t^{-n+d}q') = 1$ i.e. the elements $t^{-d}q, t^{-n+d}q'$ are inverse of each other in $R[t^{-1}]^\times$. It follows that if q' exists, it is unique. It follows that there is a natural bijection between $Q_d^{0,n}(R)$ and the subset of $Q_d(R) \times Q_{n-d}(R)$ of consisting of pairs (q, q') such that $qq' = t^n$. It follows that the functor $R \mapsto Q_d^{0,n}(R)$ is representable by a k -scheme of finite type. Since $Q_d^{0,n}$ has a unique geometric point $(q, q') = (t^d, t^{n-d})$, it is a local Artinian scheme.

We have a map $Q_d^{0,n} \rightarrow Q_d^{0,n+1}$ given by $(q, q') \mapsto (q, tq')$. Since the morphism $Q_{n-d} \rightarrow Q_{n-d+1}$ given by $q' \mapsto tq'$ is a closed immersion, the morphism $Q_d^{0,n} \rightarrow Q_d^{0,n+1}$ which derives from the latter by base change is also a closed immersion.

We have thus proved that Q_d^0 is a strict colimit of Artinian schemes based at the point t^d of Q_d . It follows that there is a canonical morphism of formal schemes $Q_d^0 \rightarrow \hat{Q}_d$. It follows from Prop. 3.6 that the morphism $Q_d^0 \rightarrow \hat{Q}_d$ is an isomorphism. \square

To summarize, the property of having a strict Weierstrass factorization is controlled by the morphism

$$\gamma : \mathcal{L}^{\beta+} \mathbb{A}^1 \rightarrow \mathcal{L}^\bullet \mathbb{A}^1 \tag{3.12}$$

which induces isomorphism on k -points. We have seen that γ is *formally étale*. On the negative side γ is *not schematic* because otherwise $\mathcal{L}^{\beta+} \mathbb{A}^1$ would be schematic and it isn't according to its explicit description in Prop. 3.7.

We will now study the geometry related to the notion of non-strict Weierstrass factorization. The natural action of \mathbb{G}_m on \mathbb{A}^1 induces on the level of arc spaces an action of $\mathcal{L} \mathbb{G}_m$ on $\mathcal{L} \mathbb{A}^1$. For every $d \in \mathbb{N}$ we have a morphism $\mathcal{L} \mathbb{A}^1 \rightarrow \mathcal{L}_d \mathbb{A}^1$ where $\mathcal{L}_d \mathbb{A}^1$ is the affine space $\text{Spec}(k[x_0, x_1, \dots, x_d])$. We will denote $\mathcal{L}^{\leq d} \mathbb{A}^1$ to be the open subscheme of $\mathcal{L} \mathbb{A}^1$ defined as

the preimage of the complement of the point 0 in \mathbb{A}^d . The set of k -points of $\mathcal{L}^{\leq d} \mathbb{A}^1$ consists in the series $x \in k[[t]]$ such that $\text{val}(x) \leq d$. We have the inclusions

$$\mathcal{L}^{\leq 0} \mathbb{A}^1 \subset \mathcal{L}^{\leq 1} \mathbb{A}^1 \subset \dots$$

The union of those open subschemes is the space $\mathcal{L}^\bullet \mathbb{A}^1$ of non-degenerate arcs.

We will consider the affine subspace Q_d of $\mathcal{L} \mathbb{A}^1$ defined by the equations $x_d = 1$ and $x_{d+1} = x_{d+2} = \dots = 0$. The action of $\mathcal{L} \mathbb{G}_m$ on $\mathcal{L} \mathbb{A}^1$ induces a morphism

$$\beta_d : Q_d \times \mathcal{L} \mathbb{G}_m \rightarrow \mathcal{L}^{\leq d} \mathbb{A}^1 \quad (3.13)$$

given by $\beta(q, u) = uq$. By definition a formal series $x \in \mathcal{L} \mathbb{A}^1(R)$ has Weierstrass factorization if it lies in the image $\beta_D(Q_d(R) \times \mathcal{L} \mathbb{G}_m(R))$ for some $d \in \mathbb{N}$.

We will see that as opposed to the morphism γ of (3.12), the Weierstrass morphism β_d of (3.13) is not formally étale but only formally smooth in a weak sense. On the other hand, it has the advantage of being schematic.

Proposition 3.8. *To simplify notation, we will write $X = Q_d \times \mathcal{L} \mathbb{G}_m$ and $Y = \mathcal{L}^{\leq d} \mathbb{A}^1$. The morphism $\beta_d : X \rightarrow Y$ is surjective on k -points. Moreover, for every $x \in X(k)$ mapping on $y \in Y(k)$, the induced mapping on formal completions $\hat{X}_x \rightarrow \hat{Y}_y$ is formally smooth.*

Proof. The surjectivity on k -points is obvious. To prove that the induced mapping on formal completions $\hat{X}_x \rightarrow \hat{Y}_y$ is formally smooth is equivalent to prove the usual lifting properties for Artinian local algebras: for R an Artinian algebra, I a ideal of R , $\bar{R} = R/I$, for every $\bar{x} \in X(\bar{R})$ mapping to $\bar{y} \in Y(\bar{R})$, if $y \in Y(R)$ reducing to \bar{y} modulo I then there exists $x \in X(R)$ mapping to $y \in Y(R)$ and reducing to $\bar{x} \in X(\bar{R}) \bmod I$.

Let $y \in Y(R)$ be a non degenerate formal series with coefficients in an Artinian local ring R . By Prop. 3.6, there exists a unique strict Weierstrass factorization $y = q_0 u_0$ where $u_0 \in R[[t]]^\times$ and q_0 is a monic polynomial of some degree e , q_0 dividing t^n for some $n \in \mathbb{N}$.

The point $\bar{x} \in X(\bar{R})$ mapping to $\bar{y} \in Y(\bar{R})$ corresponds to a Weierstrass factorization $\bar{y} = \bar{q} \bar{u}$ of \bar{y} , its reduction modulo I where \bar{q} is a monic polynomial of degree d with coefficients in \bar{R} and $\bar{u} \in \bar{R}[[t]]^\times$. By reducing $y = u_0 q_0$ modulo I we have the unique strict Weierstrass factorization $\bar{y} = \bar{u}_0 \bar{q}_0$ of \bar{y} that may be different from the Weierstrass factorization $\bar{y} = \bar{u} \bar{q}$ given by $\bar{x} \in X(\bar{R})$.

Since $t^n \in (\bar{q}_0) = (\bar{x}) = (\bar{q})$, we have

$$\bar{R}[[t]]/\bar{q} = \bar{R}[t]/(t^n, \bar{q})$$

i.e. $R[[t]]/\bar{q}$ is a quotient of $\bar{R}[t]/\bar{q}$. On the other hand, the strictness property shows that $\bar{R}[[t]]/(\bar{q}_0) = \bar{R}[t]/(\bar{q}_0)$ as in the proof of Prop. 3.3. It follows there exists a surjective R -linear homomorphism of free R -modules $\bar{R}[t]/\bar{q} \rightarrow \bar{R}[t]/\bar{q}_0$ compatible with action of t . In particular we have the inequality $d \geq e$ by comparing their ranks. Moreover by taking

characteristic polynomials of t acting on those free modules, we see that \bar{q}_0 divides \bar{q} i.e. there exists a monic \bar{p} of degree $d - e$ such that $\bar{q} = \bar{q}_0\bar{p}$. We have $\bar{p} \in Q_d(\bar{R}) \cap \bar{R}[[t]]^\times$.

