Number theory and harmonic analysis

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1 Adèles and idèles of algebraic numbers

In this section, we will review the constriction of real and p-adic numbers from the set $\mathbb{Q}$ of rational numbers by the process of completion with respect to different valuations. Completions of $\mathbb{Q}$ and more generally of number fields give rise to local fields (of characteristic zero). Local fields, which may be the field of real or complex numbers, or p-adic fields, are endowed with locally compact topology. We will also review the groups of adèles and idèles, their topology, and their connection with basic theorems in algebraic number theory.

Real numbers

We will very briefly review the construction of real numbers from rational numbers. Recall that the field of rational numbers $\mathbb{Q}$ is a totally ordered field: we write $x > y$ if $x - y \in \mathbb{Q}_+$ where $\mathbb{Q}_+$ is the semigroup of positive rational numbers. It is also equipped with the real valuation function $\mathbb{Q} \to \mathbb{Q}_+ \cup \{0\}$

$$|x| = \mathrm{sign}(x)x$$

that takes value x or $-x$ depending on whether $x \in \mathbb{Q}_+$ or $-x \in \mathbb{Q}_+$, and $|0| = 0$. For we will also use p-adic absolute values, we will write $|x|_\infty$ instead of $|x|$ and call it the the real absolute value. The absolute value of the difference $|x - y|_\infty$ defines a $\mathbb{Q}_+$-valued metric on $\mathbb{Q}$.

The field of real numbers $\mathbb{R}$ is constructed as the completion of $\mathbb{Q}$ with respect to this metric. A real number is thus defined to be an equivalence class of Cauchy sequences of rational numbers with respect to this distance. We recall that a sequence $(x_i | i \in \mathbb{N})$ of rational numbers is Cauchy if for all $\epsilon \in \mathbb{Q}_+$, for all i, j large enough the inequality $|x_i - x_j|_\infty < \epsilon$ holds. Two Cauchy sequences are said to be equivalent if by shuffling them arbitrarily we get a new Cauchy sequence (a shuffling of two sequences is a new sequence of which they are complementary subsequences).

As it makes sense to add and multiply Cauchy sequences component-wise, the set $\mathbb{R}$ of real numbers is a commutative ring. If $(x_i | i \in \mathbb{N})$ is a Cauchy sequence, which is not equivalent to 0, then $(x_i^{-1} | i \in \mathbb{N})$ is also a Cauchy sequence. It follows that $\mathbb{R}$ is a commutative field. The set of real numbers $\mathbb{R}$ constructed in this way is totally ordered by the semigroup $\mathbb{R}_+$ consisting of elements of $\mathbb{R}$ which can be represented by Cauchy sequences with only positive rational numbers. The valuation $|x|_\infty$ can be extended to $\mathbb{R}$ with range in the semigroup $\mathbb{R}_+$ of positive real numbers. The field of real numbers $\mathbb{R}$ is now complete with respect to the real valuation in the sense that every Cauchy sequences of real numbers is convergent. According to the Bolzano-Weierstrass theorem, every closed interval in $\mathbb{R}$ is compact and therefore $\mathbb{R}$ is locally compact.

p-adic numbers

We will also briefly review the construction of p-adic numbers following the same pattern. For a given prime number p, the *p-adic absolute value* of a nonzero rational number is defined by the formula

$$|m/n|_p = p^{-\mathrm{ord}_p(m) + \mathrm{ord}_p(n)}$$

where $m, n \in \mathbb{Z} - \{0\}$, and $\mathrm{ord}_p(m)$ and $\mathrm{ord}_p(n)$ are the exponents of the highest power of p dividing m and n respectively. The p-adic absolute value is *ultrametric* in the sense that it satisfies the multiplicative property and the ultrametric inequality:

$$|\alpha\beta|_p = |\alpha|_p |\beta|_p \text{ and } |\alpha + \beta|_p \leq \max\{|\alpha|_p, |\beta|_p\}. \tag{1.1}$$

This defines the p-adic metric $|x - y|_p$ of the field of rational numbers $\mathbb{Q}$.

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. Its elements are equivalence classes of Cauchy sequences of rational numbers with respect to the p-adic absolute value. If $\{\alpha_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence for the p-adic absolute value, then $\{|\alpha_i|_p\}$ is a Cauchy sequence for the real absolute value and therefore it has a limit in $\mathbb{R}_+$. This allows us to extend the p-adic absolute value to $\mathbb{Q}_p$ as a function $|.|_p : \mathbb{Q}_p \to \mathbb{R}_+$ that satisfies (**??**). If $\alpha \in \mathbb{Q}_p - \{0\}$ and if $\alpha = \lim_{i \to \infty} \alpha_i$ then the ultrametric inequality (1.1) implies that $|\alpha|_p = |\alpha_i|_p$ for i large enough. In particular, the p-adic absolute value has range in $p^{\mathbb{Z}}$, and therefore in the monoid of positive rational numbers $\mathbb{Q}_+$.

The set of p-adic integers is defined to be

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}. \tag{1.2}$$

Since $|\alpha|_p$ is ultrametric, $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$. We say that $\mathbb{Z}_p$ is the valuation ring of $\mathbb{Q}_p$ with respect to the p-adic absolute value.

We claim that every p-adic integer can be represented by a Cauchy sequence of rational integers. Indeed, we can suppose that $x \neq 0$ because the statement is obvious for $x = 0$. Let $x \in \mathbb{Z}_p - \{0\}$ be a p-adic integer represented by a Cauchy sequence $x_i = p_i/q_i$ where p_i, q_i are relatively prime nonzero integers. As discussed above, for large i, we have $|x|_p = |x_i|_p \leq 1$ so that q_i is prime to p. We infer that one can find an integer q_i' so that $q_i q_i'$ is as p-adically close to 1 as we like, for example $|q_i q_i' - 1| \leq p^{-i}$. The sequence $(x_i' = p_i q_i')$, made only of integers, is Cauchy and equivalent to the Cauchy sequence (x_i).

One can reformulate the above lemma by asserting that the ring of p-adic integers $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the p-adic absolute value. The completion of $\mathbb{Z}$ with respect to the p-adic absolute value can also described as a projective limit:

$$\mathbb{Z}_p = \lim_{\leftarrow n} \mathbb{Z}/p^n\mathbb{Z} \tag{1.3}$$

that consists of sequences of congruence classes $(x_n, n \in \mathbb{N})$ with $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ such that $x_m \equiv x_n \mod p^n$ for all $m \geq n$. It follows that $\mathbb{Z}_p$ is a local ring. Its maximal ideal is $p\mathbb{Z}_p$ and its residue field is the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

It follows also from (1.3) that $\mathbb{Z}_p$ is compact. With the definition of $\mathbb{Z}_p$ as the valuation ring (1.2), it is a neighborhood of 0 in $\mathbb{Q}_p$, and in particular, $\mathbb{Q}_p$ is a locally compact field. This is probably the main property that $\mathbb{Q}_p$ shares with its Archimedean cousin $\mathbb{R}$. On the other hand, as opposed to $\mathbb{R}$, the topology on $\mathbb{Q}_p$ is totally disconnected: every p-adic number $\alpha \in \mathbb{Q}_p$ admits a base of neighborhoods made of open and compact subsets. Indeed, the compact subgroups $p^n\mathbb{Z}_p$ form a base of neighborhoods of 0.

The product formula

It is important to observe that for every rational number $x \in \mathbb{Q}$, the product formula

$$|x|_\infty \prod_{p \in \mathscr{P}} |x|_p = 1 \tag{1.4}$$

holds. Although p runs over the set of all prime numbers $\mathscr{P}$, the formula makes sense nevertheless for $|xx|_p = 1$ for almost all primes p. This formula can be directly verified on the definition of the real and p-adic absolute values.

Ostrowski's theorem

There are essentially no other valuations of $\mathbb{Q}$ other than the ones we have already mentioned. For every prime numbers p and positive real number $t \in \mathbb{R}_+$, $|.|_p^t$ is also a valuation. For every real number $t \in (0,1]$, $|.|_\infty^t$ is also a valuation. In these two cases, a valuation of the form $|.|_v^t$ will be said to be equivalent to $|.|_v$. Equivalent valuations define the same completion of $\mathbb{Q}$.

Theorem 1.1 (Ostrowski)**.** *Every valuation of $\mathbb{Q}$ is equivalent to either the real absolute value or the p-adic absolute value for some prime number p.*

Proof. A valuation $|.| : \mathbb{Q}^\times \to \mathbb{R}_+$ is said to be *non-Archimedean* if it is bounded over $\mathbb{Z}$ and *Archimedean* otherwise. We claim that a valuation is non-Archimedean if and only if it satisfies the ultrametric inequality (**??**) is satisfied.

If (**??**) is satisfied, then for all $n \in \mathbb{N}$, we have

$$|n| = |1 + \cdots + 1| \leq 1.$$

Conversely, suppose that for some positive real number A, the inequality $|x| \leq A$ holds for all $x \in \mathbb{Z}$. For $x, y \in \mathbb{Q}$ with $|x| \geq |y|$, the binomial formula and the inequality (**??**) together imply

$$|x + y|^n \leq A(n+1)|x|^n$$

for all $n \in \mathbb{N}$. By taking n-th roots of both sides of this inequality and letting n go to ∞, we get $|x + y| \leq |x|$ and therefore (**??**) is satisfied.

Let $|.| : \mathbb{Q}^\times \to \mathbb{R}_+$ be a nonarchimedean valuation. It follows from the ultrametric inequality (**??**) that $|x| \leq 1$ for all $x \in \mathbb{Z}$. The subset $\mathfrak{p}$ of $\mathbb{Z}$ consisting of $x \in \mathbb{Z}$ such that $|x| < 1$ is then an ideal. If $|x| = |y| = 1$, then $|xy| = 1$; in other words $x, y \notin \mathfrak{p}$ implies $xy \notin \mathfrak{p}$. Thus $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}$ and must be generated by a prime number p. If t is the positive real number such that $|p| = |p|_p^t$, then for all $x \in \mathbb{Q}$ we have $|x| = |x|_p^t$.

We claim that a valuation $|.|$ is archimedean if for all integers with $y > 1$, we have $|y| > 1$. We will argue by contradiction. Assume that there is an integer $y > 1$ with $|y| \leq 1$ and we will derive that $|x| \leq 1$ holds for all natural integers $x \in \mathbb{N}$. Using Euclidean division, we can write

$$x = x_0 + x_1 y + \cdots + x_r y^r$$

with integers x_i satisfying $0 \leq x_i < y$ and $0 < x_r < y$. The triangle inequality implies $|x_i| \leq |x_i|_\infty < y$ for all i. We also have $x \geq y^r$ and therefore $r \leq \log(x)/\log(y)$

where log is the natural logarithm function. It follows from the triangle inequality (**??**) that

$$|x| \leq \left(1 + \frac{\log(x)}{\log(y)}\right) y.$$

Replacing x by x^k in the above inequality, taking k-th roots on both sides, and letting k tend to infinity, we will get $|x| \leq 1$ for all $x \in \mathbb{N}$. This contradicts our assumption that the function $|.|$ is unbounded.

We now claim that for every two natural numbers $x, y > 1$ we have

$$|x|^{1/\log x} \leq |y|^{1/\log y}. \tag{1.5}$$

One can write

$$x = x_0 + x_1 y + \cdots + x_r y^r$$

with integers x_i satisfying $0 \leq x_i \leq y-1$ and $x_r > 0$. As argued above, we then have $|x_i| < y$ and $r \leq \log(x)/\log(y)$. It follows from (**??**) and $|\beta| > 1$ that

$$|x| \leq \left(1 + \frac{\log(x)}{\log(y)}\right) |y|^{\frac{\log(x)}{\log(y)}}.$$

Replacing x by x^k in the above inequality, taking k-th roots on both sides, and letting k tend to ∞, we will get (1.5). By symmetry, we can derive the equality

$$|\alpha|^{1/\log \alpha} = |\beta|^{1/\log \beta}. \tag{1.6}$$

It follows that there exists $t > 0$ such that $|x| = |x|_\infty^t$ for all $x \in \mathbb{Z}$. The triangle inequality imposes the further constraint $0 < t \leq 1$. □

Rational adèles and idèles

We will denote by $\mathscr{P}$ the set of prime numbers. An *adèle* is a sequence

$$\left(x_\infty, x_p; p \in \mathscr{P}\right)$$

consisting of a real number $x_\infty \in \mathbb{R}$ and a p-adic number $x_p \in \mathbb{Q}_p$ for every $p \in \mathscr{P}$ such that $x_p \in \mathbb{Z}_p$ for almost all $p \in \mathscr{P}$. The purpose of the ring of adèles $\mathbb{A}$ is to simultaneously host the classical analysis on the set of real numbers and the ultrametric analysis on the set of p-adic numbers. At first time, adèle is however quite too cumbersome a structure to be imagined of as a mere number. It is thus of some use to unravel the structure of the ring $\mathbb{A}$.

A finite adèle is a sequence

$$\left(x_p; p \in \mathscr{P}\right)$$

with $x_p \in \mathbb{Q}_p$ for all prime p and $x_p \in \mathbb{Z}_p$ for almost all p. If we denote by $\mathbb{A}_{\mathrm{fin}}$ the ring of finite adèles, then $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$.

We observe that the subring $\prod_{p\in\mathscr{P}} \mathbb{Z}_p$ of $\mathbb{A}_{\mathrm{fin}}$ can be represented as the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$:

$$\prod_{p\in\mathscr{P}} \mathbb{Z}_p = \lim_{\leftarrow n} \mathbb{Z}/n = \hat{\mathbb{Z}}. \tag{1.7}$$

where the projective limit is taken over the set of nonzero integers ordered by the relation of divisibility. This is no more than a restatement of the Chinese remainder theorem.

