Arithmetic of certain integrable systems

Ngô Bao Châu

University of Chicago &
Vietnam Institute for Advanced Study in Mathematics
System of congruence equations

Let us consider a system of congruence equations

\[
\begin{cases}
P_1(x_1, \ldots, x_n) = 0 \\
\vdots \\
P_m(x_1, \ldots, x_n) = 0
\end{cases}
\]
Let us consider a system of congruence equations

\[
\begin{align*}
P_1(x_1, \ldots, x_n) &= 0 \\
\vdots \\
P_m(x_1, \ldots, x_n) &= 0
\end{align*}
\]

where \( P_1, \ldots, P_m \in \mathbb{F}_p[x_1, \ldots, x_n] \) are polynomial with coefficients in \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \).
System of congruence equations

Let us consider a system of congruence equations

\[
\begin{align*}
P_1(x_1, \ldots, x_n) &= 0 \\
\vdots \\
P_m(x_1, \ldots, x_n) &= 0
\end{align*}
\]

where \( P_1, \ldots, P_m \in \mathbb{F}_p[x_1, \ldots, x_n] \) are polynomial with coefficients in \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \).

We are interested in the number of solutions of this system with in \( \mathbb{F}_p \), and more generally in \( \mathbb{F}_{p^r} \) where \( \mathbb{F}_{p^r} \) is the finite extension of degree \( r \) of \( \mathbb{F}_p \).
If we denote \( X = \text{Spec} \mathbb{F}_p[x_1, \ldots, x_n]/(P_1, \ldots, P_m) \), the algebraic variety defined by the system of equations

\[
P_1 = 0, \ldots, P_m = 0,
\]

then \( X(\mathbb{F}_p^r) \) is the set of solutions with values in \( \mathbb{F}_p^r \).
Valued points of algebraic variety

- If we denote $X = \text{Spec} \mathbb{F}_p[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$, the algebraic variety defined by the system of equations
  \[ P_1 = 0, \ldots, P_m = 0, \]
  then $X(\mathbb{F}_p^r)$ is the set of solutions with values in $\mathbb{F}_p^r$.

- Let $X(\overline{\mathbb{F}_p}) = \bigcup_{r \in \mathbb{N}} X(\mathbb{F}_p^r)$ be the set of points with values in the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. The Galois group $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ acts on $X(\overline{\mathbb{F}_p})$. It is generated by the Frobenius element $\sigma(x) = x^p$, and $\text{Fix}(\sigma_r, X(\overline{\mathbb{F}_p})) = X(\mathbb{F}_p^r)$. 
Valued points of algebraic variety

- If we denote $X = \text{Spec} \mathbb{F}_p[x_1, \ldots, x_n]/(P_1, \ldots, P_m)$, the algebraic variety defined by the system of equations

$$P_1 = 0, \ldots, P_m = 0,$$

then $X(\mathbb{F}_p^r)$ is the set of solutions with values in $\mathbb{F}_p^r$.

- Let $X(\overline{\mathbb{F}}_p) = \bigcup_{r \in \mathbb{N}} X(\mathbb{F}_p^r)$ be the set of points with values in the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.

- The Galois group $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $X(\overline{\mathbb{F}}_p)$. It is generated by the Frobenius element $\sigma(x) = x^p$, and

$$\text{Fix}(\sigma^r, X(\overline{\mathbb{F}}_p)) = X(\mathbb{F}_p^r).$$
For a prime number $\ell \neq p$, Grothendieck defined the groups of $\ell$-adic cohomology of $X$

$$H^i(X) = H^i(X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \text{ and } H_c^i = H_c^i(X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)$$

for every algebraic variety $X$ over $\mathbb{F}_p$. 

Deligne proved that for every field isomorphism $\iota: \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$,

the inequality $|\iota(\alpha)| \leq p^{i/2}$ for all eigenvalues $\alpha$ of $\sigma$ acting on $H_c^i(X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)$. 

Grothendieck-Lefschetz formula
Grothendieck-Lefschetz formula

For a prime number $\ell \neq p$, Grothendieck defined the groups of $\ell$-adic cohomology of $X$

$$H^i(X) = H^i(X \otimes_{F_p} \bar{F}_p, \mathbb{Q}_\ell)$$

and

$$H^i_c = H^i_c(X \otimes_{F_p} \bar{F}_p, \mathbb{Q}_\ell)$$

for every algebraic variety $X$ over $\mathbb{F}_p$

and proved the Lefschetz fixed points formula

$$\#\text{Fix}(\sigma^r_p, X(\bar{\mathbb{F}}_p)) = \sum_{i=0}^{2\dim(X)} (-1)^i \text{tr}(\sigma^r_p, H^i_c(X)).$$
Grothendieck-Lefschetz formula

- For a prime number \( \ell \neq p \), Grothendieck defined the groups of \( \ell \)-adic cohomology of \( X \)
  
  \[
  H^i(X) = H^i(X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \quad \text{and} \quad H^i_c(X) = H^i_c(X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)
  \]
  
  for every algebraic variety \( X \) over \( \mathbb{F}_p \)