Now there are no problem to lift the polynomial $\bar{p} \in Q_{d-e}(\bar{R})$ to a monic polynomial $p \in Q_{d-e}(R)$ as Q_{d-e} is an affine space. Choose an arbitrary lift $p \in Q_{d-e}(R)$ of $\bar{p} \in Q_{d-e}(\bar{R})$. Since the reductio mod I of p is an invertible element $\bar{p} \in \bar{R}[[t]]^\times$, we have $p \in R[[t]]^\times$. Now we define the lift $y \in Y(R)$ of $\bar{y} = (\bar{q}, \bar{u})$ to be $y = (q_0p, p^{-1}u_0)$. \square

Proposition 3.9. *Let $\mathcal{L}_n\mathbb{G}_m$ and $\mathcal{L}_n\mathbb{A}^1$ denote the spaces of n -jets on \mathbb{G}_m and \mathbb{A}^1 for every $n \in \mathbb{N}$. If $d \leq n \in \mathbb{N}$, then the morphism*

$$\beta_{d,n} : Q_d \times \mathcal{L}_n\mathbb{G}_m \rightarrow \mathcal{L}_n^{\leq d}\mathbb{A}^1 \quad (3.14)$$

given by $(q, u) \mapsto qu$ is smooth.

Proof. If we denote $X = Q_d \times \mathcal{L}_n\mathbb{G}_m$, $X_n = Q_d \times \mathcal{L}_n\mathbb{G}_m$, $Y = \mathcal{L}_n^{\leq d}\mathbb{A}^1$ and $Y_n = \mathcal{L}_n^{\leq d}\mathbb{A}^1$ then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\beta_d} & Y \\ \mu_n \downarrow & & \downarrow \nu_n \\ X_n & \xrightarrow{\beta_{d,n}} & Y_n \end{array} \quad (3.15)$$

where μ_n and ν_n are projective limits of systems of smooth surjective maps with smooth surjective transition morphisms. For every $x_n \in X_n(k)$ mapping to $y_n \in Y_n(k)$, there exists $x \in X(k)$ over x_n mapping to $y \in Y(k)$ over y_n . We have a commutative diagram of formal completions

$$\begin{array}{ccc} \hat{X}_x & \xrightarrow{\hat{\beta}_d} & \hat{Y}_y \\ \hat{\mu}_n \downarrow & & \downarrow \hat{\nu}_n \\ \hat{X}_{n,x_n} & \xrightarrow{\hat{\beta}_{d,n}} & \hat{Y}_{n,y_n} \end{array} \quad (3.16)$$

where $\hat{\beta}_d, \hat{\mu}_n, \hat{\nu}_n$ are known to be formally smooth. It follows that $\hat{\beta}_{d,n}$ is also formally smooth. Since X_n and Y_n are k -schemes of finite type, we can derive from the smoothness of $\beta_{d,n}$ from the formal smoothness of induced morphism on formal completions of points. \square

Here we prove that a certain morphism between finite order jet spaces, which are of finite type, is smooth by proving that the morphism on the level of arc spaces is formally smooth in a weak sense. It is also possible to prove directly the smoothness of the morphism between jet spaces by calculation on their tangent spaces as we will see later in the study of the Weierstrass division theorem, see the proof of Prop. 4.1.

Let us draw two consequences of the smoothness of $\beta_{d,n}$.

Corollary 3.10. *Let R be a local Henselian ring of maximal ideal \mathfrak{m} and residue field k . Let $x \in R[[t]]$ of nonzero reduction \bar{x} in $k[[t]] \bmod \mathfrak{m}$. Let $\text{val}(\bar{x}) = d$. Then for every $n > d$ there exists a monic polynomial $q \in Q_d(R)$ with coefficients in R and $u \in R[[t]]^\times$ such that*

$$qu \equiv x \pmod{t^n}.$$

Proof. This is the Henselian lifting property for smooth morphisms. \square

Corollary 3.11. *The morphism $\beta_d : Q_d \times \mathcal{L}\mathbb{G}_m \rightarrow \mathcal{L}^{\leq d}\mathbb{A}^1$ is a projective limit of a system of smooth morphisms with possibly non-smooth transition morphisms.*

Proof. By base change we obtain smooth morphisms

$$\beta_{d,n}^\infty : (Q_d \times \mathcal{L}_n\mathbb{G}_m) \times_{\mathcal{L}_{\mathbb{A}^1}} \mathcal{L}_n\mathbb{A}^1 \rightarrow \mathcal{L}\mathbb{A}^1$$

that form a projective system whose limit is β_d . One should pay a special attention to the fact that the transition maps in this system are not smooth. Indeed, on finite levels, we have a commutative diagram

$$\begin{array}{ccc} Q_d \times \mathcal{L}_m\mathbb{G}_m & \xrightarrow{\beta_m} & \mathcal{L}_m\mathbb{A}^1 \\ p_1 \downarrow & & \downarrow p_2 \\ Q_d \times \mathcal{L}_n\mathbb{G}_m & \xrightarrow{\beta_n} & \mathcal{L}_n\mathbb{A}^1 \end{array} \quad (3.17)$$

for all integers $m \geq n$ in which all maps are smooth. The diagram is however not Cartesian and it induces a morphism

$$(Q_d \times \mathcal{L}_m\mathbb{G}_m) \rightarrow (Q_d \times \mathcal{L}_n\mathbb{G}_m) \times_{\mathcal{L}_{\mathbb{A}^1}} \mathcal{L}_m\mathbb{A}^1 \quad (3.18)$$

that is not smooth. Thus the transition morphisms in the projective system $\varprojlim_n \beta_{d,n}^\infty = \beta_d$ are not smooth. \square

Based on the discussion on the property of the Weierstrass morphism β of (3.13), we will introduce a new type of morphisms which seems to be useful in dealing with arc spaces.

Definition 3.12. *A morphism $f : X \rightarrow Y$ of affine k -schemes is said to be of Weierstrass type if it satisfies the following properties:*

- *f is a filtered limit of a projective system of smooth morphisms in which transition maps are not required to be smooth;*
- *for every k -point $x \in X(k)$ with image $y \in Y(k)$, the induced morphism on formal completions $\hat{X}_x \rightarrow \hat{Y}_y$ is formally smooth.*

If the codomain is the point $Y = \text{Spec}(k)$, we will say that X is a k -scheme of Weierstrass type.

Arbitrary base changes of a morphism of Weierstrass type are of Weierstrass type. Composition of morphisms of Weierstrass type is also of the same type. As examples, the Weierstrass maps α and β of (3.13) is of Weierstrass type.

We will introduce a much stronger notion:

Definition 3.13. • A morphism $f : X \rightarrow Y$ is said to be essentially smooth surjective if it is the filtered limit of a projective system of smooth surjective morphism in which transition maps are also smooth surjective.

- A k -scheme X is said to be essentially smooth if the structural morphism $f : X \rightarrow \text{Spec}(k)$ is essentially smooth surjective.
- A k -scheme X is said to be essentially of finite type if there exists an essentially smooth surjective morphism $f : X \rightarrow S$ where S is a k -scheme of finite type.

A k -scheme essentially of finite type is of course also of Weierstrass type. The converse is wrong. There are much more k -schemes of Weierstrass type than k -schemes essentially of finite type. We will prove an useful criterion to recognize a morphism of Weierstrass type.

Proposition 3.14. Let X and Y be k -schemes essentially of finite type. If $f : X \rightarrow Y$ be a morphism that induces formally smooth morphisms on formal completions of k -points, then $f : X \rightarrow Y$ is a morphism of Weierstrass type.