On the other hand, $\mathbb{A}_{\mathrm{fin}}$ contains $\mathbb{Q}$ for a rational number x can be represented as the "diagonal" finite adèle (x_p) with $x_p = x$. For all finite adèles $x \in \mathbb{A}_{\mathrm{fin}}$, there exists an $n \in \mathbb{N}$ so that $nx \in \hat{\mathbb{Z}}$; in other words we have

$$\mathbb{A}_{\mathrm{fin}} = \bigcup_{n\in\mathbb{N}} n^{-1}\hat{\mathbb{Z}}.$$

It follows from this relation that the natural map $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{A}_{\mathrm{fin}}$ is surjective. It is in fact an isomorphism.

The profinite completion $\hat{\mathbb{Z}}$ is a compact ring. The profinite topology on $\hat{\mathbb{Z}}$ coincides with the product topology $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ whose compactness is also asserted by the Tychonov theorem. We will equip $\mathbb{A}_{\mathrm{fin}}$ with the finest topology such that the inclusion map $\hat{\mathbb{Z}} = n^{-1}\hat{\mathbb{Z}} \to \mathbb{A}_{\mathrm{fin}}$ is continuous for all n. In other words, a subset U of $\mathbb{A}_{\mathrm{fin}}$ is open if and only if $U \cap n^{-1}\hat{\mathbb{Z}}$ is open in $n^{-1}\hat{\mathbb{Z}}$ for all n. In particular, $\mathbb{A}_{\mathrm{fin}}$ is a locally compact group, of which $\hat{\mathbb{Z}}$ is a compact open subgroup. The group of adèles $\mathbb{A} = \mathbb{A}_{\mathrm{fin}} \times \mathbb{R}$ equipped with product topology is a *locally compact group* so as each of its two factors. All open neighborhoods of 0 in $\mathbb{A}$ are of the form

$$U = U_{S,\infty} \times \prod_{p\notin S} \mathbb{Z}_p$$

where S is a finite set of primes and $U_{S,\infty}$ is an open subset of $\mathbb{R} \times \prod_{p\in S} \mathbb{Q}_p$.

Theorem 1.2. *The diagonal embedding given by $x \mapsto (x_\infty, x_p)$ with $x_p = x$ for all primes p and $x_\infty = x$ identifies $\mathbb{Q}$ with a discrete subgroup of $\mathbb{A}$. The quotient $\mathbb{A}/\mathbb{Q}$ can be identified with the pro-universal covering of $\mathbb{R}/\mathbb{Z}$*

$$\mathbb{A}/\mathbb{Q} = \lim_{\leftarrow n} \mathbb{R}/n\mathbb{Z},$$

the projective limit being taken over the set of natural integers ordered by the divisibility order. In particular, $\mathbb{A}/\mathbb{Q}$ is a compact group.

Proof. Consider the neighborhood of 0 in $\mathbb{A}$ defined by $\hat{\mathbb{Z}}\times(-1,1)$ and its intersection with $\mathbb{Q}$. If $x \in \mathbb{Q}$ lies in this intersection, then because the finite adèle part of x lies in $\hat{\mathbb{Z}}$, we must have $x \in \mathbb{Z}$; but as a real number $x \in (-1,1)$, so we must have $x = 0$ for $\mathbb{Z} \cap (-1,1) = \{0\}$. This implies that $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$.

There is an exact sequence

$$0 \to \mathbb{R} \times \hat{\mathbb{Z}} \to \mathbb{A} \to \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

where $\bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$ is the subgroup of $\prod_p \mathbb{Q}_p/\mathbb{Z}_p$ consisting of sequences (x_p) whose members $x_p \in \mathbb{Q}_p/\mathbb{Z}_p$ vanish for almost all p. Consider now the homomorphism between two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{\mathbb{Z}} \times \mathbb{R} & \longrightarrow & \mathbb{A} & \longrightarrow & \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & 0 \end{array} \tag{1.8}$$

As the middle vertical arrow is injective and the right vertical arrow is surjective with kernel $\mathbb{Z}$, the snake lemma induces an exact sequence

$$0 \to \mathbb{Z} \to \hat{\mathbb{Z}} \times \mathbb{R} \to \mathbb{A}/\mathbb{Q} \to 0 \tag{1.9}$$

$\mathbb{Z}$ being diagonally embedded in $\hat{\mathbb{Z}} \times \mathbb{R}$. In other words, there is a canonical isomorphism

$$\mathbb{A}/\mathbb{Q} \to \left(\hat{\mathbb{Z}} \times \mathbb{R}\right)/\mathbb{Z}. \tag{1.10}$$

Dividing both sides by $\hat{\mathbb{Z}}$, one gets an isomorphism

$$\mathbb{A}/(\mathbb{Q} + \hat{\mathbb{Z}}) \to \mathbb{R}/\mathbb{Z}. \tag{1.11}$$

With the same argument, for every $n \in \mathbb{N}$ one can identify the covering $\mathbb{A}/(\mathbb{Q} + n\hat{\mathbb{Z}})$ of $\mathbb{A}/(\mathbb{Q}+\hat{\mathbb{Z}})$ with the covering $\mathbb{R}/n\mathbb{Z}$ of $\mathbb{R}/\mathbb{Z}$. It follows that $\mathbb{A}/\mathbb{Q}$ is the prouniversal covering of $\mathbb{R}/\mathbb{Z}$. □

We note that, for the compactness of the quotient $\mathbb{A}/\mathbb{Q}$, one can also argue as follows. Let B denote the compact subset of $\mathbb{A}$ which is defined as follows

$$B = \{(x_\infty, x_p)_{p\in\mathscr{P}}; |x_\infty|_\infty \le 1 \text{ and } |x_p|_p \le 1\}. \tag{1.12}$$

With help of the exact sequence (1.9), we see that the map $B \to \mathbb{A}/\mathbb{Q}$ is surjective. Since B is compact, its image in $\mathbb{A}/\mathbb{Q}$ is also compact, and therefore $\mathbb{A}/\mathbb{Q}$ is compact.

Let us recall that for every prime number p, we have the p-adic absolute value $|.|_p : \mathbb{Q}_p^\times \to \mathbb{R}_+$, whose image is the discrete subgroup $p^{\mathbb{Z}}$ of $\mathbb{R}_+$. The kernel

$$\{\alpha \in \mathbb{Q}_p^\times ; |\alpha|_p = 1\}$$

is the group $\mathbb{Z}_p^\times$ of invertible elements in $\mathbb{Z}_p$. We have an exact sequence

$$0 \to \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times \to \mathbb{Z} \to 0 \tag{1.13}$$

where $\mathrm{ord}_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ is defined such that $|\alpha|_p = p^{-\mathrm{ord}_p(\alpha)}$. In particular, $\mathbb{Z}_p^\times$ is the set of p-adic numbers of order zero, and $\mathbb{Z}_p$ is the set of p-adic numbers of non-negative order.

An *idèle* is a sequence $(x_p; x_\infty)$ consisting of a nonzero p-adic number $x_p \in \mathbb{Q}_p^\times$ for each prime p such that $x_p \in \mathbb{Z}_p^\times$ for almost all p, and $x_\infty \in \mathbb{R}^\times$. The group of idèles $\mathbb{A}^\times$ is in fact the group of invertible elements in the ring of adèles $\mathbb{A}$. We will equip $\mathbb{A}^\times$ with the coarsest topology such that the inclusion map $\mathbb{A}^\times \to \mathbb{A}$ as well as the inversion map $\mathbb{A}^\times \to \mathbb{A}$ given by $x \mapsto x^{-1}$ are continuous. All neighborhoods of 1 in $\mathbb{A}^\times$ are of the form $U = U_{S,\infty} \times \prod_{p \notin S} \mathbb{Z}_p^\times$ where S is a finite set of primes and $U_{S,\infty}$ is an open subset of $\prod_{p\in S} \mathbb{Q}_p^\times \times \mathbb{R}^\times$.

We also have $\mathbb{A}^\times = \mathbb{A}_{\mathrm{fin}}^\times \times \mathbb{R}^\times$. The group

$$\prod_p \mathbb{Z}_p^\times = \hat{\mathbb{Z}}^\times = \lim_{\leftarrow n} (\mathbb{Z}/n)^\times,$$

where the projective limit is taken over the set of nonzero integers ordered by the divisibility order, is a compact open subgroup of $\mathbb{A}_{\mathrm{fin}}^\times$. It follows that $\mathbb{A}_{\mathrm{fin}}^\times$ is locally compact, and so is the group of idèles $\mathbb{A}^\times = \mathbb{A}_{\mathrm{fin}}^\times \times \mathbb{R}^\times$.

We define the absolute value of every idèle $x = (x_\infty, x_p) \in \mathbb{A}^\times$ as

$$|x| = \prod_{p\in\mathscr{P}} |x_p|_p |x_\infty|_\infty,$$

this infinite product being well defined since $|x_p|_p = 1$ for almost all prime p. If we denote by $\mathbb{A}^1$ the kernel of the idelic absolute value $x \mapsto |x|$, then we have an exact sequence

$$0 \to \mathbb{A}^1 \to \mathbb{A}^\times \to \mathbb{R}_+ \to 0. \tag{1.14}$$

The product formula (1.4) implies that the diagonal subgroup $\mathbb{Q}^\times$ is contained in $\mathbb{A}^1$.

Theorem 1.3. *The group invertible rational numbers $\mathbb{Q}^\times$ embeds diagonally as a discrete cocompact subgroup of $\mathbb{A}^1$.*

Proof. If $\alpha \in \mathbb{Q}^\times$ such that $\alpha \in \mathbb{Z}_p^\times$ for all prime p then $\alpha \in \mathbb{Z}^\times = \{\pm 1\}$. This implies that $\mathbb{Q}^\times \cap (\hat{\mathbb{Z}}^\times \times \mathbb{R}^\times) = \{\pm 1\}$ which shows that $\mathbb{Q}^\times$ is a discrete subgroup of $\mathbb{A}^\times$.

Let us consider $\hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times$ as a compact subgroup of $\mathbb{A}$ consisting of elements $(x_p; x_\infty)$ with $x_p \in \mathbb{Z}_p^\times$ and $x_\infty = 1$. Let us consider $\mathbb{R}_+$ as the subgroup of $\mathbb{A}^\times$ consisting of elements of the form $(x_p; x_\infty)$ with $x_p = 1$ and $x_\infty \in \mathbb{R}_+$ a positive real number. We claim that the homomorphism

$$\mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_+ \to \mathbb{A}^\times \tag{1.15}$$

that maps $\alpha \in \mathbb{Q}^\times, u \in \hat{\mathbb{Z}}^\times, t \in \mathbb{R}_+$ on $x = \alpha u t \in \mathbb{A}^\times$ is an isomorphism of topological groups. The image of the subgroup $\mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \{1\}$ of $\mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_+ \to \mathbb{A}^\times$ by this isomorphism is the subgroup $\mathbb{A}^1$ of $\mathbb{A}^\times$. It follows that $\mathbb{A}^1/\mathbb{Q}^\times$ isomorphic to $\hat{\mathbb{Z}}^\times$ which is a compact group.

Let us construct an inverse to (1.15). Let $x = (x_\infty, x_p)$ be an idèle. For every prime p, there is a unique way to write x_p under the form $x_p = p^{r_p} y_p$ where $y_p \in \mathbb{Z}_p^\times$ and $r_p \in \mathbb{Z}$; note that $r_p = 0$ for almost all p. We can also write $x_\infty = \epsilon |x_\infty|$ where $\epsilon \in \{\pm 1\}$ and $|x_\infty| \in \mathbb{R}_+$. Then we set $\alpha = \epsilon \prod_p p^{r_p} \in \mathbb{Q}^\times$,

$$y = (\epsilon \prod_{q \neq p} q^{-r_q} y_p, 1) \in \hat{\mathbb{Z}}^\times$$

and $t = |x|$. This defines a homomorphism from $\mathbb{A}^\times$ to $\mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_+$ which is an inverse to the multiplication map $(\alpha, u, t) \mapsto \alpha u t$. □

Integers in number fields

Number fields are finite extensions of the field of rational numbers. For every irreducible polynomial $P \in \mathbb{Q}[x]$ of degree n, the quotient ring $\mathbb{Q}[x]/P$ is a finite extension of degree n of $\mathbb{Q}$. The irreducibility of P implies that $\mathbb{Q}[x]/P$ is a domain, in other words the multiplication with every nonzero element $y \in \mathbb{Q}[x]/P$ is an injective $\mathbb{Q}$-linear map in $\mathbb{Q}[x]/P$. As $\mathbb{Q}$-vector space $\mathbb{Q}[x]/P$ is finite dimensional, all injective endomorphisms are necessarily bijective. It follows that $\mathbb{Q}[x]/P$ is a field, then a finite extension of $\mathbb{Q}$. All finite extensions of $\mathbb{Q}$ are of this form since it is known that all finite extensions of a field in characteristic zero can be generated by a single element.

Let k be a finite extension of degree n of $\mathbb{Q}$. For every element $\alpha \in k$, the multiplication by α in k can be seen as a $\mathbb{Q}$-linear transformation of k regarded as a $\mathbb{Q}$-vector space. We define $\mathrm{Tr}_{k/\mathbb{Q}}(\alpha)$ as the trace of this transformation and $\mathrm{Nm}_{k/\mathbb{Q}}(\alpha)$ as its determinant, the subscript $k/\mathbb{Q}$ can be dropped if no confusion is possible. The $\mathbb{Q}$-linear transformation induces by the multiplication by α in L has a characteristic polynomial

$$\mathrm{ch}(\alpha) = x^n - c_1 x^{n-1} + \cdots + (-1)^n c_n$$

with $c_1 = \mathrm{Tr}(\alpha)$ and $c_n = \mathrm{Nm}(\alpha)$. If $\alpha_1, \ldots, \alpha_n$ are the zeroes of $\mathrm{ch}(\alpha)$ in some field extension of $\mathbb{Q}$, then the relations $\mathrm{Tr}(\alpha) = \alpha_1 + \cdots + \alpha_n$ and $\mathrm{Nm}(\alpha) = \alpha_1 \ldots \alpha_n$ hold in that field.