- and proved the Lefschetz fixed points formula
  
  \[
  \#\text{Fix}(\sigma^r_p, X(\overline{\mathbb{F}}_p)) = \sum_{i=0}^{2\dim(X)} (-1)^i \text{tr}(\sigma^r_p, H^i_c(X)).
  \]

- Deligne proved that for every field isomorphism \( \iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C} \), the inequality \( |\iota(\alpha)| \leq p^{i/2} \) for all eigenvalues \( \alpha \) of \( \sigma \) acting on \( H^i_c(X) \).
Equality of numbers of points

- We will be concerned with proving equality of type

\[ \#X(\mathbb{F}_p) = \#X'(\mathbb{F}_p) \]

for different algebraic varieties.
Equality of numbers of points

- We will be concerned with proving equality of type
  \[ \#X(\mathbb{F}_p^r) = \#X'(\mathbb{F}_p^r) \]
  for different algebraic varieties.

- We would like to develop a principle of ”analytic continuation of equalities”: Let \( f : X \to Y \) and \( f' : X' \to Y \) be morphisms of algebraic varieties. If the equality \( \#X_y(F_{q^r}) = \#X'_y(F_{q^r}) \) holds for every point \( y \) in a dense open subset \( U \) of \( Y \), then it holds for every \( y \in Y \).

This can't be true in general. The question is to find geometric assumptions on \( f \) and \( f' \) that guarantee this principle.

The complex of \( \ell \)-adic sheaves \( f! \mathbb{Q}_\ell \) interpolates all cohomology group with compact support \( H^i_c(X_y) = H^i_c(f! \mathbb{Q}_\ell) \) for all geometric points \( y \in Y \). Geometric assumption on \( f \) give constraint on the complex \( f! \mathbb{Q}_\ell \).
Equality of numbers of points

- We will be concerned with proving equality of type
  \[ \#X(F_{p^r}) = \#X'(F_{p^r}) \]
  for different algebraic varieties.

- We would like to develop a principle of "analytic continuation of equalities": Let \( f : X \to Y \) and \( f' : X' \to Y \) be morphisms of algebraic varieties. If the equality \( \#X_y(F_{q^r}) = \#X'_y(F_{q^r}) \) holds for every point \( y \) in a dense open subset \( U \) of \( Y \), then it holds for every \( y \in Y \).

- This can't be true in general. The question is to find geometric assumptions on \( f \) and \( f' \) that guarantee this principle.
Equality of numbers of points

- We will be concerned with proving equality of type
  \[ \#X(\mathbb{F}_p) = \#X'(\mathbb{F}_p) \]
  for different algebraic varieties.
- We would like to develop a principle of "analytic continuation of equalities": Let \( f : X \to Y \) and \( f' : X' \to Y \) be morphisms of algebraic varieties. If the equality \( \#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r}) \) holds for every point \( y \) in a dense open subset \( U \) of \( Y \), then it holds for every \( y \in Y \).
- This can't be true in general. The question is to find geometric assumptions on \( f \) and \( f' \) that guarantee this principle.
- The complex of \( \ell \)-adic sheaves \( f_!\mathbb{Q}_\ell \) interpolates all cohomology group with compact support \( \mathbb{H}^i_c(X_y) \)
  \[ \mathbb{H}^i(f_!\mathbb{Q}_\ell)_y = \mathbb{H}^i_c(X_y) \]
  for all geometric points \( y \in Y \). Geometric assumption on \( f \) give constraint on the complex \( f_!\mathbb{Q}_\ell \).
The case of proper and smooth morphisms

Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset $U$ of $Y$, such that for all $y \in U(F_q)$, $\#X_y(F_q) = \#X'_y(F_q)$.
The case of proper and smooth morphisms

- Let \( f : X \rightarrow Y \) and \( f' : X' \rightarrow Y \) be proper and smooth morphisms. Assume that there exists an dense open subset \( U \) of \( Y \), such that for all \( y \in U, \#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r}) \).

- If \( f : X \rightarrow Y \) and \( f' : X' \rightarrow Y \) are proper and smooth morphisms then \( H^i(f_! \mathbb{Q}_\ell) \) and \( H^i(f'_! \mathbb{Q}_\ell) \) are \( \ell \)-adic local systems for every \( i \in \mathbb{Z} \).
The case of proper and smooth morphisms

Let \( f : X \rightarrow Y \) and \( f' : X' \rightarrow Y \) be proper and smooth morphisms. Assume that there exists an dense open subset \( U \) of \( Y \), such that for all \( y \in U(\mathbb{F}_q) \), \( \#X_y(\mathbb{F}_q) = \#X'_y(\mathbb{F}_q) \).