Proof. Let us write $X = \text{Spec}(A)$ where $A = \varinjlim (A_i)$ with A_i being k -algebras of finite type and $X_i = \text{Spec}(A_i)$. Similarly we write $Y = \text{Spec}(B)$ where $B = \varinjlim (B_j)$ with B_j being k -algebras of finite type and $Y_j = \text{Spec}(B_j)$. The restriction of $\phi : B \rightarrow A$ to B_j has image of finite type so that $\phi(B_j) \subset A_i$ for some i . We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mu_i \downarrow & & \downarrow \nu_j \\ X_i & \xrightarrow{f_{i,j}} & Y_j \end{array} \quad (3.19)$$

Let $x \in X(k)$, $y = f(x) \in Y(k)$. If we denote x_i and y_j the images of x and y in X_i and Y_j respectively then we have $f_{i,j}(x_i) = y_j$. We consider the diagram of formal completions

$$\begin{array}{ccc} \hat{X}_x & \xrightarrow{\hat{f}_x} & \hat{Y}_y \\ \hat{\mu}_{i,x} \downarrow & & \downarrow \hat{\nu}_{j,y} \\ \hat{X}_{i,x_i} & \xrightarrow{\hat{f}_{i,j,x_i}} & \hat{Y}_{j,y_j} \end{array} \quad (3.20)$$

where $\hat{f}_x, \hat{\mu}_{i,x}, \hat{\nu}_{j,y}$ are formally smooth. It follows that \hat{f}_{i,j,x_i} is also formally smooth. For X_i and Y_j are k -schemes of finite type, the morphism $f_{i,j} : X_i \rightarrow Y_j$ is smooth at x_i . Since $X \rightarrow X_i$ is surjective, this implies that $f_{i,j} : X_i \rightarrow Y_j$ is smooth. Now $f : X \rightarrow Y$ can be realized as the projective limit of smooth morphisms $X_i \times_{Y_j} Y \rightarrow Y$ as $j \rightarrow \infty$. \square

4 Variations on the Weierstrass division theorem

In the classical setting, the Weierstrass division theorem asserts that if f is a germ of holomorphic functions at the origin of \mathbb{C}^{n+1} given with coordinates x_1, \dots, x_n, t, q is a polynomial of variable t of degree d of the form $q = q_0 + q_1 t + \dots + q_{d-1} t^{d-1} + t^d$ where q_0, \dots, q_{d-1} are germs of holomorphic functions at the origin of \mathbb{C}^n equipped with coordinates x_1, \dots, x_n and vanishing at the origin, then there exist an unique expression $f = qh + a$ where h is a germ of holomorphic function at the origin of \mathbb{C}^{n+1} and a is a polynomial of variable t of degree less than d whose coefficients are germs of holomorphic functions at the origin of \mathbb{C}^n .

In this section we aim to study the Weierstrass division problem when we replace the ring of germs of holomorphic functions at the origin of \mathbb{C}^n by an arbitrary k -algebra R , f by a formal series with coefficients in R , and $q \in R[t]$ a monic polynomial of degree d .

Recall that we have denoted Q_d the d -dimensional affine space representing the functor $R \mapsto Q_d(R)$ of all monic polynomials of degree d with coefficients in R . Let us now denote A_d the d -dimensional affine space representing the functor $R \mapsto A_d(R)$ of all polynomials of degree less than d . The Weierstrass division problem for a varying test ring R consists in studying the morphism

$$\alpha_d : Q_d \times A_d \times \mathcal{L}\mathbb{A}^1 \rightarrow Q_d \times \mathcal{L}\mathbb{A}^1 \quad (4.1)$$

given by the formula:

$$\alpha_d(q, a, v) = (q, qv + a). \quad (4.2)$$

In a sense, the morphism α_d is the derivative of the morphism β_d of (3.13).

The morphism $\beta_d : X \rightarrow Y$ of (3.13) where $X = Q_d \times \mathcal{L}\mathbb{G}_m$ and $Y = \mathcal{L}^\bullet \mathbb{A}^1$ induces a morphism of tangent bundles

$$d\beta_d : TX \rightarrow TY \times_Y X. \quad (4.3)$$

where $TX = Q_d \times \mathcal{L}\mathbb{G}_m \times A_d \times \mathcal{L}\mathbb{A}^1$, $TY = \mathcal{L}^\bullet \mathbb{A}^1 \times \mathcal{L}\mathbb{A}^1$, and $TY \times_Y X = Q_d \times \mathcal{L}\mathbb{G}_m \times \mathcal{L}\mathbb{A}^1$. If $(q, u; a, v) \in Q_d \times \mathcal{L}\mathbb{G}_m \times A_d \times \mathcal{L}\mathbb{A}^1$ then

$$d\beta_d(q, u, a, v) = (q, u, ua + qv) \quad (4.4)$$

which is a point of $Q_d \times \mathcal{L}\mathbb{G}_m \times \mathcal{L}\mathbb{A}^1$. We obtain α_d from $d\beta_d$ by setting the parameter u to be 1.

Proposition 4.1. *For every integer $d \in \mathbb{N}$, the morphisms $d\beta_d$ of (4.3) and α_d of (4.1) are surjective and induces formally smooth morphisms on formal completions of points. Over the point $q = t^d$, it induces isomorphism on formal completions of points.*

Proof. Because β_d is a morphism of Weierstrass type, so is its derivative $d\beta_d$. By base change, α_d is also a morphism of Weierstrass type. The same argument applies for the property of being surjective on k -points. Over the point $q = t^d$, β_d induces an isomorphism on formal completions of points, so are $d\beta_d$ and α_d .

One could also reverse the argument and prove first that α_d induces a smooth morphism at finite levels. Indeed, for every $n \geq d$, the morphism

$$\alpha_{d,n} : Q_d \times A_d \times \mathcal{L}_n \mathbb{A}^1 \rightarrow Q_d \times \mathcal{L}_n \mathbb{A}^1 \quad (4.5)$$

is a linear morphism of finite rank vector bundles over A_d . It is enough to prove that $\alpha_{d,n}$ induces a surjective linear morphism in every fiber. It is enough to prove that for a monic polynomial $q \in Q_d(k)$ of degree d with coefficients in k , the k -linear map

$$A_d(k) \times k[[t]]/(t^n) \rightarrow k[[t]]/(t^n)$$

given by the formula (4.1) is surjective. We will prove that the k -linear map

$$A_d(k) \times k[[t]] \rightarrow k[[t]]$$

given by the formula (4.1) is surjective. This is equivalent to saying that the map

$$A_d(k) \rightarrow k[[t]]/(q)$$

is surjective. This map factors as

$$A_d(k) \rightarrow k[t]/(q) \rightarrow k[[t]]/(q)$$

where $A_d(k) \rightarrow k[t]/(q)$ is an isomorphism by the Euclidean division theorem. The map $k[t]/(q) \rightarrow k[[t]]/(q)$ is surjective as $k[[t]]/(q) = k[t]/(t^e)$ where e is the multiplicity of t dividing q . \square

We will now exhibit a strange family of finite-dimensional schemes of Weierstrass type. For every $d \in \mathbb{N}$, we denote S_d the kernel of morphism α_d of (4.1), in other words, we have the Cartesian diagram

$$\begin{array}{ccc} S_d & \longrightarrow & Q_d \times A_d \times \mathcal{L} \mathbb{A}^1 \\ \alpha_d \downarrow & & \downarrow \alpha_d \\ Q_d & \xrightarrow{0} & Q_d \times \mathcal{L} \mathbb{A}^1 \end{array} \quad (4.6)$$

where 0 is the zero section $q \mapsto (q, 0) \in Q_d \times \mathcal{L} \mathbb{A}^1$. The scheme S_d is a generalized vector bundle over Q_d in the sense that its fiber over each point $q \in Q_d(k)$ is a k -vector space. The dimension of the fibers varies however in the opposite direction of what happens with a morphism of finite type. Indeed, for every monic polynomial $q \in Q_d(k)$ of degree d , the

fiber $S_{d,q}(k)$ of S_d over q is a k -vector space of dimension $d - e$ where e is order of vanishing of q at 0. In particular, if $q(0) \neq 0$ then $\dim S_{d,q}(k) = d$, and if $q = t^d$ then $S_{d,q}(k) = 0$.