Proposition 1.4. *Let k be a finite extension of $\mathbb{Q}$ and $\alpha \in k$. The following assertions are equivalent:*

1. *The ring $\mathbb{Z}[\alpha]$ generated by α is a finitely generated $\mathbb{Z}$-module.*

2. *The coefficients $c_1, \ldots, c_n$ of the characteristic polynomial* $\mathrm{ch}(\alpha)$ *are integers.*

Proof. Assume that $\mathbb{Z}[\alpha]$ is $\mathbb{Z}$-module of finite type. It is contained in the subfield E of k generated by α. Choose a $\mathbb{Z}$-basis of $\mathbb{Z}[\alpha]$. The multiplication by α preserves $\mathbb{Z}[\alpha]$ thus can be expressed as an integral matrix in this basis. It follows that $\mathrm{ch}_{E/\mathbb{Q}}(\alpha)$ is a polynomial with integral coefficient. Now k is a E-vector space of some dimension r, the polynomial $\mathrm{ch}_{k/\mathbb{Q}}(\alpha) = \mathrm{ch}_{E/\mathbb{Q}}(\alpha)^r$ also has integral coefficients.

Assume that the coefficients $c_1, \ldots, c_n$ are integers. Since α is annihilated by its characteristic polynomial, according to the Cayley-Hamilton theorem, $\mathbb{Z}[\alpha]$ is a quotient of $\mathbb{Z}[x]/\mathrm{ch}_{k/\mathbb{Q}}(\alpha)$. Since $\mathbb{Z}[x]/\mathrm{ch}_{k/\mathbb{Q}}(\alpha)$ is a $\mathbb{Z}$-module of finite type generated by the classes of $1, x, \ldots, x^{n-1}$, so is $\mathbb{Z}[\alpha]$ which is a quotient. □

An element α of number field is called integral if it satisfies one of the above conditions. If α, β are integral then so are $\alpha + \beta$ and $\alpha\beta$ since $\mathbb{Z}[\alpha, \beta]$ is a finitely generated $\mathbb{Z}$-module as long as $\mathbb{Z}[\alpha]$ and $\mathbb{Z}[\beta]$ are. It follows that the set $\mathbb{Z}_k$ of integral elements in k is a subring of k which will be called the *ring of integers of k*.

Proposition 1.5. *Let k be a finite extension of $\mathbb{Q}$. The symmetric bilinear form given by*

$$(x, y) \mapsto \mathrm{Tr}_{k/\mathbb{Q}}(xy). \tag{1.16}$$

is a nondegenerate form on k as a $\mathbb{Q}$-vector space.

Proof. We recall that all finite extensions of field of characteristic 0 can be generated by one element. Let α be a generator of k as $\mathbb{Q}$-algebra and $P \in \mathbb{Q}[x]$ the minimal polynomial of α which is an irreducible polynomial of degree $n = \deg(k/\mathbb{Q})$, then $\{1, \alpha, \ldots, \alpha^{n-1}\}$ form a basis of k as $\mathbb{Q}$-vector space. Since P is an irreducible polynomial in $\mathbb{Q}[x]$, it has no multiple zeros and in particular $P'(\alpha) \neq 0$.

We recall the Euler formula

$$\mathrm{Tr}_{k/\mathbb{Q}}\left(\frac{\alpha^i}{P'(\alpha)}\right) = \begin{cases} 0 & \text{if } i = 0, \ldots, n-2 \\ 1 & \text{if } i = n-1 \end{cases} \tag{1.17}$$

One can derive this formula from the expansion of the polynomial fraction $1/P$ as a linear combination of simple fractions $1/(x - \alpha_j)$ where $\alpha_1, \ldots, \alpha_n$ are the zeroes of P in a field extension containing $\mathbb{Q}$ where P splits. The matrix of the bilinear form (1.16) expressed in the two basis

$$\left\{1, \alpha, \ldots, \alpha^{n-1}\right\} \text{ and } \left\{\frac{\alpha^{n-1}}{P'(\alpha)}, \frac{\alpha^{n-2}}{P'(\alpha)}, \ldots, \frac{1}{P'(\alpha)}\right\}$$

is unipotent upper triangle and therefore invertible. □

Proposition 1.6. *The ring of integers of every finite extension k of $\mathbb{Q}$ is a finitely generated $\mathbb{Z}$-module.*

Proof. The bilinear form (1.16) induces a $\mathbb{Z}$-bilinear form $\mathbb{Z}_k \times \mathbb{Z}_k \to \mathbb{Z}$ as for all $\alpha, \beta \in \mathbb{Z}_k$ we have $\mathrm{Tr}_{k/\mathbb{Q}}(\alpha, \beta) \in \mathbb{Z}$. If $\mathbb{Z}_k^{\perp}$ is the submodule of $\beta \in L$ such that $\mathrm{Tr}_{k/\mathbb{Q}}(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha \in \mathbb{Z}_k$ then $\mathbb{Z}_k \subset \mathbb{Z}_k^{\perp}$. Let us assume there is a finitely generated $\mathbb{Z}$-module M of rank n contained in $\mathbb{Z}_k$, then we have inclusions

$$M \subset \mathbb{Z}_k \subset \mathbb{Z}_k^{\perp} \subset M^{\perp}.$$

Now to construct such M we start with a generator α of k as $\mathbb{Q}$-algebra. After multiplying α by an integer, we can assume that $\alpha \in \mathbb{Z}_k$ and set M as the $\mathbb{Z}$-module generated by $1, \alpha, \ldots, \alpha^{n-1}$. This implies that $\mathbb{Z}_k$ is a finitely generated $\mathbb{Z}$-module as $M^{\perp}$ is. □

As both $\mathbb{Z}_k$ and $\mathbb{Z}_k^{\perp}$ are finitely generated $\mathbb{Z}$-module of rank n, one contained in the other, the quotient $\mathbb{Z}_k^{\perp}/\mathbb{Z}_k$ is a finite group. We define the absolute *discriminant* of k to be the order of this finite group.

Proposition 1.7. *Let k be a number field and $\mathbb{Z}_k$ its ring of integers. Then we have $\mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Q} = k$.*

Proof. Pick a $\mathbb{Z}$-basis of $\mathbb{Z}_k$. Since this basis is $\mathbb{Q}$-linearly independent, the map $\mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Q} \to L$ is injective. It remains to prove that it is also surjective. For every element $\alpha \in k$, there exists $N \in \mathbb{Z}$ so that the characteristic polynomial $\mathrm{ch}_{k/\mathbb{Q}}(\alpha)$ has integral coefficient thus $N\alpha \in \mathbb{Z}_k$ thus α belongs to the image of $\mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Q}$. □

Valuations of number fields

A *valuation* of a number field k is a homomorphism $|.| : k^\times \to \mathbb{R}_+$ such that the inequality

$$|x+y| \le |x| + |y| \tag{1.18}$$

is satisfied for all $x, y \in k^\times$. A valuation of k is said to be *nonarchimedean* if it remains bounded on the ring of integers $\mathbb{Z}_k$ and *archimedean* otherwise.

By restricting a valuation $|.|$ of k to the field of rational numbers, we obtain a valuation of $\mathbb{Q}$. By virtue of Theorem 1.1, a valuation of $\mathbb{Q}$ is equivalent to either the real absolute value or the p-adic absolute value with respect to a prime number p.

Proposition 1.8. *A valuation of k is archimedean if and only if its restriction to $\mathbb{Q}$ is equivalent to the real absolute value, and conversely, it is nonarchimedean if and only if its restriction to $\mathbb{Q}$ is equivalent to a p-adic absolute value. In the latter case, the valuation satisfies the ultrametric inequality* (**??**) *and the valuation of all integers $\alpha \in \mathbb{Z}_k$ is less than one.*

Proof. In the case where the restriction of $|.|$ to $\mathbb{Q}$ is equivalent to the real valuation, $|.|$ is unbounded on $\mathbb{Z}_k$, and therefore is archimedean. However unlike the case of rational numbers, it is generally not true that $|\alpha| \ge 1$ for all $\alpha \in \mathbb{Z}_k - \{0\}$.

In the case where the restriction of $|.|$ to $\mathbb{Q}$ is equivalent to the p-adic valuation for some prime number p, $|.|$ is bounded on $\mathbb{Z}$. We claim that it is also bounded on $\mathbb{Z}_k$, in other words this valuation is nonarchimedean. Indeed, all elements $\alpha \in \mathbb{Z}_k$ satisfies an equation of the form

$$\alpha^r + a_1\alpha^{r-1} + \cdots + a_r = 0$$

where r is the degree of the extension $k/\mathbb{Q}$ and where $a_1, \ldots, a_r \in \mathbb{Z}$. Since $|a_i| \le 1$, the triangle inequality implies that the positive real number $|\alpha|$ satisfies

$$|\alpha|^r \le |\alpha|^{r-1} + \cdots + |\alpha| + 1$$

which proves that $|\alpha|$ is bounded for $\alpha \in \mathbb{Z}_k$. The same argument as in the proof of Theorem 1.1 then shows that $|\alpha| \le 1$ for all $\alpha \in \mathbb{Z}_k$ and the ultrametric inequality (**??**) is satisfied. □

Before classifying all valuations of a number field k, we will classify them up to an equivalence relation. We will say that two valuations u and u' of a number field k are topology equivalent if the completions k_u and $k_{u'}$ of L with respect to u and u' are isomorphic as topological fields containing k. A topology equivalence class of valuation of k will be called a *place* of k. If u is a place of L, we will denote by k_u the topological field obtained as the completion of k with respect to a valuation in the topology equivalence class u; k_u can be equipped with different valuations which give rise to the same topology. We will denote by $\bar{\mathscr{P}}_k$ the set of places of k and for each $u \in \mathscr{P}_k$, k_u the completion of L with respect to a valuation in the topology equivalence class u.

In virtue of Theorem 1.1, we know that valuations of $\mathbb{Q}$ are of the form $|.|_v^t$ where $|.|_v$ is either the real valuation or the p-adic valuation. The valuation $|.|_v^t$ is equivalent to $|.|_v$ in the the above sense. We have also proved that $\mathbb{Q}_p$ is not isomorphic as topological fields neither to $\mathbb{R}$ nor to $\mathbb{Q}_\ell$ where $\ell \neq p$ is a different prime number. It follows that the set of places of $\mathbb{Q}$ is $\bar{\mathscr{P}} = \mathscr{P} \cup \{\infty\}$ where $\mathscr{P}$ is the set of primes numbers.

Let $|.|_u$ be a valuation of L at the place u and $|.|_v$ its restriction to $\mathbb{Q}$. We observe that $\mathbb{Q}_v$ can be realized as the closure of $\mathbb{Q}$ in k_u so that as topological field, $\mathbb{Q}_v$ depends only on the place u and not in a particular choice of valuation $|.|_u$ at u. We derive a map $\pi : \bar{\mathscr{P}}_k \to \bar{\mathscr{P}}$ and denote by $\mathscr{P}_k = \pi^{-1}(\mathscr{P})$ and $\mathscr{P}_\infty = \pi^{-1}(\infty)$. According to Proposition 1.8, $\mathscr{P}_k$ consists in topology equivalent classes of nonarchimedean valuations and $\mathscr{P}_\infty$ in topology equivalent classes of archimedean valuations of L.

We will give an algebraic description of the set $\pi^{-1}(v)$ of places u above a given place v of $\mathbb{Q}$. If $u \in \pi^{-1}(v)$, we will write $u|v$. In such a circumstance, the field k_u as normed $\mathbb{Q}_v$-vector space is complete, in other words, it is a Banach $\mathbb{Q}_v$-vector space. We will later prove that it is finite dimensional. Let us recall some basic facts about finite dimensional Banach spaces.

Lemma 1.9. *All linear maps between finite dimensional Banach $\mathbb{Q}_v$-vector spaces are continuous. All finite dimensional subspaces in a Banach $\mathbb{Q}_v$-vector space are closed.*

Proof. Since linear maps between finite dimensional Banach vector spaces can be expressed in terms of matrices, they are continuous. It follows that the topology on a n-dimensional Banach $\mathbb{Q}_v$-vector space is the same as product topology on $\mathbb{Q}_v^n$.

Let U be a n-dimensional subspace in a Banach $\mathbb{Q}_v$-vector space V. Let $v \in V - U$. Let U_+ be the $(n+1)$-dimensional subspace generated by U and v. Since U_+ has the same topology as $\mathbb{Q}_v^{n+1}$, U is closed in U_+. It follows that there exists a neighborhood of v in V with no intersection with U. It follows that U is closed in V. $\square$

Proposition 1.10. *Let k be a number field . Let v be a place of $\mathbb{Q}$. The $\mathbb{Q}_v$-algebra $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is a direct product of finite extensions of $\mathbb{Q}_v$. The set of factors in this product is in natural bijection with the set of places $u|v$ and the factor corresponding to u is the completion k_u of L at the place u*

$$k \otimes_{\mathbb{Q}} \mathbb{Q}_v = \prod_{u|v} k_u.$$

In particular, for all place $u|v$, k_u is a finite extension of $\mathbb{Q}_v$ and

$$\dim_{\mathbb{Q}}(k) = \sum_{u|v} \dim_{\mathbb{Q}_v}(k_u)$$

Proof. Let us write k in the form $k = \mathbb{Q}[x]/P$ where $P \in \mathbb{Q}[x]$ is an irreducible polynomial. Note that in characteristic zero, irreducible polynomial has no multiple zeroes. Let us decompose P as a product $P = P_1 \dots P_m$ of irreducible polynomials in $\mathbb{Q}_v[x]$. Since P has no multiple zeros, the P_i are mutually prime. Then we can decompose

$$k \otimes_{\mathbb{Q}} \mathbb{Q}_v = \prod_{i=1}^{m} \mathbb{Q}_v[x]/P_i \tag{1.19}$$

as product of finite extensions of $\mathbb{Q}_v$.