If \( f : X \rightarrow Y \) and \( f' : X' \rightarrow Y \) are proper and smooth morphisms then \( H^i(f_! \mathbb{Q}_\ell) \) and \( H^i(f'_! \mathbb{Q}_\ell) \) are \( \ell \)-adic local systems for every \( i \in \mathbb{Z} \).

Deligne’s theorem implies that

\[
\text{tr}(\sigma_y, H^i_c(X_y)) = \text{tr}(\sigma_y, H^i_c(X'_y)).
\]
The case of proper and smooth morphisms

- Let \( f : X \to Y \) and \( f' : X' \to Y \) be proper and smooth morphisms. Assume that there exists an dense open subset \( U \) of \( Y \), such that for all \( y \in U(\mathbb{F}_{q^r}) \), \( \#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r}) \).

- If \( f : X \to Y \) and \( f' : X' \to Y \) are proper and smooth morphisms then \( H^i(f_!\mathbb{Q}_\ell) \) and \( H^i(f'_!\mathbb{Q}_\ell) \) are \( \ell \)-adic local systems for every \( i \in \mathbb{Z} \).

- Deligne’s theorem implies that

\[
\text{tr}(\sigma_y, H^i_c(X_y)) = \text{tr}(\sigma_y, H^i_c(X'_y)).
\]

- The Chebotarev density theorem implies that the \( \ell \)-adic local systems \( H^i(f_!\mathbb{Q}_\ell) \) and \( H^i(f'_!\mathbb{Q}_\ell) \) are isomorphic.
The case of proper and smooth morphisms

Let $f : X \to Y$ and $f' : X' \to Y$ be proper and smooth morphisms. Assume that there exists an dense open subset $U$ of $Y$, such that for all $y \in U(\mathbb{F}_{q^r})$, $\#X_y(\mathbb{F}_{q^r}) = \#X'_y(\mathbb{F}_{q^r})$.

If $f : X \to Y$ and $f' : X' \to Y$ are proper and smooth morphisms then $H^i(f_*\mathbb{Q}_\ell)$ and $H^i(f'_*\mathbb{Q}_\ell)$ are $\ell$-adic local systems for every $i \in \mathbb{Z}$.

Deligne’s theorem implies that

$$\text{tr}(\sigma_y, H^i_c(X_y)) = \text{tr}(\sigma_y, H^i_c(X'_y)).$$

The Chebotarev density theorem implies that the $\ell$-adic local systems $H^i(f_*\mathbb{Q}_\ell)$ and $H^i(f'_*\mathbb{Q}_\ell)$ are isomorphic.

A local system is determined by its restriction to any dense open subset.
To obtain interesting cases, one has to drop the smoothness assumption.
Singularities

- To obtain interesting cases, one has to drop the smoothness assumption.
- Goresky-MacPherson’s theory of perverse sheaves is very efficient in dealing with singularities of algebraic maps.
Singularities

To obtain interesting cases, one has to drop the smoothness assumption.

Goresky-MacPherson’s theory of perverse sheaves is very efficient in dealing with singularities of algebraic maps.

For every algebraic variety $Y$, the category $P(Y)$ of perverse sheaves of $Y$ is an abelian categories. For every morphism $f : X \to Y$, one can define perverse cohomology

$$pH^i(f_! Q_\ell) \in P(Y)$$

in similar way as usual cohomology $H^i(f_! Q_\ell)$ are usual $\ell$-adic sheaves.
Purity and semi-simplicity

Let $f : X \to Y$ be a proper morphism where $X$ is a smooth variety. Then according to Deligne, $f_! \mathbb{Q}_\ell$ is a pure complex of sheaves.

As an important consequence of Deligne's purity theorem, Beilinson, Bernstein, Deligne and Gabber proved that after base change to $Y \otimes \overline{\mathbb{F}}_p$, $H^i(f_! \mathbb{Q}_\ell)$ is a direct sum of simple perverse sheaves.

There exists over $Y \otimes \overline{\mathbb{F}}_p$ a decomposition in direct sum $f_! \mathbb{Q}_\ell = \bigoplus_{\alpha \in A} K_\alpha [n_\alpha]$ where $K_\alpha$ are simple perverse sheaves and $n_\alpha \in \mathbb{Z}$. 
Purity and semi-simplicity

- Let $f : X \rightarrow Y$ be a proper morphism where $X$ is a smooth variety. Then according to Deligne, $f_!\mathbb{Q}_\ell$ is a pure complex of sheaves.

- As an important consequence of Deligne’s purity theorem, Beilinson, Bernstein, Deligne and Gabber proved that after base change to $Y \otimes \overline{\mathbb{F}}_p$, $^pH^i(f_!\mathbb{Q}_\ell)$ is a direct sum of simple perverse sheaves.
Let $f : X \to Y$ be a proper morphism where $X$ is a smooth variety. Then according to Deligne, $f_! \mathbb{Q}_\ell$ is a pure complex of sheaves.

