# Endoscopy theory of automorphic forms

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- In that case, the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the group $E[\ell^n] = \ell^{-n}\Lambda/\Lambda \sim (\mathbb{Z}/\ell^n\mathbb{Z})^2$ for every prime number $\ell$ and $n \in \mathbb{N}$.

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- All together, these actions define a system of 2-dimensional $\ell$-adic representations

$$\rho_E : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_\ell).$$

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- for unramified $p$, $\rho_E(\mathrm{Fr}_p)$ is a well defined conjugacy class in $\mathrm{GL}_2(\mathbb{Q}_\ell)$.
- The number of $\mathbb{F}_p$-points on $E_p$ can be calculated by the formula $1 + p - \mathrm{tr}(\rho_E(\mathrm{Fr}_p))$.

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- This is a theorem of Wiles, Taylor-Wiles, Breuil-Conrad-Diamond-Taylor.

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- In the number field case, it is more difficult even to state the non abelian reciprocity law because there are automorphic forms that do not correspond to Galois representations.
- Since classical automorphic forms are functions on hermitian symmetric domain, it is natural to consider automorphic forms on classical groups as well.

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- Unramified representations of $G(F_v)$, $F_v$ being a nonarchimedean local field are classified by semisimple conjugacy classes of $\hat{G}(\mathbb{C})$. Local components of automorphic representations are unramified almost everywhere.

# Automorphic representations



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- They are of the form $\pi = \bigotimes_v \pi_v$. For every finite prime $v$, $\pi_v$ is an unitary admissible representation of $G(F_v)$. For all but finitely many $v$, $\pi_v$ are unramified.
- It is important to understand in which circumstances, the tensor product of local representation $\pi_v$ is automorphic. Langlands' prediction is based on an elaborated form of the local and the global Galois groups.

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- For global field, there should be an extension $L_F$ of the global Weil group $W_F$ with homomorphism $L_{F_v} \to L_F$ also well defined up to conjugacy.
- Automorphic representations $\pi$ should be parametrized by homomorphisms $\phi : L_F \to \hat{G}$. Local parameters $\phi_v : L_{F_v} \to \hat{G}$ are obtained by restricting $\phi$ from $L_F$ to $L_{F_v}$.

# Langlands' functoriality conjecture



*Let $\rho : \hat{H} \to \hat{G}$ be a homomorphism. For every automorphic representation $\pi_H = \bigotimes_v \pi_{H,v}$ of $H$, there exists an automorphic representation $\pi = \bigotimes_v \pi_v$ of $G$ such that if at an unramified place $v$, $\pi_{H,v}$ is parametrized by a semisimple conjugacy class $s_v \in \hat{H}$, then $\pi_v$ is also unramified and parametrized by the conjugacy class $\rho(s_v)$.*

# Functoriality's consequences



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- Many deep conjectures are also consequences of the functoriality : general form of the Ramanujan conjecture, the Sato-Tate conjecture, the Artin conjecture ...

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- $p$-adic method : recent proof due to Taylor and collaborators of the Sato-Tate conjecture via a weak form of the functoriality.
- Endoscopy theory via the stabilization of the trace formula.

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- They belong to the same **packet** of representations.
- Rotations of angle $\theta$ and $-\theta$ are not conjugate in $\mathrm{SL}_2(\mathbb{R})$ but become conjugate in either $\mathrm{GL}(2, \mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$. They are **stably conjugate**.

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- Let $S_{\phi_v}$ denote the centralizer of $\phi_v$ in $\hat{G}$ and let $\mathcal{S}_{\phi_v} = S_{\phi_v}/S^0_{\phi_v} Z_{\hat{G}}$, where $S^0_{\phi_v}$ is the neutral component of $S_{\phi_v}$ and $Z_{\hat{G}}$ is the center of $\hat{G}$ ($G$ is supposed split).

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- There should be a natural bijection between the packet $\Pi_{\phi_v}$ and the set of irreducible representations of the finite group $\mathcal{S}_{\phi_v}$.