Let u be a valuation of k that restrict to the valuation v of $\mathbb{Q}$. By construction, k_u is a complete $\mathbb{Q}_v$-algebra containing k as a dense subset. We derive a homomorphism of $\mathbb{Q}_v$-algebras

$$\phi_u : k \otimes_{\mathbb{Q}} \mathbb{Q}_v \to k_u.$$

Since k is a finite-dimensional $\mathbb{Q}$-vector space, its image is a finite dimensional $\mathbb{Q}_v$-vector subspace of the normed vector space k_u. Because $\mathrm{im}(\phi_u)$ is finite dimensional, it is necessarily a closed subspace of the normed vector space k_u. Moreover, as it contains the dense subset L, it is equal to k_u. It follows that ϕ_u is surjective and k_u is a finite extension of $\mathbb{Q}_v$, therefore a factor $\mathbb{Q}_v[x]/P_i$ of $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$.

Two valuations of k are equivalent if and only if k_u and $k_{u'}$ are isomorphic as topological fields containing k. In that case they are the same factor $\mathbb{Q}_v[x]/P_i$ of $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$. It follows that the set of equivalence class of valuations $u|v$ is a subset of the set of factors $\mathbb{Q}_v[x]/P_i$ of $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$.

It remains to prove that every factor of $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$ can be obtained as the completion of L with respect to a valuation. Let F be a factor of $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$. F is a finite dimensional $\mathbb{Q}_v$-vector space equipped with a $\mathbb{Q}_v$-norm

$$|\alpha|_F = |\mathrm{Nm}_{F/\mathbb{Q}_v}(\alpha)|_v^{1/r_u}. \tag{1.20}$$

We claim that k is dense in the Banach $\mathbb{Q}_v$-vector space k. Indeed, this follows from the fact that k is dense in $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$ and the surjective map $k \otimes_{\mathbb{Q}} \mathbb{Q}_v \to F$ is continuous. In only remains to prove that (1.20) satisfies the triangle inequality. We will check this statement in real and p-adic cases separately. □

Assuming that (1.20) satisfies the triangle inequality, we have proved that the topology equivalence classes of valuations of a number field L lying over a given valuation u of $\mathbb{Q}$ are classified by the factors of $k \otimes_{\mathbb{Q}} \mathbb{Q}_v$. In order to complete the classification of valuations of K, it remains to determine valuations of local fields. In these notes, by *local fields* we mean finite extensions of $\mathbb{R}$ or $\mathbb{Q}_p$.

Let F be a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$. A *positive norm of F* is continuous homomorphism $\|.\| : F^\times \to \mathbb{R}_+$ such that the family of subsets parametrized by $c \in \mathbb{R}_+$

$$B_c = \{x \in F\, , \|x\| < c\}$$

form a base of neighborhood of 0 in F. A *valuation of F* is a positive norm satisfying the triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|. \tag{1.21}$$

We say that it is a *ultrametric norm* if it satisfies the ultrametric inequality

$$\|x + y\| \leq \max(\|x\|, \|y\|) \tag{1.22}$$

for all $x, y \in F$.

Proposition 1.11. *Let $v \in \bar{\mathcal{P}}$ be a place of $\mathbb{Q}$ and $\mathbb{Q}_v$ the corresponding complete field. For every finite extension F of $\mathbb{Q}_v$, (1.20) is the unique valuation $|.|_F : F^\times \to \mathbb{R}_+$, extending the valuation $|.|_v$ on $\mathbb{Q}_v$. Moreover, all positive norms of F are of the forms $\|.\| = |.|_F^t$ for some positive real number t. This positive norm further satisfies the triangle inequality if and only if either v is nonarchimedean, or v is archimedean and $0 < t \leq 1$.*

The proof of this proposition is a case by case analysis and consists in a series of Lemmas 1.12, 1.13, 1.14 and 1.15. First, if F is an archimedean local field then F is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$ for $\mathbb{C}$ is the only nontrivial extension of $\mathbb{R}$.

Lemma 1.12. *All positive norms of $\mathbb{R}$ are of the forms $\|.\| = |.|_{\mathbb{R}}^t$ for some positive real number t. This positive norm further satisfies the triangle inequality if and only if $0 < t \leq 1$.*

Proof. We claim that all continuous homomorphism $\chi : \mathbb{R}^\times \to \mathbb{R}_+$ is of the form $x \mapsto |x|^t$ for some $t \in \mathbb{R}$. Since $\chi(-1)^2 = 1$, we have $\chi(-1) = 1$, and we only have to determine the restriction of χ to $\mathbb{R}_+$. Now since the exponential defines a isomorphism of topological groups $\mathbb{R} \to \mathbb{R}_+$, we only need to prove that all continuous homomorphism $a : \mathbb{R} \to \mathbb{R}$ is of the form $x \mapsto \alpha x$ for some $\alpha \in \mathbb{R}$. Indeed, If $a(1) = \alpha$ then $a(q) = \alpha q$ for all rational number q. It follows by continuity that $a(x) = \alpha x$ for all $x \in \mathbb{R}$.

Now the family of subsets $B_c = \{x, |x|^t < c\}$ for $c \in \mathbb{R}_+$ form a base of neighborhood of 0 if and only if $t > 0$, and $\|x\| = |x|^t$ satisfies the triangle inequality if and only if $0 < t \leq 1$. □

Lemma 1.13. *As for $\mathbb{C}$, the formula* (1.20) *gives rise to the usual absolute value $|z|_{\mathbb{C}} = \sqrt{\mathfrak{R}(z)^2 + \mathfrak{I}(z)^2}$. All positive norms of $\mathbb{C}$ are of the forms $\|.\| = |.|_{\mathbb{C}}^t$ for some positive real number t. All valuations of $\mathbb{C}$ are of the form $z \mapsto |z|^t$ for some real number $0 < t \leq 1$.*

Proof. Let $\chi : \mathbb{C}^\times \to \mathbb{R}_+$ be a continuous homomorphism. It maps the unit circle on a compact subgroup of $\mathbb{R}_+$. However, $\mathbb{R}_+$ has no compact subgroup but the trivial one. It follows that χ factorizes through the absolute value. It follows from 1.12 that $\chi(z) = |z|^t$ for some $t \in \mathbb{R}_+$. If χ extends the usual absolute value on $\mathbb{R}^\times$ then $t = 1$. It satisfies triangle inequality if and only if $t \leq 1$. □

Lemma 1.14. *All positive norms of $\mathbb{Q}_p$ are of the form $\|x\| = |\mathrm{Nm}_{F/\mathbb{Q}_p}(x)|_v^t$ for some positive real number t, and satisfy the ultrametric inequality* (1.22).

Proof. We use similar arguments as for $\mathbb{C}$. Since $\mathbb{Z}_p^\times$ is a compact subgroup of $\mathbb{Q}_p^\times$, the restriction of $\|.\|$ to $\mathbb{Z}_p^\times$ is trivial. If t is the real number such that such that $\|p\| = p^{-t}$ then $\|x\| = |x|_p^t$ for all $x \in \mathbb{Q}_p$. The homomorphism $x \mapsto |x|_p^t$ defines a positive norm of F if and only if $t > 0$, and in this case it satisfies the ultrametric inequality (1.22). □

Lemma 1.15. *Let $F/\mathbb{Q}_p$ be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ of degree r. Then $|.|_F$ defined in* (1.20) *is a valuation of F, and it is the unique valuation on F whose restriction to $\mathbb{Q}_p$ is the p-adic valuation. All positive norms of F is of the form $\|x\| = |x|_F^t$ for some positive real number t, and satisfy the ultrametric inequality* (1.22).

Proof. Let r denote the dimension of F as $\mathbb{Q}_p$-vector space. For every $\alpha \in F$, the multiplication by α defines a $\mathbb{Q}_p$-linear transformation of F and thus has a characteristic polynomial

$$\mathrm{ch}(\alpha) = x^r - c_1 x^{r-1} + \cdots + (-1)^r c_r \in \mathbb{Q}_p[x].$$

An element $\alpha \in F$ is called integral if all coefficients of its characteristic polynomial are in p-adic integers. As in 1.6, we prove that the set $\mathcal{O}_F$ of all integral elements in F is a $\mathbb{Z}_p$-algebra and as a $\mathbb{Z}_p$-module, it is finitely generated. We claim that $\mathcal{O}_F$ is the set of elements $\alpha \in F$ such that $\|\alpha\| \leq 1$.

First we prove that for all $\alpha \in \mathcal{O}_F$, we have $\|\alpha\| \leq 1$. Indeed, $\mathcal{O}_F$ being a finitely generated $\mathbb{Z}_p$-module, is a compact subset of F. The restriction of $\|.\|$ to $\mathcal{O}_F$ is therefore bounded. Since $\mathcal{O}_F$ is stable under multiplication, if the restriction of $\|.\|$ to $\mathcal{O}_F$ is bounded, it is bounded by 1.

Second we prove that of $\|\alpha\| \leq 1$ then α is an integral element of F. Let us choose an arbitrary basis $v_1, \ldots, v_r$ of F as $\mathbb{Q}_p$ vector space. The linear mapping $\mathbb{Q}_p^r \to F$ given by this basis is then a homeomorphism. In particular, the $\mathbb{Z}_p$-module generated by $v_1, \ldots, v_n$ is a neighborhood of 0. By definition of the topology on F, the subsets $B_c = \{x \in F, \|x\| < c\}$ form a system of neighborhood of 0 as $c \to 0$. For c small enough, B_c is contained in $\bigoplus_{i=1}^r \mathbb{Z}_p v_i$. Now, B_c being a $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p^n$, it has to be finitely generated. Because it is open, it has to be of rank r. For all $\alpha \in F$ with $\|\alpha\| \leq 1$, the multiplication by α preserves B_c for all t. Thus the multiplication by α preserves a $\mathbb{Z}_p$-submodule of rank r. It follows that its characteristic polynomial of α has coefficients in $\mathbb{Z}_p$.

The set of integral elements $\mathcal{O}_F$ in F is a local ring with maximal ideal

$$\mathfrak{m}_F = \{x \in F, \|x\| < 1\}$$

since elements $x \in \mathcal{O}_F - \mathfrak{m}_F$ have norm one and are obviously invertible elements of $\mathcal{O}_F$. For $\mathcal{O}_F$ is finitely generated as $\mathbb{Z}_p$-module, so is its maximal ideal $\mathfrak{m}_F$. In particular, it is a compact subset of F. We claim $\mathfrak{m}_F$ is generated as $\mathcal{O}_F$-module by a single element.

Indeed, for $\mathfrak{m}_F$ is a compact subset of F, the range of the the positive norm restricted to $\mathfrak{m}_F$ is a compact subset of $\mathbb{R}$. The norm $\|.\|$ reaches its maximum on some element $\varpi \in \mathfrak{m}$. For all $x \in \mathfrak{m}$, we have $\|\varpi\| \geq \|x\|$ and therefore $x = \varpi y$ for some $y \in \mathcal{O}_F$, in other words, ϖ is a generator of $\mathfrak{m}_F$.

Finally, we claim that for all $x \in F^\times$, $\|x\| = \|\varpi\|^n$ for some integer $n \in \mathbb{Z}$. Indeed, if this is not the case, one can form a product of the form $x^m \varpi^n$ with $m, n \in \mathbb{Z}$ such that

$$\|\varpi\| < \|x^m \varpi^n\| < 1$$

contradicting the very definition of ϖ.

It follows that every element $x \in F^\times$ is of the form $x = \varpi^n y$ for some integer n and $y \in \mathcal{O}_F^\times$. If $\|p\| = p^{-t}$ then we will have $\|x\| = |x|_F^t$ for all $x \in F^\times$. For $|x|_F^t$ to be a positive norm of F, the necessary and sufficient condition is $t > 0$. Moreover it satisfies the ultrametric inequality for all $t > 0$. □

A generator ϖ of the maximal ideal $\mathfrak{m}_F$ is called *uniformizing parameter*. There exists a unique integer $e \in \mathbb{N}$, the *ramification index* of F, such that $p = \varpi^e y$ with $y \in \mathcal{O}_F^\times$. Since $|p| = p^{-1}$, we must have

$$|\varpi|_F = p^{-1/e}. \tag{1.23}$$

Let us denote by $\mathfrak{f}_F = \mathcal{O}_F/\mathfrak{m}_F$ the residue field. The residue field $\mathfrak{f}_F = \mathcal{O}_F/\mathfrak{m}_F$ is a finite extension of $\mathbb{F}_p$ of degree d which will be called the *residual degree* of F.