As an important consequence of Deligne’s purity theorem, Beilinson, Bernstein, Deligne and Gabber proved that after base change to $Y \otimes \overline{\mathbb{F}}_p$, $p^i \mathcal{H}(f_! \mathbb{Q}_\ell)$ is a direct sum of simple perverse sheaves.

There exists over $Y \otimes \overline{\mathbb{F}}_p$ a decomposition in direct sum

$$f_! \mathbb{Q}_\ell = \bigoplus_{\alpha \in \mathcal{A}} K_\alpha [n_\alpha]$$

where $K_\alpha$ are simple perverse sheaves and $n_\alpha \in \mathbb{Z}$. 
Simple perverse sheaves

Let \( i : Z \to Y \otimes_{F_p} \overline{F}_p \) be the immersion of an irreducible closed irreducible subscheme. Let \( j : U \to Z \) be the immersion of a nonempty open subscheme. Let \( L \) be an irreducible local system on \( U \), then

\[
K = i_* j_! L[\dim Z]
\]

is a simple perverse sheaf on \( Y \otimes_{F_p} \overline{F}_p \).
Simple perverse sheaves

- Let $i : Z \to Y \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ be the immersion of an irreducible closed irreducible subscheme. Let $j : U \to Z$ be the immersion of a nonempty open subscheme. Let $L$ be an irreducible local system on $U$, then

$$K = i_! j^! L[\dim Z]$$

is a simple perverse sheaf on $Y \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$.

- According to Goresky and MacPherson, every simple perverse sheaf is of this form.
Simple perverse sheaves

- Let \( i : Z \to Y \otimes_{F_p} \bar{F}_p \) be the immersion of an irreducible closed irreducible subscheme. Let \( j : U \to Z \) be the immersion of a nonempty open subscheme. Let \( L \) be an irreducible local system on \( U \), then

\[
K = i_! j_! L [\dim Z]
\]

is a simple perverse sheaf on \( Y \otimes_{F_p} \bar{F}_p \).

- According to Goresky and MacPherson, every simple perverse sheaf is of this form.

- The definition of the intermediate extension functot \( j_! \) is complicated. For us, what really matters is that the perverse sheaf is completely determined by the local system \( L \), more generally, it is determined by the restriction of \( L \) to any nonempty open subscheme of \( U \).
Support

- If $K$ is a simple perverse sheaf on $Y \otimes \overline{F}_p$, it is of the form $K = i_*j!_* L[\dim Z]$. In particular, $\text{supp}(K) := Z$ is completely determined.
If $K$ is a simple perverse sheaf on $Y \otimes \bar{F}_p$, it is of the form $K = i_*j_!j^!L[\dim Z]$. In particular, $\text{supp}(K) := Z$ is completely determined.

Let $f : X \to Y$ be a proper morphism where $X$ is a smooth variety. Then, $f_*\mathbb{Q}_\ell$ can be decomposed into a direct sum

$$f_*\mathbb{Q}_\ell = \bigoplus_{\alpha \in \mathcal{A}} K_\alpha[n_\alpha]$$

of simple perverse sheaves. The finite set

$$\text{supp}(f) = \{Z_\alpha | Z_\alpha = \text{supp}(K_\alpha)\}$$

is well determined. This is an important topological invariant of $f$. 
Let $f : X \to Y$ and $f' : X' \to Y$ be proper morphisms with $X, X'$ smooth varieties. If

$$\text{supp}(f) = \text{supp}(f') = \{Y\},$$

then the analytic continuation principle applies as $f_!\mathbb{Q}_\ell$ and $f'_!\mathbb{Q}_\ell$ are determined by their restrictions to any nonempty open subset.
Let $f : X \to Y$ and $f' : X' \to Y$ be proper morphisms with $X, X'$ smooth varieties. If

$$\text{supp}(f) = \text{supp}(f') = \{ Y \},$$

then the analytic continuation principle applies as $f_! \mathbb{Q}_\ell$ and $f'_! \mathbb{Q}_\ell$ are determined by their restrictions to any nonempty open subset.

This is true if $f$ and $f'$ are proper and smooth.
Let $f : X \to Y$ and $f' : X' \to Y$ be proper morphisms with $X, X'$ smooth varieties. If

$$\text{supp}(f) = \text{supp}(f') = \{Y\},$$

then the analytic continuation principle applies as $f_! Q_\ell$ and $f'_! Q_\ell$ are determined by their restrictions to any nonempty open subset.

This is true if $f$ and $f'$ are proper and smooth.

Are there more interesting cases?
Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$. 

Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if $X$ is smooth, then $f^! \mathcal{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.

In particular $\text{supp}(f) = \{Y\}$.

Argument: play the Poincaré duality against the cohomological amplitude.
Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if \( \dim(\mathcal{X} \times Y X - \Delta_X) < \dim(X) \).

- Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if $X$ is smooth, then $f_!\mathcal{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.
$f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$.

Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if $X$ is smooth, then $f_! \mathbb{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.