Let us compute explicitly the ring of coordinates of S_1 . We will write $Q_1 = \text{Spec}(k[q_0])$ with the universal monic polynomial of degree one $q = q_0 + t \in Q_1(k[q_0])$. We will denote $A_1 = \text{Spec}(k[a_0])$ the space of polynomial of degree less than one and $\mathcal{L}\mathbb{A}^1 = k[x_0, x_1, \dots]$. The scheme S_1 is defined as the closed subscheme of $Q_1 \times A_1 \times \mathcal{L}\mathbb{A}^1$ defined by the system of equations

$$\begin{aligned} a_0 + q_0 x_0 &= 0 \\ x_0 + q_0 x_1 &= 0 \\ x_1 + q_0 x_2 &= 0 \\ &\dots \end{aligned}$$

In other words, we have $S_1 = \text{Spec}(R_1)$ where

$$R_1 = k[q_0, a_0, x_0, x_1, \dots] / (a_0 + q_0 x_0, x_0 + q_0 x_1, \dots). \quad (4.7)$$

The fiber of S_1 over a point with coordinate $q_0 \neq 0$ in Q_1 is the affine line since the coordinate a_0 will determine other coordinates by $x_0 = -a_0^{-1} a_0$, $x_1 = -a_0^{-1} x_0 \dots$. On the other hand, the fiber of S_1 over the point $q_0 = 0$ in Q_1 is reduced to the point of coordinates $a_0 = 0$ and $x_0 = x_1 = \dots = 0$.

We may look at S_1 from three different angles. First, from the equations defining S_1 we see that S_1 is a projective limit of affine planes, in other words its ring of coordinates R_1 is an inductive limit $R_1 = \varinjlim A_n$ with $A_n = k[q_0, x_n]$ with transition maps $A_n \mapsto A_{n+1}$ given by $x_n \mapsto -q_0 x_{n+1}$. The morphism $\text{Spec}(A_{n+1}) \rightarrow \text{Spec}(A_n)$ can be realized as the blow-up of the affine plane $\mathbb{A}^2 = \text{Spec}(A_n)$ at the origin $(q_0, x_n) = (0, 0)$ with the strict transform of the x_n -axis deleted.

Second, set theoretically speaking, S_1 looks like a "sawed plane" i.e. the plane of coordinates (q, x) with the x -axis $(0, x)$ deleted and the origin $(0, 0)$ added back in. This constructible subset of the plane can be thus realized as the set of points of a scheme which is of course not of finite type.

Third, the sawed plane S_1 may be seen in some sense as the glueing of the plane (q, x) with the x -axis deleted and with the q -axis $(q, 0)$. Indeed, if A is a k -algebra of finite type and $\phi : R_1 \rightarrow A$ be a A -point of S_1 , mapping some prime ideal \mathfrak{p}_A of A to the prime ideal of R_1 generated by $\mathfrak{p}_0 = (q_0, a_0, x_0, \dots)$ corresponding to the origin $(0, 0)$ of S_1 , then the morphism $\text{Spec}(A) \rightarrow S_1$ must factor through the q -axis. Indeed, under this assumption we have $\phi(q_0) \in \mathfrak{p}_A$. Since the variables a_0, x_0, \dots are divisible by every power of q_0 , we have $\phi(a_0), \phi(x_0), \dots \in \mathfrak{p}_A^n$ for every $n \in \mathbb{N}$. Since A is a k -algebra of finite type, it follows that $\phi(a_0) = \phi(x_0) = \dots = 0$.

Let us identify the q -axis with the subscheme of S_1 defined by the equations $a_0 = x_0 = x_1 = \dots = 0$. We can show that that the completion of S_1 at origin is isomorphic to the

completion of the q -axis at the origin defined by the maximal ideal \mathfrak{q} of R_1 . Indeed, for every n , we have $a_0, x_0, x_1, \dots \in \mathfrak{q}^n$ and therefore $R_1/\mathfrak{p}_0^n = k[q_0]/\mathfrak{q}_0^n$.

Proposition 4.2. *For every $d \in \mathbb{N}$, let S_d be the kernel of the homomorphism α_d of (4.1) defined by the Cartesian diagram (4.6).*

1. S_d is the projective limit of kernels of homomorphisms $\alpha_{d,n}$ of (4.5) which are vector bundles of rank d over Q_d as $n \rightarrow \infty$. In particular S_d is a scheme of Weierstrass type.
2. As topological space S_d is the constructible subspace of $Q_d \times A_d$ whose fiber over $q \in Q_d(k)$ is a k -vector space of rank $d-e$ where e is the order of vanishing of q at $t = 0$. In particular, S_d contains the zero section $Q_d \times \{0\}$ of $Q_d \times A_d$.
3. For every k -algebra of finite type A and $\phi : \text{Spec}(A) \rightarrow S_d$ a A -point of S_d whose image contains the point $(t^d, 0) \in Q_d \times A_d$. Then x factors through the zero section $Q_d \times \{0\}$.
4. The completion of S_d at the point $(t^d, 0)$ is isomorphic to the completion of the zero section at the same point.

Proof. The proof proceeds in the same way as the calculations in the case $d = 1$ that we have just outlined. \square

We will now consider another family of strange schemes springing out of Weierstrass division problem. For every $d \in \mathbb{N}$, we consider the homomorphism

$$\mu_d : Q_d \times \mathcal{L}\mathbb{A}^1 \rightarrow Q_d \times \mathcal{L}\mathbb{A}^1 \quad (4.8)$$

defined by $(q, x) \mapsto (q, qx)$, and define Z_d to be the kernel of μ_d . For every k -algebra R , the set $Z_d(R)$ consists of pairs (q, x) where $q \in R[[t]]$ is a monic polynomial of degree d and $x \in R[[t]]$ a formal series with coefficients in R satisfying $qx = 0$.

We may also define Z_d as the closed subscheme of S_d consisting of triples $(q, a, x) \in Q_d \times A_d \times \mathcal{L}\mathbb{A}^1$ with $a = 0$. Set theoretically, there is no difference between Z_d and the zero section $Q_d \times \{0\}$ as for every $q \in Q_d(k)$, the multiplication by q is injective in $k[[t]]$. These spaces are however different as schemes, in other words Z_d is an infinitesimal thickening of $Q_d \times \{0\}$.

Let us consider again the case $d = 1$ and describe explicitly the coordinate ring of Z_1 . Using the equations (4.7) of S_1 , we have $Z_1 = \text{Spec}(V_1)$ where

$$V_1 = k[q_0, x_0, x_1, \dots,] / (q_0 x_0, x_0 + q_0 x_1, x_1 + q_0 x_2, \dots). \quad (4.9)$$

The algebra V_1 can be realized as the inductive limit of $V_{1,n} = k[q_0, x_n] / (q_0^{n+1} x_n)$ with respect to transition maps $V_{1,n} \rightarrow V_{1,n+1}$ given by $x_n \mapsto -q_0 x_{n+1}$. We note that the transition maps $V_{1,n} \rightarrow V_{1,n+1}$ are all injective so that $x_n \neq 0$ in V_1 . The scheme Z_1 is therefore the projective

limit of schemes $Z_{1,n} = \text{Spec}(V_{1,n})$ which topologically is a union of the q_0 -axis and the x_n -axis as $n \rightarrow \infty$. We see that in the limit, the x_n -axis all contract topologically to the origin although they don't algebraically in the sense that $x_n \neq 0$. The ideal of V_1 generated by x_0, x_1, \dots defined the closed subscheme $Q_1 \times \{0\}$ of Z_1 .