We claim that the ramification index and the residual degree satisfy the formula

$$r = de \tag{1.24}$$

where r is the degree of the extension $F/\mathbb{Q}_p$. Indeed since all element $x \in \mathcal{O}_F$ can be written in the form $x = \varpi^n y$ with $n \in \mathbb{N}$ and $y \in \mathcal{O}_F^\times$, for every n, the quotient $\mathfrak{m}_F^n/\mathfrak{m}_F^{n+1}$ is one dimensional $\mathfrak{f}_F$-vector space. As $\mathbb{F}_p$-vector space, $\mathfrak{m}_F^n/\mathfrak{m}_F^{n+1}$ has dimension d. It follows that $\mathcal{O}_F/p\mathcal{O}_F$ is a $\mathbb{F}_p$-vector space of dimension de. On the other hand, $\mathcal{O}_F$ is a free $\mathbb{Z}_p$-module of rank r, thus $\mathcal{O}_F/p\mathcal{O}_F$ is a $\mathbb{F}_p$-vector space of rank r and therefore $r = de$.

We will record the following statement that derives from the description of positive norms on local fields archimedean or nonarchimedean.

Corollary 1.16. *Let F be a local field equipped with a positive norm $x \mapsto \|x\|$. For all $c > 0$, the set $\{x \in F; \|x\| \leq c\}$ is a compact subset of F. Moreover the map $\|.\| : F^\times \to \mathbb{R}_+$ is proper.*

Let L be a number field and u an archimedean place of L. We have $u|\infty$ where ∞ is the infinite place of $\mathbb{Q}$. Then k_u can be either $\mathbb{R}$ or $\mathbb{C}$ and we will say that u is a real or complex place correspondingly. By Proposition 1.19, we have $L \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{u|\infty} k_u$. In particular if r is the degree of extension $k/\mathbb{Q}$, r_1 the number of real places and r_2 the number of complex places of L, then we have

$$r = r_1 + 2r_2. \tag{1.25}$$

Let L be an extension of degree r of the field of rational numbers $\mathbb{Q}$. For each valuation u of L dividing a prime number p, we will denote by d_u the residual degree of the extension $k_v/\mathbb{Q}_p$ and e_u the ramification index. Then we have the formula

$$r = \sum_{u|p} d_u e_u. \tag{1.26}$$

We are now going to generalize the product formula (1.4) to number field L. Instead of the valuation $|.|_u$ defined in (1.20) which turns out to be unfit for this purpose, we set

$$\|x\|_F = |\mathrm{Nm}_{F/\mathbb{Q}_v}(x)|_v, \tag{1.27}$$

in other words, if F is a finite extension of $\mathbb{Q}_v$ of degree r, we have $\|x\|_F = |x|_F^r$. In particular as for $\mathbb{C}$, we have

$$\|z\|_{\mathbb{C}} = |z|_{\mathbb{C}}^2 = \Re(z)^2 + \Im(z)^2. \tag{1.28}$$

If F is a finite extension of $\mathbb{Q}_p$ of degree r, of ramification index e and of residual degree d, we have

$$\|\varpi\|_F = |\varpi|_F^r = p^{-d} = |\mathfrak{f}_F|^{-1} \tag{1.29}$$

where ϖ is an uniformizing parameter and $|\mathfrak{f}_F|$ is the cardinal of the residue field $\mathfrak{f}_F$.

Proposition 1.17. *For all $\alpha \in k^\times$, we have $\|\alpha\|_u = 1$ for almost all places u of L, and the product formula*

$$\prod_{u \in \bar{\mathscr{P}}_k} \|\alpha\|_u = 1 \tag{1.30}$$

holds.

Proof. We claim that for all places v of $\mathbb{Q}$, we have

$$|\mathrm{Nm}_{k/\mathbb{Q}}(\alpha)|_v = \prod_{u|v} \|\alpha\|_u. \tag{1.31}$$

Recall that $L \otimes_{\mathbb{Q}} \mathbb{Q}_v = \prod_{u|v} k_u$. The multiplication by α defines a $\mathbb{Q}_v$-linear endomorphism of $L \otimes_{\mathbb{Q}} \mathbb{Q}_v$ preserving each factor k_u. Its determinant $\mathrm{Nm}_{k/\mathbb{Q}}(\alpha)$ is therefore equal to a product

$$\mathrm{Nm}_{k/\mathbb{Q}}(\alpha) = \prod_{u|v} \mathrm{Nm}_{k_u/\mathbb{Q}_v}(\alpha)$$

from which we derive (1.31).

In virtue of (1.31), the product formula for L can be reduced to the product formula for $\mathbb{Q}$. □

Dedekind domains

Let us recall that a *Dedekind domain* is a noetherian integrally closed domain of which every nonzero prime ideal is maximal.

Proposition 1.18. *The ring of integers $\mathbb{Z}_k$ in a finite extension L of $\mathbb{Q}$ is a Dedekind domain.*

Proof. Every ideal of $\mathbb{Z}_k$ is finitely generated as $\mathbb{Z}$-module thus a fortiori as $\mathbb{Z}_k$-module. It follows that $\mathbb{Z}_k$ is a noetherian ring. Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathbb{Z}_k$, $\mathfrak{f} = \mathbb{Z}_k/\mathfrak{p}$ is a domain. The intersection $\mathfrak{p} \cap \mathbb{Z}$ is a nonzero prime ideal thus it is generated by a prime number p. Now $\mathfrak{f}$ is a domain over $\mathbb{F}_p$ and finite dimensional as $\mathbb{F}_p$-vector space. For every $\alpha \in \mathfrak{f}^\times$, the multiplication by α is an injective $\mathbb{F}_p$-linear transformation and therefore surjective. It follows that $\mathfrak{f}$ is a field, in other words $\mathfrak{p}$ is a maximal ideal. $\square$

It is obvious from definition that localization of Dedekind domain is still a Dedekind domain. Local Dedekind domains have very simple structure: they are *discrete valuation ring*. A local ring R is said to be a discrete valuation ring if there exists an element $\varpi \in R$ such that every ideal of R is generated by a power of ϖ.

Proposition 1.19. *Local Dedekind domains are discrete valuation rings.*

Proof. Let R be a local Dedekind domain, L its field of fraction and $\mathfrak{m}$ its maximal ideal. Assume that $\mathfrak{m} \neq 0$ i.e. $R \neq L$ because otherwise the statement would be vacuous. Because R is a Dedekind domain, it has exactly two prime ideals, namely 0 and $\mathfrak{m}$.

First we claim that for all $x \in \mathfrak{m}$, $R[x^{-1}] = L$ where $R[x^{-1}]$ is the subring of L generated by R and x^{-1}. Let $\mathfrak{p}$ be a prime ideal of $R[x^{-1}]$. The intersection $\mathfrak{p} \cap R$ is a prime ideal of A which does not contain x, thus $\mathfrak{p} \cap R \neq \mathfrak{m}$ in other words $\mathfrak{p} \cap R = 0$. It follows that $\mathfrak{p} = 0$ since if there is a nonzero element $y \in \mathfrak{p}$, yx^n will be a nonzero element of $R \cap \mathfrak{p}$ for some integer n. Since all prime ideals of $R[x^{-1}]$ are zero, $R[x^{-1}]$ is a field. As a field containing R and contained in the field of fractions L of R, the only possibility is $R[x^{-1}] = L$.

Second we claim that for all nonzero element $a \in \mathfrak{m}$, there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n \subset (a)$ where (a) is the ideal generated by a. Since R is noetherean, $\mathfrak{m}$ is finitely generated. Let $x_1, \ldots, x_n$ be aset of generators of $\mathfrak{m}$. Since $a \in R[x_i^{-1}]$, there exists $n_i \in \mathbb{N}$ such that $a = y/x_i^n$ for some $y \in R$. It follows that $x_i^{n_i} \in (a)$. Now for $n \geq \sum_i n_i$, we have $\mathfrak{m}^n \subset (a)$.

Let $n \in \mathbb{N}$ be the smallest integer such that $\mathfrak{m}^n \subset (a)$, and let $b \in \mathfrak{m}^{n-1} - (a)$. We claim that a/b is a generator of $\mathfrak{m}$. Let us consider the R-submodule $M = (b/a)\mathfrak{m}$ of L. Since $b\mathfrak{m} \subset (a)$, we have $M \subset R$. If $(b/a)\mathfrak{m} = \mathfrak{m}$, then b/a is an integral element over R as $\mathfrak{m}$ is a finitely generated R-module. As consequence, $b/a \in R$ since R is integrally closed. We derive the inclusion $b \in (a)$ that contradicts the initial assumption on b. Hence M is a R-submodule of R which is not contained in the maximal ideal $\mathfrak{m}$. The only possibility left is $(b/a)\mathfrak{m} = M = R$ i.e. $\mathfrak{m} = (a/b)R$.

Let ϖ be a generator of the maximal ideal $\mathfrak{m}$. We claim that for all $x \in R$, there exist $n \in \mathbb{N}$ and $y \in R^\times$ such that $x = \varpi^n y$. If $x \notin \mathfrak{m}$ then $x \in \mathbb{R}^\times$ and we are done. If $x \in \mathfrak{m}$ then we can write $x = \varpi x_1$ for some $x_1 \in R$, and we reiterate with x_1 instead of x. If this iteration does not stop, then for all $n \in \mathbb{N}$ there exists $x_n \in R$ such that $y = \varpi^n x_n$. In that case, we have an increasing chain of ideals $(x) \subset (x_1) \subset (x_2) \subset \cdots$ which must eventually stop because R is noetherean. If $(x_n) = (x_{n+1})$ with $x_n = \varpi x_{n+1}$ then $\varpi \in R^\times$ and cannot be a generator of $\mathfrak{m}$ as we have assumed. Thus the iteration process has to stop i.e. there exists $n \in \mathbb{N}$ with $x_n \in R^\times$ and we have $x = \varpi^n x_n$.

Now we prove that every ideal I of R is principal. Since R is noetherean, I is finitely generated. Let $x_1, \ldots, x_r$ be a system of generators of I and let us write $x_i = \varpi^{n_i} y_i$ with $y_i \in R^\times$. We can assume that $n_1 \leq n_2 \leq \cdots \leq n_r$. Then I is generated by x_1. □

Corollary 1.20. *Let k be a number field, $\mathbb{Z}_k$ its ring of integers. There is a canonical bijection between the set of nonarchimedean places of L and the set of maximal ideals of $\mathbb{Z}_k$.*

Proof. If $\mathfrak{p}$ is a maximal ideal of $R = \mathbb{Z}_k$, then the local ring $R_\mathfrak{p}$ is of discrete valuation i.e. there exists an uniformizing parameter ϖ in the maximal ideal $\mathfrak{m}_\mathfrak{p}$ of $R_\mathfrak{p}$ such that every nonzero element $x \in R_\mathfrak{p}$ can be written uniquely in the form $x = \varpi^n y$ for some $n \in \mathbb{N}$ and $y \in R_\mathfrak{p}^\times$. Thus every element $x \in k^\times$ can be uniquely written in the form $x = \varpi^n y$ with $n \in \mathbb{Z}$ and $y \in R_\mathfrak{p}^\times$. The map $x \mapsto q^{-n}$ where q is any real number greater than 1 defines a ultrametric valuation of R. These valuations are topology equivalent and give rise to a place of L to be denoted $u_\mathfrak{p}$.

Conversely if u is a nonarchimedean place of L lying over a a prime number p, then the completion k_u is a finite extension of $\mathbb{Q}_p$, its ring of integers $\mathcal{O}_u$ contains the ring of integers R of L. Now, $\mathcal{O}_u$ is also a complete discrete valuation ring with maximal ideal $\mathfrak{m}_u$. If we set $\mathfrak{p}_u = \mathfrak{m}_u \cap R$ then $\mathfrak{p}_u$ is a prime ideal of R. Since $R/\mathfrak{p}_u$ is a subring of the residual field $\mathfrak{f}_u$ of $\mathcal{O}_u$ which is finite, $R/\mathfrak{p}_u$ is also finite and therefore $\mathfrak{p}_u$ is a maximal ideal.

The two maps $\mathfrak{p} \mapsto u_\mathfrak{p}$ and $u \mapsto \mathfrak{p}_u$ that just have been defined between the set of maximal ideals of $\mathbb{Z}_k$ and the set of nonarchimedean valuations of L, are inverse one of each other. □

We recall that a module M over a commutative ring R is said to be locally free of rank r if for every prime ideal $\mathfrak{p}$ of R, the localization $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank r where $R_{\mathfrak{p}}$ is the localization of R at $\mathfrak{p}$. An *invertible R-module* is a shorthand for locally free R-module of rank one.

Corollary 1.21. *A finitely generated module M over a Dedekind domain R is locally free if and only if it is torsion-free.*

Proof. One can be reduced to prove the local statement that a finitely generated module over a local Dedekind domain is free if and only if it is torsion free. As we know from Propostion 1.19 that local Dedekind domains are principal, this statement derive from the classification of modules over principal ideal domain, see [**Lang**] for more details. □

Proposition 1.22. *Let R be a noetherean ring. Then set $\mathrm{Cl}(R)$ of isomorphism classes of invertible R-modules is a commutative group under tensor product.*

Proof. We only need to prove the existence of an inverse. The inverse of an invertible R-module M is given by $M' = \mathrm{Hom}_R(M, R)$. Since the operation $M \mapsto M'$ commutes with localization, M' is an invertible module as long as M is. For the natural map $M \otimes_R M' \to R$ is an isomorphism after localization, it is an isomorphism. □

In what follows, we will consider a Dedekind domain R of field of fractions L. This discussion applies to the ring of integers $\mathbb{Z}_k$ in any number field k or its localizations.

If M is an invertible R-module, then $M \otimes_R L$ is one-dimensional L-vector space. We will denote by $\mathrm{Cl}^+(R)$ the group of isomorphism classes of invertible R-modules M equipped with an isomorphism $\iota : M \otimes L \to L$.

The group $\mathrm{Cl}^+(R)$ can be conveniently described as the group of fractional ideals where a *fractional ideal* of R is a nonzero finitely generated R-submodule of L.