In particular $\text{supp}(f) = \{Y\}$.
Small map

- $f : X \to Y$ is small in the sense of Goresky and MacPherson if $\dim(X \times_Y X - \Delta_X) < \dim(X)$.
- Goresky and MacPherson proved that if $f : X \to Y$ is a small proper map and if $X$ is smooth, then $f_!\mathbb{Q}_\ell$ is a perverse sheaf which is the intermediate extension of its restriction to any dense open subset.
- In particular $\text{supp}(f) = \{Y\}$
- Argument: play the Poincaré duality against the cohomological amplitude.
Let \( f : X \to Y \) be a relative curve such that \( X \) is smooth, \( f \) is proper, for generic \( y \in Y \), \( X_y \) is smooth and for every \( y \in Y \), \( X_y \) is irreducible.

Then according to Goresky and MacPherson, \( \text{supp}(f) = \{Y\} \). Argument: play the Poincaré duality against the cohomological amplitude.
Relative curve

- Let $f : X \to Y$ be a relative curve such that $X$ is smooth, $f$ is proper, for generic $y \in Y$, $X_y$ is smooth and for every $y \in Y$, $X_y$ is irreducible.
- Then according to Goresky and MacPherson, $\text{supp}(f) = \{Y\}$. 
Let $f : X \to Y$ be a relative curve such that $X$ is smooth, $f$ is proper, for generic $y \in Y$, $X_y$ is smooth and for every $y \in Y$, $X_y$ is irreducible.

Then according to Goresky and MacPherson, $\text{supp}(f) = \{Y\}$.

Argument: play the Poincaré duality against the cohomological amplitude.
Assume for simplicity \( \dim(X) = 2 \) and \( \dim(Y) = 1 \).
Assume for simplicity $\dim(X) = 2$ and $\dim(Y) = 1$.

The cohomological amplitude of a relative curve:

$H^i(f_!Q_\ell[2]) = 0$ for $i \notin \{-2, -1, 0\}$
Assume for simplicity \( \dim(X) = 2 \) and \( \dim(Y) = 1 \).

The cohomological amplitude of a relative curve: \( H^i(f_!\mathbb{Q}_\ell[2]) = 0 \) for \( i \notin \{-2, -1, 0\} \).

Assume there exists a simple perverse sheaf \( K_\alpha \) such that \( K_\alpha[n_\alpha] \) is a direct factor of \( f_!\mathbb{Q}_\ell[-2] \) and \( \dim(Z_\alpha) = 0 \) where \( Z_\alpha = \text{supp}(K_\alpha) \).
Assume for simplicity \( \dim(X) = 2 \) and \( \dim(Y) = 1 \).

The cohomological amplitude of a relative curve:
\[
H^i(f_! Q_\ell[2]) = 0 \text{ for } i \not\in \{-2, -1, 0\}
\]

Assume there exists a simple perverse sheaf \( K_\alpha \) such that \( K_\alpha[n_\alpha] \) is a direct factor of \( f_! Q_\ell[-2] \) and \( \dim(Z_\alpha) = 0 \) where \( Z_\alpha = \text{supp}(K_\alpha) \).

\( H^0(K_\alpha) \neq 0 \), the cohomological amplitude implies that \( n_\alpha \geq 0 \).
Poincaré duality versus cohomological amplitude

- Assume for simplicity \( \dim(X) = 2 \) and \( \dim(Y) = 1 \).
- The cohomological amplitude of a relative curve:
  \[ H^i(f_! \mathbb{Q}_\ell [2]) = 0 \text{ for } i \notin \{-2, -1, 0\} \]
- Assume there exists a simple perverse sheaf \( K_\alpha \) such that
  \( K_\alpha[n_\alpha] \) is a direct factor of \( f_! \mathbb{Q}_\ell [-2] \) and \( \dim(Z_\alpha) = 0 \) where \( Z_\alpha = \text{supp}(K_\alpha) \).
- \( \text{H}^0(K_\alpha) \neq 0 \), the cohomological amplitude implies that \( n_\alpha \geq 0 \).
- By Poincaré duality \( K_\alpha^\vee[-n_\alpha] \) is also a direct factor of \( f_! \mathbb{Q}_\ell [2] \)
  where \( \text{supp}(K_\alpha^\vee) = \text{supp}(K_\alpha) \). It follows that \( n_\alpha \leq 0 \).
Poincaré duality versus cohomological amplitude