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- Conjectural multiplicity formula for a representation $\pi = \otimes \pi_v$ in the global packet $\Pi_\phi$

$$m(\pi, \phi) = |\mathcal{S}_\phi|^{-1} \sum_{\epsilon \in \mathcal{S}_\phi} \prod \langle \epsilon_v, \pi_v \rangle.$$

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- For real groups, Shelstad proved remarkable identities between the sum of characters of representations of an endoscopic group $H$ in a packet and certain linear combination of character of representations of $G$ in the corresponding packet.
- Langlands suggested a strategy for proving the endoscopic functoriality by the stabilization of the trace formula.

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- $\mathrm{SO}(2n) \to GL(2n)$, $\mathrm{Sp}(2n) \to GL(2n+1)$, $\mathrm{SO}(2n+1) \to GL(2n)$.
- It also includes the theory of base change which is important in number theory.

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- He also proved the multiplicity formula in the global packet.

# Construction of Galois representations attached to a selfdual automorphic representation

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- the component at infinity has the same infinitesimal character as some algebraic representation satisfying some regularity condition.

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- Clozel, Harris, Taylor, Yoshida, Labesse, Morel, Shin.

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- The hidden parts contain the contribution of the continuous of the spectrum as well as the contribution of hyperbolic conjugacy classes.
- The test function is of the form $f = \bigotimes_v f_v$ where $f_v$ are smooth functions with compact support on $G(F_v)$. For almost all $v$, $f_v$ is the characteristic function of $G(\mathcal{O}_v)$.

# Stable conjugacy classes



- $\gamma \in G(F)$ is strongly regular if its centralizer is a torus. Strongly regular elements $\gamma, \gamma' \in G(F)$ are said to be stably conjugate if they are conjugate in $G(\bar{F})$.

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- The conjugacy classes within the stable conjugacy class of $\gamma$ are parametrized by a subset $A_\gamma$ of $\mathrm{H}^1(F, I_\gamma)$, $I_\gamma$ being the centralizer of $\gamma$.
- If $F = F_v$ is a local nonarchimedean field, $A_\gamma$ is a finite abelian group.

# Stable distribution



- The integral over a stable conjugacy class

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- Stable distribution is a weak limit of finite combinations of stable orbital integrals.

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- Let $\gamma \in G(F)$ be a strongly regular element. For each place $v$, let $\gamma'_v \in G(F_v)$ be stably conjugate to $\gamma$. There might not be $\gamma' \in G(F)$ such that $\gamma'$ in conjugate to $\gamma'_v \in G(F_v)$.

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- The main purpose of the stabilization is to compare the above errors terms with the stable term in the trace formula of its endoscopic groups.
- The problem can be reduced to a comparison between $\kappa$-orbital integrals on $G$ with stable orbital integrals on endoscopic groups $H$.

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- The reason is that $G$ and $H$ share a maximal torus and there is an inclusion of their Weyl groups $W_H \subset W$.

# Transfer conjecture and the fundamental lemma



- *Transfer conjecture* : For every smooth function $f$ with compact support on $G(F_v)$, there exists a smooth function $f^H$ with compact support on $H(F_v)$ such that

$$\Delta(\gamma_H, \gamma) O^{\kappa}_{\gamma}(f) = SO_{\gamma_H}(f^H)$$

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  where $\Delta(\gamma_H, \gamma)$ is the Langlands-Shelstad transfer factor, independent of $f$.
- *The fundamental lemma* : If both $G$ and $H$ are unramified at $v$, the above identity holds for $f = 1_{G(\mathcal{O}_v)}$ and $f^H = 1_{H(\mathcal{O}_v)}$.

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- Waldspurger proved that the transfer conjecture follows from the fundamental lemma.
- Waldspurger, Cluckers-Hales-Loeser proved by different methods that the $p$-adic case of the fundamental lemma is equivalent to the case of formal series case, more accessible to the geometric methods.

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# Hitchin system



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- The endoscopy theory can be entirely translated to properties of the action of this group scheme.

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- The fundamental lemma follows.