Proposition 4.3. *Let $Z_d = \text{Spec}(V_d)$ and I_d the ideal of V_d defining the zero section $Q_d \times \{0\}$. Then we have $I_d^2 = 0$. Moreover every element $f \in I_d$ vanishes at every point $z \in Z_d$ at infinite order.*

Proof. Let us restrict ourselves to the case $d = 1$ as the general case is completely similar. We first prove that $x_m x_n = 0$ for all $m, n \in \mathbb{N}$. Since $x_m = (-1)^m q_0^{m+1} x_{m+1}$, we have $x_m x_n = (-1)^{m+n+1} x_{m+n+1} q_0^{m+n+1} x_n = 0$.

We next prove that x_n vanishes to an infinite order at every point of $\text{Spec}(V_1) = \text{Spec}(k[q_0])$. At a point z of coordinate $q_0 \neq 0$, the equation $q_0^{n+1} x_n = 0$ implies that $x_n = 0$ in the local ring $V_{1,z}$ in which q_0 is invertible. At the point z of coordinate $q_0 = 0$, as q_0 belongs to the maximal ideal \mathfrak{m}_z , x_n belongs to every power of \mathfrak{m}_z . \square

Definition 4.4. *For every commutative ring R , we will denote*

$$\text{nil}_\infty(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \bigcap_{n \in \mathbb{N}} \mathfrak{p}^n, \quad (4.10)$$

the infinite nilradical of R , the intersection of all powers of all prime ideals of R .

Let us first collect some easy facts about the infinite nilradical. First, the infinite nilradical is obviously contained in the nilradical

$$\text{nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p},$$

which consists of all nilpotent elements of R . In particular, if R is reduced then both $\text{nil}(R)$ and $\text{nil}_\infty(R) = 0$. Second, as we can determine whether an element $x \in R$ belongs to the nilradical by evaluating it via the homomorphisms $\nu_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$ for all prime ideal $\mathfrak{p} \in \text{Spec}(R)$, we can also determine whether an element $x \in R$ belongs to the infinite nilradical by evaluating it via homomorphisms $\nu_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}^n$ for all $\mathfrak{p} \in \text{Spec}(R)$ and $n \in \mathbb{N}$. In particular, $\text{nil}_\infty(R) = 0$ if and only if for every non-zero element $x \in R$, there exists a ring homomorphism $\nu : R \rightarrow A$ from R to an Artinian local ring such that $\nu(x) \neq 0$.

Proposition 4.5. *For a Noetherian ring R , we have $\text{nil}_\infty(R) = 0$.*

Proof. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ denote the set of associated primes of R . By the primary decomposition theorem [5, Thm. 6.8], there exist primary ideals I_1, \dots, I_r with $\sqrt{I_i} = \mathfrak{p}_i$ such that $I_1 \cap \dots \cap I_r = 0$. As $\sqrt{I_i} = \mathfrak{p}_i$, for every i , there exists m_i such that $\mathfrak{p}_i^{m_i} \subset I_i$. It follows that $\text{nil}_\infty(R) \subset I_1 \cap \dots \cap I_r$ and therefore $\text{nil}_\infty(R) = 0$. \square

Here is an example of commutative ring R with $\text{nil}_\infty(R) \neq 0$ deriving from our study of Weierstrass division theorem. By Prop. 4.3, $\text{nil}_\infty(V_d)$ is a nonzero ideal generated by x_0, x_1, \dots . Here is another example familiar in the p -adic Hodge theory. Let \mathcal{O} denote the ring of integers of \mathbb{C}_p the p -adic completion of the algebraic closure of the field \mathbb{Q}_p of p -adic numbers. Then \mathcal{O} is a local ring with maximal ideal \mathfrak{m} satisfying $\mathfrak{m} = \mathfrak{m}^2$. In particular $\mathfrak{m} \subset \text{nil}_\infty(\mathcal{O})$ and the latter is nonzero.

Definition 4.6. *A morphism of k -schemes $f : X \rightarrow Y$ is said to be a nil_∞ -quasi-isomorphism if it induces a bijection $X(R) \rightarrow Y(R)$ on every k -algebra R with $\text{nil}_\infty(R) = 0$.*

For instance, for every commutative ring, the closed embedding $\text{Spec}(R/\text{nil}_\infty(R)) \rightarrow \text{Spec}(R)$ is a nil_∞ -quasi-isomorphism. In particular, the morphism $Q_d \rightarrow Z_d$ in Prop. 4.3 is a nil_∞ -quasi-isomorphism. Here is a related and very instructive example of nil_∞ -quasi-isomorphism which is not an isomorphism.

Let $X = \text{Spec}(k[x, y]/(xy))$ denote the union of the x -axis and y -axis in the plane of coordinates (x, y) . Let $X' = X - \{0\} = \text{Spec}(k[x^{\pm 1}]) \sqcup \text{Spec}(k[y^{\pm 1}])$ denote the complement of the origin in X . We consider the space $\mathcal{L}^\bullet X$ of arcs on X which are non-degenerate with respect to the open subset X' . The inclusion of the x -axis in X induces a morphism $\mathcal{L}^\bullet \mathbb{A}^1 \rightarrow \mathcal{L}^\bullet X$ and so does the y -axis. We have thus a morphism

$$\mathcal{L}^\bullet \mathbb{A}^1 \sqcup \mathcal{L}^\bullet \mathbb{A}^1 \rightarrow \mathcal{L}^\bullet X \quad (4.11)$$

which induces a bijection on k -points. In fact it induces a bijection on R -points for all k -algebras R with $\text{nil}_\infty(R) = 0$ by 4.7 but not in general by 4.3.

Lemma 4.7. *Let R be a commutative ring with $\text{nil}_\infty(R) = 0$, $x \in R[[t]]$ a non-degenerate formal series. Then the multiplication by x in $R[[t]]$ is injective.*

Proof. First we assume that R is an Artinian local ring of maximal ideal \mathfrak{m} and residue field k_R . In this case, the non-degeneracy condition means $\bar{x} \in k_R[[t]]$ is non zero in other words $\bar{x} \in k_R[[t]] \cap k_R((t))^\times$. It follows that $x \in R[[t]] \cap R((t))^\times$. Because the map $R[[t]] \rightarrow R((t))$ is injective, and the multiplication by x is a bijection on $R((t))$, the multiplication by x is injective on $R[[t]]$.

Let R be a ring with zero infinite radical and $x \in R[[t]]$ a non-degenerate formal series. Let $y \in R[[t]]$ be a formal series such that $xy = 0$. Then for every ring homomorphism $\nu : R \rightarrow A$ from R to Artinian local ring, we have $\nu(y) = 0$. If $\text{nil}_\infty(R)$ this implies $y = 0$. \square

We will use the Newton method to analyze the arc space of certain complete intersection following Drinfeld's paper [3]. The difference with [3] is that we will have to work over all k -algebras R while Drinfeld restricts himself to Artinian rings. We will soon realize that Drinfeld's arguments generalize well only under the assumption $\text{nil}_\infty(R) = 0$ for some familiar linear algebra facts only hold over rings with zero infinite radical.

Lemma 4.8. *Let R be a commutative ring with $\text{nil}_\infty(R) = 0$. Let $\phi \in \text{Mat}_n(R[[t]])$ a $n \times n$ -matrix with coefficients in $R[[t]]$ whose determinant $\xi = \det(\phi) \in R[[t]]$ in a non-degenerate formal series as defined in 3.1. Then the associated linear operator*

$$\phi : R[[t]]^n \rightarrow R[[t]]^n$$

is injective, and its image consists of vectors $v \in R[[t]]^n$ such that

$$\phi'v \equiv 0 \pmod{\xi}$$

where ϕ' is the adjugate matrix of ϕ .

Proof. By the Cramer formula we have $\phi' \phi = \phi \phi' = \xi \text{id}_n$ where id_n is the identity matrix. By Lem. 4.7 the multiplication by ξ in $R[[t]]^n$ is injective. It follows that both linear operators ϕ and ϕ' are injective.