- Assume for simplicity \( \dim(X) = 2 \) and \( \dim(Y) = 1 \).
- The cohomological amplitude of a relative curve: 
  \[ H^i(f_!\mathbb{Q}_\ell[2]) = 0 \text{ for } i \notin \{-2, -1, 0\} \]
- Assume there exists a simple perverse sheaf \( K_\alpha \) such that 
  \( K_\alpha[n_\alpha] \) is a direct factor of \( f_!\mathbb{Q}_\ell[-2] \) and \( \dim(Z_\alpha) = 0 \) where 
  \( Z_\alpha = \text{supp}(K_\alpha) \).
- \( H^0(K_\alpha) \neq 0 \), the cohomological amplitude implies that 
  \( n_\alpha \geq 0 \).
- By Poincaré duality \( K_\alpha^\vee[-n_\alpha] \) is also a direct factor of \( f_!\mathbb{Q}_\ell[2] \) 
  where \( \text{supp}(K_\alpha^\vee) = \text{supp}(K_\alpha) \). It follows that \( n_\alpha \leq 0 \).
- It follows that \( n_\alpha = 0 \). But then \( H^0(K_\alpha) \) is a direct factor of 
  \( H^2(f_!\mathbb{Q}_\ell) = \mathbb{Q}_\ell(-1) \). This is not possible.
Goresky-MacPherson’s inequality

Let $f : X \to Y$ be a proper morphism with fiber of dimension $d$. Assume $X$ smooth. Let $Z \in \text{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_\ell$.

Moreover, if the geometric fibers of $f$ are irreducible, then $\text{codim}(Z) < d$.

For abelian fibration, Goresky-MacPherson’s inequality can be used to establish the full support theorem.
Goresky-MacPherson’s inequality

- Let $f : X \to Y$ be a proper morphism with fiber of dimension $d$. Assume $X$ smooth. Let $Z \in \text{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_\ell$.

- Then $\text{codim}(Z) \leq d$. For abelian fibration, Goresky-MacPherson’s inequality can be used to establish the full support theorem.
Let $f : X \to Y$ be a proper morphism with fiber of dimension $d$. Assume $X$ smooth. Let $Z \in \text{supp}(f)$ be the support of a perverse direct factor of $f_!\mathbb{Q}_\ell$.

Then $\text{codim}(Z) \leq d$.

Moreover, if the geometric fibers of $f$ are irreducible, then $\text{codim}(Z) < d$. 
Let $f : X \to Y$ be a proper morphism with fiber of dimension $d$. Assume $X$ smooth. Let $Z \in \text{supp}(f)$ be the support of a perverse direct factor of $f_! \mathbb{Q}_{\ell}$.

Then $\text{codim}(Z) \leq d$.

Moreover, if the geometric fibers of $f$ are irreducible, then $\text{codim}(Z) < d$.

For abelian fibration, Goresky-MacPherson’s inequality can be used to establish the full support theorem.
Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension $d$, we assume that the action has affine stabilizers: for every geometric point $s \in S$, for every $m \in M_s$, the stabilizer $P_m$ is affine.
- We assume that the Tate modules of $P$ is polarizable.
Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension $d$, 
- $P$ acts on $M$ relatively over $S$. 

We assume that the action has affine stabilizers: for every geometric point $s \in S$, for every $m \in M_s$, the stabilizer $P_m$ is affine.

We assume that the Tate modules of $P$ is polarizable.
Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension $d$,
- $P$ acts on $M$ relatively over $S$.
- We assume that the action has affine stabilizers: for every geometric point $s \in S$, for every $m \in M_s$, the stabilizer $P_m$ is affine.
Weak abelian fibration

- $f : M \to S$ is a proper morphism, $g : P \to S$ is a smooth commutative group scheme, both of relative dimension $d$,
- $P$ acts on $M$ relatively over $S$.
- We assume that the action has affine stabilizers: for every geometric point $s \in S$, for every $m \in M_s$, the stabilizer $P_m$ is affine.
- We assume that the Tate modules of $P$ is polarizable.
Tate module in family

- Assume $P$ has connected fibers, for every geometric point $s \in S$, there exists a canonical exact sequence

$$0 \to R_s \to P_s \to A_s \to 0$$

where $A_s$ is an abelian variety and $R_s$ is a connected affine group. This induces an exact sequence of Tate modules

$$0 \to T_{\mathbb{Q}_\ell}(R_s) \to T_{\mathbb{Q}_\ell}(P_s) \to T_{\mathbb{Q}_\ell}(A_s) \to 0.$$
Tate module in family

Assume $P$ has connected fibers, for every geometric point $s \in S$, there exists a canonical exact sequence

$$0 \to R_s \to P_s \to A_s \to 0$$

where $A_s$ is an abelian variety and $R_s$ is a connected affine group. This induces an exact sequence of Tate modules

$$0 \to T_{\mathbb{Q}_\ell}(R_s) \to T_{\mathbb{Q}_\ell}(P_s) \to T_{\mathbb{Q}_\ell}(A_s) \to 0.$$  

The Tate modules can be interpolated into a single $\ell$-adic sheaf

$$H^1(P/S) = H^{2d-1}(g_! \mathbb{Q}_\ell)$$

with fiber $H^1(P/S)_s = T_{\mathbb{Q}_\ell}(P_s)$. Polarization of the Tate module of $P$ is an alternating form on $H^1(P/S)$ vanishing on $T_{\mathbb{Q}_\ell}(R_s)$ and induces a perfect pairing on $T_{\mathbb{Q}_\ell}(A_s)$. 
For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$ the dimension of the affine part of $P_s$. 
\( \delta \)-regularity