Proposition 1.23. *By mapping the isomorphism class of a pair (M, ι) where M is an invertible R-module and $\iota : M \otimes_R L \to L$ is an isomorphism of L-vector spaces, on the fractional ideal $m = \iota(M)$, we obtain a map from $\mathrm{Cl}^+(R)$ on the set of fractional ideals.*

The induced group law on the set of fractional ideals is given as follows: if $m, m' \subset L$ are fractional ideals then mm' is the module generated by elements of the form $\alpha\alpha'$ where $\alpha \in m$ and $\alpha' \in m'$. The inverse m^{-1} is the submodule generated be elements $\beta \in L$ such that $\alpha\beta \in A$ for all $\alpha \in m$.

Proof. For locally free R-modules are torsion-free, the map ι induces an isomorphism from M on its image m. It follows that the map from $\mathrm{Cl}^+(R)$ to the set of nonzero finitely generated R-submodules of L is injective. All R-submodules of L are torsion free, nonzero finitely generated R-submodules m of L are automatically locally free of rank one. It follows that the above mentioned map is also surjective.

If (M, ι) and (M', ι') are elements of $\mathrm{Cl}^+(R)$ with $m = \iota(M)$, $m' = \iota(M')$ corresponding submodules in L, $(M \otimes_R M', \iota \otimes_R \iota')$ will correspond to the submodule $(\iota \otimes \iota')(M \otimes_R M')$ generated by elements of the form $\alpha\alpha'$ where $\alpha \in m$ and $\alpha' \in m'$.

If $m \subset L$ is a finitely generated R-submodule of L then a R-linear map $M \to A$ is just a map $m \to L$ with range in R. Now a R-linear map $m \to L$ extends uniquely to a L-linear map $m \otimes_R L \to L$ which must be necessarily given by an element $\beta \in L$. This is equivalent to an element $\beta \in L$ such that $\beta m \subset A$. □

Let $\mathscr{P}(R)$ denote the set of maximal ideals in R and $\mathbb{Z}\mathscr{P}(R)$ the free abelian group generated by this set. As $\mathscr{P}(R)$ can naturally be embedded into the set of all nonzero finitely generated R-submodules of L, there is a canonical homomorphism of abelian groups $\mathbb{Z}\mathscr{P}(R) \to \mathrm{Cl}^+(R)$.

Theorem 1.24. *The inclusion $\mathscr{P}(R) \subset \mathrm{Cl}^+(R)$ induces an isomorphism between the free abelian group $\mathbb{Z}\mathscr{P}(R)$ generated by $\mathscr{P}(R)$ and the group $\mathrm{Cl}^+(R)$ of fractional ideals of R.*

Proof. First we prove that the map $\mathbb{Z}\mathscr{P}(R) \to \mathrm{Cl}^+(R)$ is injective. If it is not, there exists an element $\sum_{i \in I} r_i \mathfrak{p}_i \in \mathbb{Z}\mathscr{P}(R)$ such that $\prod_{i \in I} \mathfrak{p}_i^{r_i} = A$. Here I is a finite set of indices, $\mathfrak{p}_i$ are distinct maximal ideals indexed by I and $r_i \in \mathbb{Z}$. Let us separate $I = I_+ \cup J$ such that $r_i > 0$ for all $i \in I_+$ and $r_j \leq 0$ for all $j \in J$. We can assume that I_+ is non empty. Under this notation, we have

$$\prod_{i \in I_+} \mathfrak{p}_i^{r_i} = \prod_{j \in J} \mathfrak{p}_j^{-r_j}.$$

Fix an element $i \in I_+$. One can choose for each $j \in J$ an element $\alpha_j \in \mathfrak{p}_j$ which does not belong to $\mathfrak{p}_i$ and form the product $\prod_{j \in J} \alpha_j^{r_j}$ that belongs to the right hand side but does not belong to $\mathfrak{p}_i$ a fortiori to the left hand side.

Now we prove that $\mathbb{Z}\mathscr{P}(R) \to \mathrm{Cl}^+(R)$ is surjective. For every finitely generated R-submodule m of L, we claim that m can be written as $m = m_1 m_2^{-1}$ where m_1, m_2 are R-submodules of R. Indeed, one can tale $m_2 = m^{-1} \cap A$ which is a nonzero R-submodule of R and $m_1 = m m_2$.

Now we prove that every nonzero ideal of R lies in the image of the homomorphism $\mathbb{Z}\mathscr{P}(R) \to \mathrm{Cl}^+(R)$. Assume that there exists a nonzero R-submodule m of R which does not lie in this image. We can assume that m is maximal with this property. Now m is an ideal, there exists a maximal ideal $\mathfrak{p} \in \mathscr{P}(R)$ such that $m \subset \mathfrak{p}$. Since $\mathfrak{p}^{-1} m$ contains strictly m, it does lie in the image of $\mathbb{Z}\mathscr{P}(R) \to \mathrm{Cl}^+(R)$, and then so does m. We reached a contradiction that shows indeed all non zero ideals of R lie in the image of $\mathbb{Z}\mathscr{P}(R) \to \mathrm{Cl}^+(R)$. □

The homomorphism $\mathrm{Cl}^+(R) \to \mathrm{Cl}(R)$ mapping the class of isomorphism of (M,ι) on the class of isomorphism of M is surjective. The group $k^\times$ of invertible elements of L acts on $\mathrm{Cl}^+(R)$ by mapping the class of isomorphism of (M,ι) on the class of isomorphism of $(M,\alpha\iota)$ for all $\alpha \in k^\times$. In terms on R-submodules of L, α maps a fractional $m \subset L$ on $\alpha m \subset L$.

The orbits of $k^\times$ in $\mathrm{Cl}^+(R)$ are exactly the fibers of the map $\mathrm{Cl}^+(R) \to \mathrm{Cl}(R)$ so that $\mathrm{Cl}(R)$ can be identified with the quotient set of $\mathrm{Cl}^+(R)$ by the action of $k^\times$. However, the action of $k^\times$ on $\mathrm{Cl}^+(R)$ contains more information than the mere quotient set.

Proposition 1.25. *The orbits of $k^\times$ in $\mathrm{Cl}^+(R)$ are exactly the fibers of the map $\mathrm{Cl}^+(R) \to \mathrm{Cl}(R)$ so that $\mathrm{Cl}(R)$ can be identified with the quotient set of $\mathrm{Cl}^+(R)$ by the action of $k^\times$. The stabilizer of $k^\times$ at any point $m \in \mathrm{Cl}^+(R)$ is equal to $R^\times$ for all $m \in \mathrm{Cl}^+(R)$.*

Proof. Let $m \in \mathrm{Cl}^+(R)$ be a nonzero finitely generated R-submodule of L. An automorphism of m induces a L-linear automorphism of $m \otimes L = L$ thus an element $\alpha \in k^\times$ which has to satisfies $\alpha m = m$.

If $\mathfrak{p}$ is a maximal ideal of R, we will denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$ and $m_\mathfrak{p}$ the localization of m. If $\alpha \in k^\times$ such that $\alpha m = m$ then $\alpha m_\mathfrak{p} = m_\mathfrak{p}$ for all $\mathfrak{p} \in \mathscr{P}(R)$. Since $m_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank one, this implies that $\alpha \in R_\mathfrak{p}^\times$. Since R is normal, an element $\alpha \in k^\times$ such that $\alpha \in R_\mathfrak{p}$ lies necessarily in R. For the same argument applies to α^{-1}, we have $\alpha \in R^\times$. □

Let k be a number field and $\mathbb{Z}_k$ its ring of integers in L. Let $\bar{\mathscr{P}}_k = \mathscr{P}_k \cup \mathscr{P}_\infty$ be the set of places of k, where the set of nonarchimedean places $\mathscr{P}_k$ can be identified with the set of maximal ideals of $\mathbb{Z}_k$, and $\mathscr{P}_\infty$ is the set of archimedean places. Let $S \subset \mathscr{P}_k$ be a finite set of nonarchimedean places. Let $\mathbb{Z}_{k,S}$ denote the localization of $\mathbb{Z}_k$ away from S i.e. $\mathbb{Z}_{k,S}$ is generated by elements of k of the form $\alpha\beta^{-1}$ where $\alpha, \beta \in \mathbb{Z}_k$ and $\beta \notin \mathfrak{p}$ for all maximal ideals $\mathfrak{p} \notin S$. In particular the set $\mathscr{P}(\mathbb{Z}_{k,S})$ of maximal ideals in $\mathbb{Z}_{k,S}$ is $\mathscr{P}(\mathbb{Z}_k) - S$.

We now state together two finiteness theorems whose proof can be found in every textbook of algebraic number theory. The first statement is concerned with the finiteness of class number and the second with the Dirichlet unit theorem. We will later reformulate these theorems and their proof in the language of adèles.

Theorem 1.26 (Dirichlet)**.** *The group $\mathrm{Cl}(\mathbb{Z}_{k,S})$ of isomorphism classes of invertible $\mathbb{Z}_{k,S}$-modules is finite. The group of invertible elements $\mathbb{Z}_{k,S}^\times$ is finitely generated of free rank equal to $|S| + |S_\infty| - 1$.*

Adèles and idèles for number fields

We will denote by $\bar{\mathscr{P}}_k = \mathscr{P}_k \cup \mathscr{P}_\infty$ the set of all places of a number field k; $\mathscr{P}_k$ is the of nonarchimedean places and $\mathscr{P}_\infty$ the set of archimedean places. If we denote by $\mathbb{Z}_k$ the ring of integers in L, the set of nonarchimedean places of k can be identified with the set $\mathscr{P}_k = \mathscr{P}(\mathbb{Z}_k)$ of maximal ideals in $\mathbb{Z}_k$. For every $v \in \mathscr{P}_k$, we will denote by k_v the completion of k at v and $\mathscr{O}_v$ its ring of integers. The set of archimedean places of a number field L is in bijection with the set of factors in the decomposition of $k_\mathbb{R} = l \otimes_\mathbb{Q} \mathbb{R}$ as product of fields, see 1.19

$$k_\mathbb{R} = \prod_{u \in \mathscr{P}_\infty} k_u,$$

where the completion of k at the place $u \in S_\infty$ is denoted by k_u; it can be either $\mathbb{R}$ or $\mathbb{C}$. The real vector space $k_\mathbb{R}$ is called the Minkowski space of k.

An *adèle* of k is a sequence $(x_v; v \in \bar{\mathscr{P}}_k)$ where $x_v \in k_v$ for all places and $x_v \in \mathscr{O}_v$ for almost all nonarchimedean places. The *ring of adèles* $\mathbb{A}_k$ of k can be factorized as a direct product

$$\mathbb{A}_k = k_\mathbb{R} \times \mathbb{A}_{k,\text{fin}}$$

where the ring of finite adèles $\mathbb{A}_{k,\text{fin}}$ is the ring of all sequences $(x_v; v \in \mathscr{P}_k)$ where $x_v \in k_v$ for all $v \in \mathscr{P}_k$ and $x_v \in \mathscr{O}_v$ for almost all v. The ring of finite adèles can be seen as a direct limit of smaller subrings

$$\mathbb{A}_{k,\text{fin}} = \varinjlim \mathbb{A}_{k,S} \tag{1.32}$$

over all finite subsets $S \subset \mathscr{P}_k$ where $\mathbb{A}_{k,S}$ is the subring of $\mathbb{A}_k$ of adeles (x_v) such that $x_v \in \mathscr{O}_v$ for all nonarchimedean place $v \notin S$. We have

$$\mathbb{A}_{k,S} = \prod_{v \in \mathscr{P}_k - S} \mathscr{O}_v \times \prod_{v \in S} k_v.$$

If we denote by $\mathbb{Z}_{k,S}$ the localization of $\mathbb{Z}_k$ away from S, then the set of maximal ideals $\mathscr{P}(\mathbb{Z}_{k,S})$ is $\mathscr{P}_k - S$, and we have

$$\prod_{v \in \mathscr{P}_k - S} \mathscr{O}_v = \lim_{\leftarrow N} \mathbb{Z}_{k,S}/N = \hat{\mathbb{Z}}_{k,S}$$

where the projective limit is taken over the set of nonzero ideals N of R_S ordered by inclusion relation. We note that $\hat{\mathbb{Z}}_{k,S}$ is compact as a projective limit of finite sets. We have

$$\mathbb{A}_{k,S} = \hat{\mathbb{Z}}_{k,S} \times \prod_{v \in S} k_v.$$

For all finite subsets $S \subset \mathscr{P}_k$, let us equip $\mathbb{A}_{k,S}$ with the product topology which is the coarsest one such that the projection to every factor is continuous. We will equip $\mathbb{A}_k$ with the finest topology on $\mathbb{A}_k$ such that for every finite subset S of $\mathscr{P}_k$, the inclusion $\mathbb{A}_{k,S} \to \mathbb{A}_k$ is continuous. Open neighborhoods of 0 in $\mathbb{A}_k$ are of the form $U_{S,\infty} \times \prod_{v \notin S} \mathscr{O}_v$ where S is a finite subset of $\mathscr{P}_k$ and $U_{S,\infty}$ is an open subsets of $\prod_{v \in S \cup \mathscr{P}_\infty} k_v$.

We claim that $\mathbb{A}_k$ is locally compact. Indeed we can construct a system of compact neighborhoods of 0 with boxes which will also be useful for other purposes. The size of a box is given by a sequence of positive real integers $c = (c_v; v \in \bar{\mathscr{P}}_k)$ with $c_v = 1$ for almost all v. The *box* B_c of size c around 0 is the subset of $\mathbb{A}_k$ of all sequences $(x_v; v \in \bar{\mathscr{P}}_k)$ with $x_v \in k_v$ satisfying $|x_v| \leq c_v$ for every place $v \in \bar{\mathscr{P}}_k$. Since the subset of k_v defined by the inequality $|x_v| \leq c_v$ is compact, the product B_c is compact according to the Tychonov theorem.