For $u, v \in R[[t]]^n$, the injectivity of ϕ, ϕ' and the Cramer rule together imply that $\phi(v) = u$ if and only if $\xi v = \phi'(u)$. In other words $u \in \text{im}(\phi)$ if and only if $\phi'(u) \equiv 0 \pmod{\xi}$. \square

5 Variations on the Newton method

As in [3], we will assume that X is defined as the central fiber of a morphism $p : \mathbb{A}^m \rightarrow \mathbb{A}^n$ with $m \geq n$. The coordinates of $f = (f_1, \dots, f_n)$ are regular functions on \mathbb{A}^m . We will use the coordinates $x = (x_1, \dots, x_{m-n}, y_1, \dots, y_n)$ for $x \in \mathbb{A}^n$. We will consider the $n \times n$ -matrix

$$B(x) = \left(\frac{\partial f_i}{\partial y_j} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \quad (5.1)$$

whose coefficients are regular functions on \mathbb{A}^n . We will assume that

$$\xi(x) = \det(B(x)) \quad (5.2)$$

is a non-zero regular function on \mathbb{A}^m . Let Z denote the closed subscheme of \mathbb{A}^m defined by the equation $\xi = 0$ and U its complement open subscheme. We will denote

$$\mathcal{L}^\bullet \mathbb{A}^m = \mathcal{L} \mathbb{A}^m - \mathcal{L} Z$$

the non-degenerate arc space of \mathbb{A}^m with respect to the open subset U .

The Taylor expansion of f with respect to the variables y_1, \dots, y_n is of the form

$$f(x + v) = f(x) + B(x; v) + B_2(x; v) + B_3(x; v) + \dots \quad (5.3)$$

for $x = (x_1, \dots, x_{m-n}, y_1, \dots, y_n) \in \mathbb{A}^m$ and $v = (v_1, \dots, v_n) \in \mathbb{A}^n$ and

$$x + v = (x_1, \dots, x_{m-n}, y_1 + v_1, \dots, y_n + v_n). \quad (5.4)$$

Here $B(x; v)$ is a function linear with respect to the variable v given by the matrix $B(x)$, and more generally $B_n(x; v)$ is a homogenous polynomial of degree n of variables v whose coefficients are polynomials of x .

We consider the Cartesian diagram

$$\begin{array}{ccc} N_0 & \longrightarrow & \mathcal{L}^\bullet \mathbb{A}^m \times \mathcal{L} \mathbb{A}^n \\ \alpha_0 \downarrow & & \downarrow \alpha \\ \mathcal{L}^\bullet X & \longrightarrow & \mathcal{L} \mathbb{A}^m \end{array} \quad (5.5)$$

with $\alpha(x, v_0) = x + t\xi(x)v_0$ where the addition is defined as in (5.4). For α is a surjective morphism of Weierstrass type, so is α_0 . For every k -algebra R , $N_0(R)$ is the set of $(x, v_0) \in R[[t]]^m \times R[[t]]^n$ such that

$$f(x + t\xi(x)v_0) = 0. \quad (5.6)$$

and $\xi(x)$ is a non-degenerate formal series.

We will replace N_0 by another infinite dimensional scheme defined by congruence equations. Let N_1 be the affine k -scheme that whose R -points are

$$N_1(R) = \{(x, v_1) \in \mathcal{L}^\bullet \mathbb{A}^m(R) \times \mathcal{L} \mathbb{A}^n(R) \mid \hat{B}(x, f(x)) = t\xi(x)^2 v_1\} \quad (5.7)$$

As the equations $\hat{B}(x, f(x)) = t\xi(x)^2 v_1$ defining N_1 are essentially equivalent to the congruence equation

$$\hat{B}(x; f(x)) \equiv 0 \pmod{t\xi(x)^2} \quad (5.8)$$

one may say that N_1 is defined in the arc space of the affine space of \mathbb{A}^m by finitely many congruence equations.

Proposition 5.1. *There exists a nil_∞ -quasi-isomorphism $\nu_0 : N_0 \rightarrow N_1$.*

Proof. We consider the Taylor expansion (5.3) of $f(x + t\xi(x)v_0)$ to rewrite the equation (5.6) in the form

$$f(x) + t\xi(x)B(x; v_0) + t^2\xi(x)^2H(x; v_0) = 0. \quad (5.9)$$

By applying the linear form $\hat{B}(x)$, we get the equation

$$\hat{B}(x; f(x)) + t\xi(x)^2 v_0 + t^2\xi(x)^2 \hat{B}(x; H(x; v_0)) = 0 \quad (5.10)$$

where \hat{B} is the adjugate matrix of B . We then derive the equation

$$\hat{B}(x; f(x)) + t\xi^2(x)v_1 = 0 \quad (5.11)$$

with $v_1 = v_0 + t\hat{B}(x; H(x; v_0))$. Our desired morphism $N_0 \rightarrow N_1$ is given by the formula $(x, v_0) \mapsto (x, v_0 + t\hat{B}(x; H(x; v_0)))$.

We will now prove that the morphism $N_0 \rightarrow N_1$ constructed above induces a bijection $N_0(R) \rightarrow N_1(R)$ for every k -algebra with zero infinite radical. For an arbitrary k -algebra R , and an arbitrary vector $v_1 \in R[[t]]^n$, there exists a unique $v_0 \in R[[t]]^n$ such that $v_1 = v_0 + t\hat{B}(x; H(x; v_0))$ by the following lemma 5.2. Since $v_1 \in N_1(R)$ the relation (5.10) holds. We derives (5.9) from Prop. 4.8 under the assumption $\text{nil}_\infty(R) = 0$. \square

Lemma 5.2. *Let $h : \mathbb{A}^n \rightarrow \mathbb{A}^n$ an arbitrary polynomial self-map. Then for every commutative ring R and an arbitrary, the map $R[[t]]^n \rightarrow R[[t]]^n$ given by $v_0 \mapsto v_0 + th(v_0)$ is bijective.*

Proof. Let $v_1 \in R[[t]]^n$. We will prove that there exists a unique v_0 such that $v_1 = v_0 + th(v_0)$. It is equivalent to say that the self-map $\phi(v) = v_1 - th(v)$ of $R[[t]]^n$ admits a unique fixed point. This is just an application of the Banach fixed point theorem for contraction map.

Let us consider the norm on $R[[t]]^n$ defined as follows. For $v = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \dots \in R[[t]]^n$ with $\alpha_i \in R^n$, we set $|v| = 2^{-d}$ where d is the smallest integer such that $\alpha_d \neq 0$. By construction of $R[[t]]$ as a projective limit of $R[[t]]/(t^d)$, $R[[t]]^n$ is complete metric space with respect to this norm.

Now for h is polynomial, we have $|h(v) - h(w)| \leq |v - w|$. As $\phi(v) = v_1 - th(v)$, we have $|\phi(v) - \phi(w)| \leq |v - w|/2$. In other words, the self-map $v \mapsto \phi(v)$ is contraction map with contraction ratio $\leq 1/2$. It therefore admits a unique fixed point by the Banach fixed point theorem. \square

6 Local structure of arc space

We will simplify notations in setting $c(x) = \xi^2(x)$ and $a(x) = \hat{B}(x, f(x))$ in the definition of N_1 :

$$N_1(R) = \{(x, v) \in \mathcal{L}^\bullet \mathbb{A}^m(R) \times \mathcal{L} \mathbb{A}^n(R) \mid a(x) = tc(x)v\}. \quad (6.1)$$

By definition c is a polynomial function on \mathbb{A}^m and a is a polynomial vector $a : \mathbb{A}^m \rightarrow \mathbb{A}^n$.