- For every geometric point \( s \in S \), we define \( \delta(s) = \dim(R_s) \) the dimension of the affine part of \( P_s \).
- For every \( \delta \in \mathbb{N} \),

\[
S_\delta = \{ s \in S | \delta(s) = \delta \}
\]

is locally closed.
$\delta$-regularity

- For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$, the dimension of the affine part of $P_s$.
- For every $\delta \in \mathbb{N}$,

$$S_\delta = \{ s \in S | \delta(s) = \delta \}$$

is locally closed.
- $P \to S$ is said to be $\delta$-regular if $\text{codim}(S_\delta) \geq \delta$ for every $\delta \in \mathbb{N}$.
For every geometric point $s \in S$, we define $\delta(s) = \dim(R_s)$, the dimension of the affine part of $P_s$.

For every $\delta \in \mathbb{N}$,

$$S_\delta = \{ s \in S | \delta(s) = \delta \}$$

is locally closed.

$P \to S$ is said to be $\delta$-regular if $\text{codim}(S_\delta) \geq \delta$ for every $\delta \in \mathbb{N}$.

In particular, for $\delta = 1$, the $\delta$-regularity means $P$ is generically an abelian variety.
\(\delta\)-regularity

- For every geometric point \(s \in S\), we define \(\delta(s) = \dim(R_s)\) the dimension of the affine part of \(P_s\).
- For every \(\delta \in \mathbb{N}\),
  \[
  S_\delta = \{ s \in S | \delta(s) = \delta \}
  \]
  is locally closed.
- \(P \to S\) is said to be \(\delta\)-regular if \(\text{codim}(S_\delta) \geq \delta\) for every \(\delta \in \mathbb{N}\).
- In particular, for \(\delta = 1\), the \(\delta\)-regularity means \(P\) is generically an abelian variety.
- One can prove \(\delta\)-regularity for all Hamiltonian completely integrable system.
\(\delta\)-regularity

- For every geometric point \(s \in S\), we define \(\delta(s) = \dim(R_s)\), the dimension of the affine part of \(P_s\).
- For every \(\delta \in \mathbb{N}\),
  \[
  S_\delta = \{s \in S | \delta(s) = \delta\}
  \]
  is locally closed.
- \(P \to S\) is said to be \(\delta\)-regular if \(\text{codim}(S_\delta) \geq \delta\) for every \(\delta \in \mathbb{N}\).
- In particular, for \(\delta = 1\), the \(\delta\)-regularity means \(P\) is generically an abelian variety.
- One can prove \(\delta\)-regularity for all Hamiltonian completely integrable system.
- \(\delta\)-regularity is harder to prove in characteristic \(p\).
Theorem of support for abelian fibration

Theorem: Let \((f : M \to S, g : P \to S)\) be a \(\delta\)-regular abelian fibration. Assume that \(M\) is smooth, the fibers of \(f : M \to S\) are irreducible. Then

\[
\text{supp}(f) = \{S\}.
\]
Theorem of support for abelian fibration

- **Theorem:** Let \((f : M \to S, g : P \to S)\) be a \(\delta\)-regular abelian fibration. Assume that \(M\) is smooth, the fibers of \(f : M \to S\) are irreducible. Then

\[
\text{supp}(f) = \{S\}.
\]

- **Corollary:** Let \((P, M, S)\) and \((P', M', S)\) be \(\delta\)-regular abelian fibrations as above (in particular, \(M_s\) and \(M'_s\) are irreducible). If the generic fibers of \(P\) and \(P'\) are isogenous abelian varieties, then for every \(s \in S(\mathbb{F}_q)\), 

\[
#M_s(\mathbb{F}_q) = #M'_s(\mathbb{F}_q).
\]
Theorem of support for abelian fibration

- **Theorem:** Let \((f : M \to S, g : P \to S)\) be a \(\delta\)-regular abelian fibration. Assume that \(M\) is smooth, the fibers of \(f : M \to S\) are irreducible. Then

\[ \text{supp}(f) = \{S\}. \]

- **Corollary:** Let \((P, M, S)\) and \((P', M', S)\) be \(\delta\)-regular abelian fibrations as above (in particular, \(M_s\) and \(M'_s\) are irreducible). If the generic fibers of \(P\) and \(P'\) are isogenous abelian varieties, then for every \(s \in S(\mathbb{F}_q)\), \(\#M_s(\mathbb{F}_q) = \#M'_s(\mathbb{F}_q)\).