For S being the empty set, we have $A_\emptyset = \mathbb{Z}_k$, and therefore $\mathbb{A}_{k,\emptyset} = \hat{\mathbb{Z}}_k$. The ring of finite adèles can also be described as

$$\mathbb{A}_{k,\mathrm{fin}} = \hat{\mathbb{Z}}_k \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.33}$$

by the same argument as in the particular case of the field of rational numbers.

Theorem 1.27. *For every number fields k, the ring $\mathbb{A}_k$ of adèles of k is locally compact. Moreover k embeds diagonally in $\mathbb{A}_k$ as a discrete cocompact subgroup.*

Proof. Let us consider the box B_c of size (c_v) with $c_v = 1$ for all nonarchimedean places v. This is a compact neighborhood of 0 in $\mathbb{A}_k$ whose intersection with k is finite. Indeed, if $\alpha \in k \cap B_c$ then $\alpha \in \mathbb{Z}_k$ for $\alpha \in \mathscr{O}_v$ for all $v \in \mathscr{P}_k$. It follows that α satisfies the equation $\alpha^r + a_1\alpha^{r-1} + \cdots + a_r = 0$ where r is the degree of $k/\mathbb{Q}$ and $\alpha_i \in \mathbb{Z}$. Now the integers a_i can be expressed as elementary symmetric functions of variables $\phi(\alpha)$ where $\phi : k \to \mathbb{C}$ runs over the set of embeddings of k into the field of complex numbers. The real absolute value of a_i can be therefore bounded by a quantity depending on $(c_v; v \in \mathscr{P}_\infty)$. There are thus only finitely many possible integers a_i, and this infers the finiteness of $k \cap B_c$. The same argument shows that $k \cap B_c = \{0\}$ for small enough $(c_v; v \in \mathscr{P}_\infty)$. It follows that k is a discrete subgroup of $\mathbb{A}_k$.

Let $\alpha_1, \ldots, \alpha_r$ be a basis of the $\mathbb{Q}$-vector space k which induces an isomorphism of $\mathbb{Q}$-vector spaces $\mathbb{Q}^r \to k$. For each place v of Q, it induces an isomorphism of topological $\mathbb{Q}_v$-vector spaces

$$\mathbb{Q}_v^r \to k \otimes_{\mathbb{Q}} \mathbb{Q}_v = \prod_{u|v} k_u$$

which gives rises to an isomorphism $\mathbb{Z}_v^r \to \prod_{u|v} \mathcal{O}_u$ for almost all non-archimedean places u of k. It follows that the basis $\{\alpha_1, \ldots, \alpha_r\}$ induces an isomorphism of topological groups $\mathbb{A}^r \to \mathbb{A}_k$. It follows that

$$\mathbb{A}_k / k = (\mathbb{A}/\mathbb{Q})^r$$

and therefore $\mathbb{A}_k / k$ is compact. □

Let us now introduce the notion of *idèles in number fields*. An idèle of k is a sequence $(x_v, v \in \bar{\mathcal{P}}_k)$, v being finite or infinite place of k, consisting of $x_v \in k_v^\times$ with $x_v \in \mathcal{O}_v^\times$ for almost all finite places. The group of idèles $\mathbb{A}_k^\times$ is nothing but the group of invertible elements in the ring of adèles $\mathbb{A}_k$. It is equipped with the coarsest topology permitting the inclusion $\mathbb{A}_k^\times \subset \mathbb{A}_k$ as well as the inversion $\mathbb{A}_k^\times \to \mathbb{A}_k, x \mapsto x^{-1}$ to be continuous. We have a system of open neighborhoods of 1 in $\mathbb{A}_k^\times$ of the form $U_{S \cup \mathcal{P}_\infty} \times \prod_{\mathcal{P}_k - S} \mathcal{O}_v^\times$ where S is a finite subset of $\mathcal{P}_k$ and $U_{S \cup \mathcal{P}_\infty}$ is an open subset in $\prod_{v \in S \cup \mathcal{P}_\infty} k_v^\times$.

The group of idèles $\mathbb{A}_k^\times$ is equipped with a norm

$$\|.\|_k : \mathbb{A}_k^\times \to \mathbb{R}_+$$

which is a continuous homomorphism defined by

$$\|x\|_k = \prod_{v \in \mathcal{P}_k \cup \mathcal{P}_\infty} \|x_v\|_v$$

for all idèles $x = (x_v; v \in \bar{\mathcal{P}}_k) \in \mathbb{A}_k^\times$. For almost all finite places v, $\|x_v\|_v = 1$ so that the infinite product is well defined. Let denote by $\mathbb{A}_k^1$, the group of norm one idèles, the kernel this homomorphism. The product formula (1.30) implies that $k^\times$ embeds diagonally as a subgroup of $\mathbb{A}_k^1$

Theorem 1.28. *The group $k^\times$ embeds diagonally as a discrete cocompact subgroup of $\mathbb{A}_k^1$.*

Proof. The discreteness of $k^\times$ as subgroup of $\mathbb{A}_k^\times$ can be proved in the same manner as the discreteness of k in $\mathbb{A}_k$ in Theorem 1.27.

For proving the discreteness of k as subgroup of $\mathbb{A}_k$, we used the fact that the intersection of k with a box B_c is a finite set which is even reduced to 0 if c is set to be small enough. There is a converse to this namely if c is set to be large then $B_c \cap k^\times \neq \{0\}$. The following lemma is an adelic variant of Minskowski's theorem on symmetrical convex bodies in Euclidean spaces.

Lemma 1.29 (Minkowski)**.** *There exists a constant $C > 0$, depending only on the discriminant of k, such that for all sequences $c = (c_v; v \in \bar{\mathscr{P}}_k)$ of positive real numbers c_v with $c_v = 1$ for almost all v, satisfying $\prod_v c_v > C$, the intersection $k^\times$ with $B_c = \{x = (x_v) \in \mathbb{A}_k, \|x_v\| \leq c_v\}$ is not empty.*

We will also need the following lemma.

Lemma 1.30. *For all sequences $c = (c_v; v \in \bar{\mathscr{P}}_k)$ of positive real numbers c_v with $c_v = 1$ for almost all v, the intersection $B_c \cap \mathbb{A}_k^1$ is a compact subset of $\mathbb{A}_k^1$.*

Proof. Let $(x_v; v \in \bar{\mathscr{P}}_k)$ be an idèle of norm one such that $\|x_v\|_v \leq c_v$ for all v. For all v, we have upper and lower bounds for $\|x_v\|_v$

$$c_\bullet^{-1} c_v \leq x_v \leq c_v$$

where $c_\bullet = \prod_{u \in \bar{\mathscr{P}}_k} c_u$. Let K_v denote the compact subset of $k_v^\times$ defined by these lower and upper bounds. For almost all v, $c_v = 1$ so that K_v is defined by the inequality $c_\bullet^{-1} \leq \|x_v\|_v \leq 1$. For v such that the cardinal number q_v of the residue field $\mathfrak{f}_v$ is greater than $c_\bullet$, these inequalities imply $\|x_v\|_v = 1$, in other words $K_v = \mathscr{O}_v^\times$. It follows that $\prod_{v \in \bar{\mathscr{P}}_k} K_v$ is a compact subset of $\mathbb{A}_k^\times$, and therefore

$$B_c \cap \mathbb{A}_k^1 = \prod_{v \in \bar{\mathscr{P}}_k} K_v \cap \mathbb{A}_k^1$$

is a compact subset of $\mathbb{A}_k^1$. □

We will now complete the proof of Theorem 1.28. Let B_c be the box associated to a sequence $c = (c_v; v \in \bar{\mathscr{P}}_k)$ of positive real numbers satisfying $\prod_v c_v > C$ as above. We claim that

$$\mathbb{A}_k^1 \subset \bigcup_{\alpha \in k^\times} \alpha B_c. \tag{1.34}$$

Indeed if $x \in \mathbb{A}_k^1$, $x^{-1} B_c$ is also a box $B_{c'}$ with $c'_v = \|x_v\|_v^{-1} c_v$ for all $v \in \bar{\mathscr{P}}_k$ satisfying $\prod_v c'_v = \prod_v c_v$. By Minkowski's lemma, we know that there exists an element $\alpha \in B_{c'} \cap k^\times$. But if $\alpha \in x^{-1} B_c$ then $x \in \alpha^{-1} B_c$ and therefore the inclusion (1.34) is proved. Now the compactness of $\mathbb{A}_k^1 / k^\times$ follows from the compactness of $B_c \cap \mathbb{A}_k^1$, as asserted by Lemma 1.30. □

Let us discuss the relation between Theorem 1.28 and Theorem 1.26. First let us set $S = \emptyset$ in the statement of Theorem 1.26. The group $\hat{\mathbb{Z}}_k^\times = \prod_{v \in \mathscr{P}_k} \mathscr{O}_v^\times$ can be embedded as a compact subgroup of $\mathbb{A}_k^\times$ by setting all component at archimedean places to be one. We have an exact sequence

$$0 \to k_{\mathbb{R}}^\times \times \hat{\mathbb{Z}}_k^\times \to \mathbb{A}_k^\times \to \bigoplus_{v \mathscr{P}_k} k_v^\times / \mathscr{O}_v^\times \to 0$$

where $\bigoplus_{v\in\mathscr{P}_k} k_v^\times/\mathscr{O}_v^\times$ is the set of sequences $(n_v, v \in \mathscr{P}_k)$ with $n_v \in k_v^\times/\mathscr{O}_v^\times$, n_v vanish for almost all v. We observe that there is a canonical isomorphism between $k_v^\times/\mathscr{O}_v^\times$ and $\mathbb{Z}$ so that the n_v can be seen as integers.

Restricted to the norm one idèles, we have an exact sequence

$$0 \to k_{\mathbb{R}}^1 \times \hat{\mathbb{Z}}_k^\times \to \mathbb{A}_k^1 \to \bigoplus_{v\in\mathscr{P}_k} k_v^\times / \mathscr{O}_v^\times \to 0$$

where $k_{\mathbb{R}}^1 = k_{\mathbb{R}}^\times \cap \mathbb{A}_k^1$ is the subgroup of $k_{\mathbb{R}}^\times$ consisting of elements $(x_v; v \in \mathscr{P}_\infty)$, such that $\prod_{v\in\mathscr{P}_\infty} \|x_v\|_v = 1$. There is homomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & k^\times \arrow[r] \arrow[d] & k^\times \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k_{\mathbb{R}}^1 \times \hat{\mathbb{Z}}_k^\times \arrow[r] & \mathbb{A}_k^1 \arrow[r] & \bigoplus_{v\in\mathscr{P}_k} k_v^\times/\mathscr{O}_v^\times \arrow[r] & 0
\end{tikzcd} \tag{1.35}$$

of which middle vertical arrow is injective. We observe that $\bigoplus_{v\in\mathscr{P}_k} k_v^\times/\mathscr{O}_v^\times$ is nothing but the group $\mathrm{Cl}^+(\mathbb{Z}_k)$ of fractional ideals of $\mathbb{Z}_k$. According to 1.25, the kernel of the right vertical map has kernel $k^\times \cap \hat{\mathbb{Z}}_k^\times = \mathbb{Z}_k^\times$ and cokernel

$$\mathrm{Cl}_k = \Big(\bigoplus_{v\in\mathscr{P}_k} k_v^\times / \mathscr{O}_v^\times\Big)/k^\times.$$

the group of ideal classes $\mathrm{Cl}_k = \mathrm{Cl}(\mathbb{Z}_k)$ of k.

We derive from the above diagram an exact sequence

$$0 \to \left(\hat{\mathbb{Z}}_k^\times \times k_{\mathbb{R}}^1\right)/\mathbb{Z}_k^\times \to \mathbb{A}_k^1/k^\times \to \mathrm{Cl}_k \to 0. \tag{1.36}$$

The group $\mathbb{A}_k^1/k^\times$ is compact of and only if both $\left(\hat{\mathbb{Z}}_k^\times \times k_{\mathbb{R}}^1\right)/\mathbb{Z}_k^\times$ and Cl_k are compact. As Cl_k is discrete, it is compact if and only if it is finite. Since $\hat{\mathbb{Z}}_k^\times$ is compact, $\left(\hat{\mathbb{Z}}_k^\times \times k_{\mathbb{R}}^1\right)/\mathbb{Z}_k^\times$ if and only if $k_{\mathbb{R}}^1/\mathbb{Z}_k^\times$ is compact. Now, we have a homomorphism with compact kernel

$$\log_{\mathscr{P}_\infty} : k_{\mathbb{R}}^\times \to \prod_{u\in\mathscr{P}_\infty} \mathbb{R}$$

mapping $(x_u; u \in \mathscr{P}_\infty)$ to $(y_u; u \in \mathbb{R})$ with $y_u = \log \|x_u\|_u$. The subgroup k_R^1 is the preimage of the hyperplane $H_{\mathbb{R}}$ in $\prod_{u\in\mathscr{P}_\infty} \mathbb{R}$ defined by the equation

$$\sum_{u\in\mathscr{P}_\infty} y_u = 0.$$

Lemma 1.31. *The restriction of* $\log_{\mathscr{P}_\infty}$ *to* $\mathbb{Z}_k^\times$ *has finite kernel. Its image is a discrete subgroup of* $H_{\mathbb{R}}$.