In the goal of this section is to prove that after base change by a morphism of Weierstrass type $N \rightarrow N_1$, there exists an essentially smooth morphism $N \rightarrow Y$ where Y is a scheme of finite type over the n -fold Cartesian product of S_d over Q_d

$$S_d^n = S_d \times_{Q_d} S_d \times_{Q_d} \dots \times_{Q_d} S_d \quad (6.2)$$

where S_d is the "strange" vector bundle over Q_d defined in (4.6). Recall that as d varies S_d form family of finite dimensional schemes, which are not of finite type, in which the first member S_1 is the "sawed plane".

Proposition 6.1. *There exists a cover of N_1 by morphisms of Weierstrass type $N_{1,d} \rightarrow N_1$ such that for every d , there exists an isomorphism*

$$N_{1,d} = T_d \times \mathbb{A}^\infty$$

where T_d is a scheme of finite type over S_{d+1}^{n+1} .

Proof. For every $d \in \mathbb{N}$, we consider the Cartesian diagram

$$\begin{array}{ccc} N_{1,d} & \xrightarrow{\xi} & Q_d \times \mathcal{L}\mathbb{G}_m \times A_{d+1}^m \times \mathcal{L}\mathbb{A}^m \\ \alpha_1 \downarrow & & \downarrow (\beta, \alpha) \\ N_1 & \xrightarrow{\gamma} & \mathcal{L}\mathbb{A} \times \mathcal{L}\mathbb{A}^m \end{array} \quad (6.3)$$

where γ is essentially the graph of c

$$\gamma(x, v) = (c(x), x)$$

and (β, α) is the morphism of Weierstrass type

$$(\beta, \alpha)(q, u, \bar{x}, x') = (uq, \bar{x} + tqx'). \quad (6.4)$$

By base change, $N_{1,d} \rightarrow N_1$ is a morphism of Weierstrass type.

Let us denote M the space of quintuples (q, u, \bar{x}, x', v) :

$$M = Q_d \times \mathcal{L}\mathbb{G}_m \times A_{d+1}^m \times \mathcal{L}\mathbb{A}^m \times \mathcal{L}\mathbb{A}^n$$

By construction $N_{1,d}$ is the closed subscheme of M consisting of quintuples

$$(q, u, \bar{x}, x', v) \in M \quad (6.5)$$

satisfying the equations:

$$a(\bar{x} + tqx') = tuqv \quad (6.6)$$

$$c(\bar{x} + tqx') = uq \quad (6.7)$$

The idea is now to expand $a(\bar{x} + tqx')$ and $c(\bar{x} + tqx')$ in the form

$$a(\bar{x} + tqx') = \bar{a}(q, \bar{x}) + tqa'(q, \bar{x}, x') \quad (6.8)$$

$$c(\bar{x} + tqx') = \bar{c}(q, \bar{x}) + tqc'(q, \bar{x}, x') \quad (6.9)$$

where $\bar{a} : Q_d \times A_{d+1}^m \rightarrow Q_d \times A_{d+1}^n$ and $\bar{c} : Q_d \times A_{d+1}^m \rightarrow Q_d \times A_{d+1}^n$ are deduced from a and c by the general procedure ??, $a'(q, \bar{x}, x') \in \mathcal{L}\mathbb{A}^n$ and $c'(q, \bar{x}, x') \in \mathcal{L}\mathbb{A}^1$ are respectively a n -vector of formal series and a single formal series whose coefficients are polynomial functions of coefficients of q, \bar{x}, x' .

We will now construct the morphisms $\bar{a} : Q_d \times A_{d+1}^m \rightarrow Q_d \times A_{d+1}^n$ and $\bar{c} : Q_d \times A_{d+1}^m \rightarrow Q_d \times A_{d+1}^n$ and the expansions (6.8) and (6.9).

Let X be an affine k -scheme of finite type. The functor X_{Q_d} on the category of k -algebras

$$X_{Q_d}(R) = \{(q, x) \mid q \in Q_d(R), x : \text{Spec}(R[t]/q) \rightarrow X\}. \quad (6.10)$$

is representable by a flat affine Q_d -scheme X_{Q_d} . The fiber of X_{Q_d} over separable polynomial $q \in Q_d(k)$ is isomorphic to X^d . The fiber of X_{Q_d} over the polynomial $q = t^d$ is isomorphic to the jet space $\mathcal{L}_{d-1}X$ of X . If $X = \mathbb{A}^1$ is the affine line, we have

$$\mathbb{A}_{Q_d}^1(R) = \{(q, x) \mid q \in Q_d(R), x \in R[t]/q\} \quad (6.11)$$

which is a vector bundle of rank r over Q_d . As the canonical map $A_d(R) \rightarrow R[t]/q$ from the space $A_d(R)$ of polynomials of degree less than d to $R[t]/q$ is an isomorphism by Euclidian division theorem, the latter vector bundle can be canonically trivialized

$$Q_d \times A_d = \mathbb{A}_{Q_d}^1. \quad (6.12)$$

For every morphism $f : X \rightarrow Y$, we have an induced morphism

$$f_{Q_d} : X_{Q_d} \rightarrow Y_{Q_d} \quad (6.13)$$

Now the morphism $a : \mathbb{A}^m \rightarrow \mathbb{A}^n$ induces a morphism $a_{Q_d} : \mathbb{A}_{Q_d}^m \rightarrow \mathbb{A}_{Q_d}^n$. By identifying $\mathbb{A}_{Q_d}^m = Q_d \times A_d^m$ and $\mathbb{A}_{Q_d}^n = Q_d \times A_d^n$ as in (6.12), we get the desired morphism

$$\bar{a} : Q_d \times A_d^m \rightarrow Q_d \times A_d^n. \quad (6.14)$$

Similarly, we have a morphism

$$\bar{c} : Q_d \times A_d^m \rightarrow Q_d \times A_d. \quad (6.15)$$

induced by c .

Now N can be the space of quintuples (q, u, \bar{x}, x', v) as in (6.5) satisfying two equations

$$\bar{a}(q, \bar{x}) + tq(a'(q, \bar{x}, x') - uv) = 0 \quad (6.16)$$

$$(\bar{c}(q, \bar{x}) - u_0q) + tq(c'(q, \bar{x}, x') - u') = 0 \quad (6.17)$$

where u_0 is the constant coefficient of u and $u = u_0 + tu'$ where u' is a formal series.

We "solve" this system of equations by the following diagram in which all square are Cartesian:

$$\begin{array}{ccccccc} N_{1,d} & \xrightarrow{g_N} & T_d & \longrightarrow & S_{d+1}^{n+1} & \longrightarrow & Q_{d+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow 0 \\ M & \xrightarrow{g_M} & \mathbb{G}_m \times Q_d \times A_{d+1}^m \times \mathcal{L}\mathbb{A}^{n+1} & \longrightarrow & Q_{d+1} \times A_{d+1}^{n+1} \times \mathcal{L}\mathbb{A}^{n+1} & \xrightarrow{\alpha_{d+1}} & Q_{d+1} \times \mathcal{L}\mathbb{A}^{n+1} \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{G}_m \times Q_d \times A_{d+1}^m & \xrightarrow{\psi} & Q_{d+1} \times A_{d+1}^{n+1} & & \end{array}$$

We will describe the arrows in this diagram from right to left, and then move bottom up in each column

- the morphism $0 : Q_{d+1} \rightarrow Q_{d+1} \times \mathcal{L}\mathbb{A}^{n+1}$ is the zero section $q_{d+1} \mapsto (q_{d+1}, 0)$;
- α_{d+1} is given by the formula (4.1);
- the upper right square is Cartesian by the very definition of $S_{d+1, n+1}$;
- the morphism ψ on the third line is given by $\psi(u_0, q, \bar{x}) = (tq, (\bar{a}(q, \bar{x}), \bar{c}(q, \bar{x}) - u_0q))$;
- the two down arrows from the second line to the third are obvious projections, the lower square between those lines is obviously Cartesian;
- T_d is defined so that the upper middle square is Cartesian;
- the morphism $g_M : M \rightarrow \mathbb{G}_m \times Q_d \times A_{d+1}^m \times \mathcal{L}\mathbb{A}^{n+1}$ maps a point $(q, u, \bar{x}, x', v) \in M$ with $u = u_0 + tu'$ on the point

$$g_M(q, u, \bar{x}, x', v) = (u_0, q, \bar{x}, (a'(q, \bar{x}, x') - uv, c'(q, \bar{x}, x') - u')).$$

- as by definition $N = M \times_{Q_{d+1} \times \mathcal{L}\mathbb{A}^{n+1}} Q_{d+1}$, there exists a unique morphism $g_N : N \rightarrow S$ making the upper left square Cartesian.