- **Remark:** In practice, one has to drop the condition \(M_s\) irreducible and \(P_s\) connected. In these cases, the formulations of the support theorem and the numerical equality are more complicated.
Theorem of support for abelian fibration

- **Theorem:** Let \((f : M \to S, g : P \to S)\) be a \(\delta\)-regular abelian fibration. Assume that \(M\) is smooth, the fibers of \(f : M \to S\) are irreducible. Then

\[
\text{supp}(f) = \{S\}.
\]

- **Corollary:** Let \((P, M, S)\) and \((P', M', S)\) be \(\delta\)-regular abelian fibrations as above (in particular, \(M_s\) and \(M'_s\) are irreducible). If the generic fibers of \(P\) and \(P'\) are isogenous abelian varieties, then for every \(s \in S(\mathbb{F}_q)\), \(\# M_s(\mathbb{F}_q) = \# M'_s(\mathbb{F}_q)\).

- **Remark:** In practice, one has to drop the condition \(M_s\) irreducible and \(P_s\) connected. In these cases, the formulations of the support theorem and the numerical equality are more complicated.

This theorem is the key geometric ingredient in the proof of Langlands’ fundamental lemma.
For every closed irreducible subscheme $Z$ of $S$, we set $\delta_Z = \min_{s \in S} \delta(s)$. 

We prove that if $Z \in \text{supp}(f)$, then $\text{codim}(Z) \leq \delta_Z$ and if the geometric fibers of $f: M \to S$ are irreducible then $\text{codim}(Z) < \delta_Z$ unless $Z = S$. 

By the $\delta$-regularity, we have the inequality $\text{codim}(Z) \geq \delta_Z$. The only possibility is $Z = S$. 
For every closed irreducible subscheme $Z$ of $S$, we set $\delta_Z = \min_{s \in S} \delta(s)$.

We prove that if $Z \in \text{supp}(f)$, then $\text{codim}(Z) \leq \delta_Z$ and if the geometric fibers of $f : M \to S$ are irreducible then $\text{codim}(Z) < \delta_Z$ unless $Z = S$. By the $\delta$-regularity, we have the inequality $\text{codim}(Z) \geq \delta_Z$. The only possibility is $Z = S$. 
For every closed irreducible subscheme $Z$ of $S$, we set $\delta_Z = \min_{s \in S} \delta(s)$.

We prove that if $Z \in \text{supp}(f)$, then $\text{codim}(Z) \leq \delta_Z$ and if the geometric fibers of $f : M \to S$ are irreducible then $\text{codim}(Z) < \delta_Z$ unless $Z = S$.

By the $\delta$-regularity, we have the inequality $\text{codim}(Z) \geq \delta_Z$. The only possibility is $Z = S$. 
Topological explanation

- Let \( s \in Z \) such that \( \delta(s) = \delta_Z \).
Topological explanation

- Let $s \in Z$ such that $\delta(s) = \delta_Z$.
- Assume there is a splitting $A_s \to P_S$ of the Chevalley exact sequence
  \[
  0 \to R_s \to P_s \to A_s \to 0
  \]
Let $s \in Z$ such that $\delta(s) = \delta_Z$.

Assume there is a splitting $A_s \to P_S$ of the Chevalley exact sequence

$$0 \to R_s \to P_s \to A_s \to 0$$

Assume there exists an étale neighborhood $S'$ of $s$, an abelian scheme $A' \to S'$ of special fiber $A_s$, and a homomorphism $A' \to P'$ extending the splitting $A_s \to P_s$. 

Then over $S'$, $A'$ acts almost freely on $M'$ and one can factorize $M' \to S'$ as $M' \to [M'/A'] \to S'$ where the morphism $M' \to [M'/A']$ is proper and smooth, and the morphism $[M'/A'] \to S'$ is of relative dimension $\delta_s$. Thus our inequality can be reduced to the Goresky-MacPherson inequality.
Let $s \in Z$ such that $\delta(s) = \delta_Z$.

Assume there is a splitting $A_s \rightarrow P_S$ of the Chevalley exact sequence

$$0 \rightarrow R_s \rightarrow P_s \rightarrow A_s \rightarrow 0$$

Assume there exists an étale neighborhood $S'$ of $s$, an abelian scheme $A' \rightarrow S'$ of special fiber $A_s$, and a homomorphism $A' \rightarrow P'$ extending the splitting $A_s \rightarrow P_s$.

Then over $S'$, $A'$ acts almost freely on $M'$ and on can factorize $M' \rightarrow S'$ as $M' \rightarrow [M'/A'] \rightarrow S'$ where the morphism $M' \rightarrow [M'/A']$ is proper and smooth, and the morphism $[M'/A'] \rightarrow S'$ is of relative dimension $\delta_s$. Thus our inequality can be reduced to the Goresky-MacPherson inequality.
Implement the argument

- The assumptions are not satisfied in general. The generic fiber of $P$ is usually an irreducible abelian variety and does not admit a factor of smaller dimension i.e the lifting $A' \rightarrow P'$ can’t exist.
Implement the argument

- The assumptions are not satisfied in general. The generic fiber of $P$ is usually an irreducible abelian variety and does not admit a factor of smaller dimension i.e. the lifting $A' \to P'$ can’t exist.

- To overcome this difficulty, one need to reformulate the above argument in terms of homological algebra instead of topology.