By construction, T_d is a scheme of finite type over S_{d+1}^{n+1} as ψ is a morphism between k -schemes of finite type. Since the upper left corner is Cartesian, in order to prove that there exists an isomorphism $N_{1,d} = T_d \times \mathbb{A}^\infty$, it is enough to prove that there exists an isomorphism

$$M = (\mathbb{G}_m \times Q_d \times A_{d+1}^m \times \mathcal{L}\mathbb{A}^{n+1}) \times \mathbb{A}^\infty \quad (6.18)$$

compatible with the projection to $\mathbb{G}_m \times Q_d \times A_{d+1}^m \times \mathcal{L}\mathbb{A}^{n+1}$.

Given a point

$$(u_0, q, (tq, \bar{x}), w) \in (\mathbb{G}_m \times Q_d \times_{Q_{d+1}} \mathbb{A}_{Q_{d+1}}^m \times \mathcal{L}\mathbb{A}^{n+1})(R)$$

we need to solve the equation

$$g_M(q, u, \bar{x}, x', v) = (u_0, q, (tq, \bar{x}), w).$$

Let us write the vector $w \in \mathcal{L}\mathbb{A}^{n+1}(R)$ as (w_n, w_1) with $w_n \in R[[t]]^n$ and $w_1 \in R[[t]]$ and $u = u_0 + u't$ where $u_0 \in R^\times$ and $u' \in R[[t]]$. We need to determine the set of

$$(u', x', v) \in R[[t]] \times R[[t]]^m \times R[[t]]^n$$

such that

$$\begin{aligned} a'(q, \bar{x}, x') - uv &= w_n \\ c'(q, \bar{x}, x') - u' &= w_1 \end{aligned}$$

The set of solutions is uniquely parametrized by $x' \in R[[t]]^m$ because the above system of equations is equivalent to

$$\begin{aligned} v &= u^{-1}(a'(q, \bar{x}, x') - w_n) \\ u' &= c'(q, \bar{x}, x') - w_1. \end{aligned}$$

These equations yield the desired isomorphism (6.18). □

As consequence, we obtain the following diagram describing local structures of arc spaces

$$\begin{array}{ccccc} N_{1,d} = T_d \times \mathbb{A}^\infty & \xrightarrow{\alpha_1} & N_1 & \xleftarrow{\nu_0} & N_0 & \xrightarrow{\alpha_0} & \mathcal{L}X \\ \text{pr}_1 \downarrow & & & & & & \\ T_d & \xrightarrow{\gamma} & S_{d+1}^{n+1} & & & & \end{array} \quad (6.19)$$

where ν_0 is a nil_∞ -quasi-isomorphism, α_0, α_1 are morphisms of Weierstrass type, and γ is a morphism of finite type. We note that the positions of ν_0 and γ_1 may be exchanged so that $N_{1,d} \rightarrow \mathcal{L}X$ is the composition of a nil_∞ -quasi-isomorphism with a morphism of Weierstrass type. On the lower line, T_d is of of finite type over a the finite-dimensional scheme S_{d+1}^{n+1} which is of Weierstrass type but not of finite type. It doesn't seem possible to invert orders i.e. T_d is likely not being of Weierstrass type over a scheme of finite type.

Although, the local structure of arc spaces as described by this diagram is far from being pleasant, we can derive the existence of good slices.

7 Construction of good slices

We keep the notations in force in the previous section. In particular, we have a isomorphism $N_{1,d} \simeq T_d \times \mathbb{A}^\infty$ where T_d is a S_{d+1}^{n+1} -scheme of finite type. Recall that S_{d+1}^{n+1} , finite-dimensional but not of finite type, is a strange vector bundle over Q_{d+1} . We consider the subscheme Y_d of T_d constructed by the Cartesian diagram:

$$\begin{array}{ccc} Y_d & \longrightarrow & Q_{d+1} \\ \downarrow & & \downarrow 0 \\ T_d & \longrightarrow & S_{d+1}^{n+1} \end{array} \quad (7.1)$$

where the right vertical map is the zero section. By base change, the morphism $Y_d \rightarrow Q_{d+1}$ is of finite type and therefore Y_d is a k -scheme affine of finite type. Let $Y_d = \text{Spec}(R_d)$ where R_d is a k -algebra of finite type.

For every R_d -point $r \in \mathbb{A}^\infty(R_d)$, we consider the composed morphism

$$\phi_1(r) : Y_d \rightarrow T_d \rightarrow T_d \times \mathbb{A}^\infty \simeq N_{1,d} \rightarrow N_1$$

defines a R_d -point of N_1 . Since the morphism $N_0 \rightarrow N_1$ is a nil_∞ -quasi-isomorphism after Prop. 5.1, the induced map on R_d -points $N_0(R_d) \rightarrow N_1(R_d)$ is a bijection. We can thus lift $f_1(r) : Y_d \rightarrow N_1$ to a morphism

$$\phi_0(r) : Y_d \rightarrow N_0$$

and thus a morphism

$$\phi(r) : Y_d \rightarrow \mathcal{L}X.$$

Proposition 7.1. *For every $r \in \mathbb{A}^\infty(R_d)$, for every k -field k' , $y' \in Y(k')$ mapping to $x' \in \mathcal{L}X(k')$, there exists an isomorphism*

$$\hat{Y}_{y'} \hat{\times} \hat{D}^\infty \simeq \mathcal{L}X_{x'} \hat{\times} \hat{D}^\infty.$$

Proof. A morphism $f : M \rightarrow N$ will be said to be a formal equivalence on formal completions if for every $m \in M(k)$ mapping to $n \in N(k)$, there exists an isomorphism

$$\hat{M}_m \hat{\times} \hat{D}^\infty \simeq \hat{N}_n \hat{\times} \hat{D}^\infty.$$

It is clear that the composition of two morphisms which are formal equivalences on formal completions morphisms, is also a formal equivalence on formal completions. It is also clear that a morphism of Weierstrass type is a formal equivalence on formal completions.

We claim that the morphism $Y_d \rightarrow T_d$ is a formal equivalence on formal completions. It follows that $\phi_1(r) : Y_d \rightarrow N_1$ is a formal equivalence on formal completions for the morphism $N_{1,d} \rightarrow N_1$ is of Weierstrass type.

The morphism $N_0 \rightarrow N_1$ being a nil_∞ -quasi-isomorphism, induces isomorphism on formal completions of points, it follows that $\phi_0(r) : Y_d \rightarrow N_0$ is a formal equivalence on formal completions. Finally the morphism $N_0 \rightarrow \mathcal{L}X$ is of Weierstrass type, and therefore a formal equivalence on formal completions. We derive that the morphism $\phi(r) : Y_d \rightarrow \mathcal{L}X$ is a formal equivalence on formal completions. \square

Lemma 7.2. *The morphism $Y_d \rightarrow T_d$ constructed in the diagram (7.1) is a formal equivalence on formal completions.*

Proof. This follows from the fact the morphism $Q_{d+1} \rightarrow S_{d+1}^{n+1}$ is a formal equivalence on formal completions. This property is preserved by base change by a morphism of finite type. \square

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