Proof. Both statement follows from 1.16. First, the kernel of $\log_{\mathscr{P}_\infty} : k_{\mathbb{R}}^1 \to H_{\mathbb{R}}$ is a compact group whose intersection with the discrete subgroup $\mathbb{Z}_k^\times$ is necessarily finite. Second, let U be a compact neighborhood of the neutral element of $L_{\mathbb{R}}^1$ such that $U \cap \mathbb{Z}_k^\times = \{1\}$. Because $\log_{\mathscr{P}_\infty}$ is proper, $\log_{\mathscr{P}_\infty}^{-1}(\log_{\mathscr{P}_\infty}(U))$ is a compact subset of $k_{\mathbb{R}}^1$ whose intersection with $\mathbb{Z}_k^\times$ is necessarily finite. It follows that $\log_{\mathscr{P}_\infty}(\mathbb{Z}_k^\times)$ has finite intersection with $\log_{\mathscr{P}_\infty}(U)$ which is a compact neighborhood of 0 in $H_{\mathbb{R}}$. □

Now, the quotient $k_{\mathbb{R}}^1/\mathbb{Z}_k^\times$ is compact if and only if $H_{\mathbb{R}}/\log_{\mathscr{P}_\infty}(\mathbb{Z}_k^\times)$ is compact. According to the lemma, $\log_{\mathscr{P}_\infty}(\mathbb{Z}_k^\times)$ is a discrete subgroup of $H_{\mathbb{R}}$, and therefore its ranks is at most the real dimension of $H_{\mathbb{R}}$, namely $|\mathscr{P}_\infty| - 1$. The compactness of this quotient implies

$$\mathrm{rk}(\log_{\mathscr{P}_\infty}(\mathbb{Z}_k^\times)) = |\mathscr{P}_\infty| - 1.$$

Thus the compactness of $\mathbb{A}_k^1/k^\times$ implies both the finiteness of class number of k and Dirichlet's theorem on the rank of $\mathbb{Z}_k$. Conversely, if we know that Cl_k is finite and $L_R^1/\mathbb{Z}_k^\times$ is compact, then we also know that $\mathbb{A}_k^1/k^\times$ is compact.

We also observe that the same argument applies when we replace $\mathbb{Z}_k$ by its localization R_S away from a finite set of places $S \subset \mathscr{P}_k$. We also have an exact sequence

$$0 \to \left(\hat{\mathbb{Z}}_{k,S}^\times \times \prod\nolimits_{v\in S\cup\mathscr{P}_\infty}^1 k_v^\times\right)/\mathbb{Z}_{k,S}^\times \to \mathbb{A}_k^1/k^\times \to \mathrm{Cl}(\mathbb{Z}_{k,S}) \to 0$$

where $\prod_{v\in S\cup\mathscr{P}_\infty}^1 k_v^\times$ is the subgroup of $\prod_{v\in S\cup\mathscr{P}_\infty} k_v^\times$ consisting of elements $(x_v; v \in S\cup\mathscr{P}_\infty)$ such that $\prod_{v\in S\cup\mathscr{P}_\infty} \|x_v\|_v = 1$. For all finite subset S of $\mathscr{P}_k$, the compactness of $\mathbb{A}_k^1/k^\times$ is equivalent to the finiteness of $\mathrm{Cl}(R_S)$ and the compactness of the quotient $(\prod_{v\in S\cup\mathscr{P}_\infty} k_v^\times)/\mathbb{Z}_{k,S}^\times$ combined. We note that the compactness of the latter implies that $\mathbb{Z}_{k,S}^\times$ is a finitely generated abelian group of rank $|S \cup \mathscr{P}_\infty| - 1$.

2 Pontryagin duality

Measure theory

We recall some elements of Lebesgue's integration theory. A basic idea on Lebesgue's theory is that it is usually not possible to assign a measure to all subsets of a given set. In order to build a theory of integration, one must first specify a family of subsets of X which can be given a measure. Such a family is a σ-algebra. Given a set X, a σ-algebra is a collection of subsets of X which is closed under countable unions, countable intersections, and complements. An element of this family is called a measurable set with respect to the theory of integration we consider.

A measure on a space X equipped with a σ-algebra $\mathscr{M}$ is a function

$$\mu : \mathscr{M} \to \mathbb{R}_+ \cup \{\infty\}$$

taking non-negative real values or infinity, satisfying the property that if $Y_1, Y_2, \ldots \in \mathscr{M}$ is a countable family is disjoint measurable sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} Y_i\right) = \sum_{i=1}^{\infty} \mu(Y_i).$$

We observe that the set $\mathbb{R}_+ \cup \{\infty\}$ is equipped with a positive linear structure i.e. we can add two elements of this set as well as multiply an element of this set with a non-negative real number. Addition rule in $\mathbb{R}_+ \cup \{\infty\}$ is obvious as well as multiplication rule except that the value to be assigned to the element ∞ multiplied by the scalar 0. In this setting, we decide that the result of this multiplication is 0. We also observe that the set $\mathbb{R}_+ \cup \{\infty\}$ is linearly ordered and every increasing sequence has a limit.

Let $A(\mathscr{M})$ denote the space of finite linear combination characteristic functions of measurable subsets. Elements of $A(\mathscr{M})$ are called step functions. The space $A(\mathscr{M})$ is a $\mathbb{R}$-vector space. Let $A^+(\mathscr{M})$ denote the space of step functions with non-negative real values. Elements of $A^+(\mathscr{M})$ are finite linear combinations of characteristic functions of measurable sets with positive coefficients. Indeed, for every $a \in A(\mathscr{M})$ there is a canonical way to write a as a linear combination of characteristic functions of disjoint measurable subsets, and if moreover $a \in A^+(\mathscr{M})$, all coefficients entering in this canonical expression are positive.

Proposition 2.1. *There exists a unique map $\ell_\mu : A^+(\mathscr{M}) \to \mathbb{R}_+ \cup \{\infty\}$ satisfying the following properties:*

- *for all $Y \in \mathscr{M}$, we have $\ell_\mu(1_Y) = \mu(Y)$;*

- *for all* $a_1, a_2 \in A^+(\mathscr{M})$*, we have* $\ell_\mu(a_1 + a_2) = \ell_\mu(a_1) + \ell_\mu(a_2)$*;*
- *for all* $a \in A^+(\mathscr{M})$ *and* $\alpha \in \mathbb{R}_+$*, we have* $\ell_\mu(\alpha a) = \alpha \ell_\mu(a)$*.*

Proof. The uniqueness is clear as if $a = \sum_{i=1}^n \alpha_i 1_{Y_i}$ with $\alpha_i \in \mathbb{R}_+$ and $Y_i \in \mathscr{M}$, then $\ell_\mu(a) = \sum_{i=1}^n \alpha_i \mu(Y_i)$. For the existence, we need to check that the number $\sum_{i=1}^n \alpha_i \mu(Y_i)$ depends only on a, and not on the way we write a as positive linear combination of step functions. For this, we can use the canonical expression of a as positive linear combination of characteristic functions of disjoints measurable sets. □

A real valued function $\phi : X \to \mathbb{R}$ is said to measurable if for every $a \in \mathbb{R}$, the set $\phi^{-1}((-\infty, a)) = \{x \in X | \phi(x) < a\}$ is measurable. Let $B(\mathscr{M})$ denote the space of real valued measurable functions on X with respect to the σ-algebra $\mathscr{M}$. Let $B^+(\mathscr{M})$ denote the measurable functions with non-negative values.

Lemma 2.2. *For every non-negative valued measurable function* $b \in B^+(\mathscr{M})$*, there exists a sequence of non-negative step functions* $a_1, a_2, \ldots \in A^+(X)$ *such that for every* $x \in X$*, the sequence* $a_1(x) \leq a_2(x) \leq \cdots$ *is increasing and converge to* $b(x)$*.*

Proof. For every $n \in \mathbb{N}$ we define the step function b_n as follow:

$$\begin{aligned} b_n &= \sum_{i=1}^{n^2} \frac{1_{\{x \in X | b(x) \geq i/n\}}}{n} \\ &= \begin{cases} \lfloor b(x)n \rfloor / n & \text{if } b(x) \leq n \\ n & \text{if } b(x) \geq n \end{cases} \end{aligned}$$

It is clear that is a positive step function bounded from above by b. Moreover, if we set $a_n = b_{2^n}$, then the sequence of positive step function $a_1, a_2, \ldots$ satisfy all the required properties. □

Proposition 2.3. *The map* $\ell_\mu : A^+(\mathscr{M}) \to \mathbb{R}_+ \cup \{\infty\}$ *can be extended uniquely as a map* $\ell_\mu : B^+(\mathscr{M}) \to \mathbb{R}_+ \cup \{\infty\}$ *such that for if an increasing sequence of positive step functions* $a_1 \leq a_2 \leq \cdots$ *converge point wise to a function* $b \in B^+(\mathscr{M})$ *then*

$$\ell_\mu(b) = \lim \ell_\mu(a_i).$$

Proof. The uniqueness follows from the preceding lemma since for every $b \in B^+(\mathscr{M})$ is a point wise limit of an increasing sequence of positive step functions, the value $\ell_\mu(b)$ has to be the limit of the increasing sequence $\ell_\mu(a_i)$. We only need to prove that $\ell_\mu(b)$ defined as the limit of the increasing sequence $\ell_\mu(a_i)$ depends only on b, not on the way of writing b an the limit of the increasing sequence a_i. □

If $f : X \to \mathbb{R}$ is a measurable function, the function $|f|(x) = |f(x)|$ is a non-negative measurable function. We say that f is integrable if $\ell_\mu(|f|) < \infty$. The subset of $B(\mathcal{M})$ of integrable function is a $\mathbb{R}$-linear subspace to be denoted $L^1(X,\mu)$. For every $f \in L^1(\mu)$, we will write $f = f_+ - f_-$ where $f_+(x) = \max(f(x),0)$ and $f_-(x) = \max(-f(x),0)$ where $f_+, f_- \in L^1(X,\mu) \cap B^+(X)$. By setting

$$\ell_\mu(f) = \ell_\mu(f_+) - \ell_\mu(f_-)$$

we define a $\mathbb{R}$-linear map $\ell_\mu : L^1(X,\mu) \to \mathbb{R}$. For mnemonic reason, we shall write the linear form ℓ_μ with integral notation

$$\ell_\mu(f) = \int_X f(x)\mu(x)$$

implying that $\ell_\mu(f)$ is the integration of the function f against the measure μ.

Radon measures

For a topological space X, the smallest σ-algebra containing all open subsets is called the Borel σ-algebra $\mathcal{B}$, and its elements are called the Borel subsets. Every continuous function $f : X \to \mathbb{R}$ is then measurable for the open set $f^{-1}((-\infty, a))$ is automatically a Borel set.

A topological space is said to be *locally compact* if each point $x \in X$ admits a compact neighborhood. The idea of Radon measure is to restrict attention to those measures for which compact subset have finite measures. More precisely, a measure $\mu : \mathcal{B} \to \mathbb{R}_+ \cup \{\infty\}$ on a locally compact Hausdorff topological space X is said to be Radon if the following requirements are satisfied:

- $\mu(K)$ if finite for compact subset $K \subset X$;
- for every Borel set $Y \in \mathcal{B}$, $\mu(Y)$ is the supremum of $\mu(K)$ for K ranging over compact subsets K contained in Y (inner regularity);
- for every Borel set $Y \in \mathcal{B}$, $\mu(Y)$ is the infimum of $\mu(U)$ for U ranging over open subsets U containing Y (outer regularity).

Continuous functions with compact support are integrable with respect to a Radon measure μ. Indeed, let $f : X \to \mathbb{R}$ be a continuous function with support contained in a compact set K. For K is compact, $|f|$ is bounded from above over K and attains it maximum. If $a : X \to \mathbb{R}$ is a non-negative step function with $a(x) \le |f(x)|$ for all $x \in X$, we have

$$\ell_\mu(a) \le \max_{x\in K} |f(x)|\mu(K) < \infty.$$

It follows that $\ell_\mu(|f|)$ define as the supremum of $\ell_\mu(a)$ for all non-negative step function a bounded from above by $|f|$, is finite, therefore f is integrable. We have thus a linear functional ℓ_μ on the space $C_c(X)$ of continuous functions with compact support.

This functional is continuous with respect to the natural topology of $C_c(x)$ which we are about to recall. For every compact subset K of X, we will denote $C_K(X)$ is the space of continuous functions on X with support in K. The space of all continuous functions with compact support $C_c(X)$ can be realized as the union of all $C_K(X)$. For every compact subset K, $C_K(X)$ is equipped with the uniform norm:

$$||f-g||_\infty = \sup_{x\in K} |f(x)-g(x)|. \tag{2.1}$$

and its induced topology. If $K \subset K'$ then $C_K(X)$ is naturally a closed subspace of $C_{K'}(X)$. The inductive limit $C_c(X)$ may be thought of as the union of the spaces $C_K(X)$. We consider the inductive limit on $C_c(X)$ which is the finest topology such that all inclusion maps $C_K(X) \to C_c(X)$ are continuous. The linear form $\ell_\mu : C_c(X) \to \mathbb{R}$ is continuous.

A continuous linear functional $\ell : C_c(X) \to \mathbb{R}$ is said to be positive if for all non-negative continuous function with compact support $f : X \to \mathbb{R}$, $\ell(f) \ge 0$. A Radon measure μ on X defines thus a positive linear functional $\ell_\mu : C_c(X) \to \mathbb{R}$. Conversely, for every positive linear functional $\ell : C_c(X) \to \mathbb{R}$, there exists a unique Radon measure μ on X such that $\ell = \ell_\mu$. This is the Riesz-Markov-Kakutani representation theorem.

For locally compact topological space, instead of Radon measures, we can use the equivalent concept of positive continuous linear functional $C_c(X) \to \mathbb{R}$. We define Radon complex measures as a continuous linear functional $C_c(X) \to \mathbb{C}$.

Proposition 2.4. *Let X be a locally compact topological space, $C_c(X)$ the space of all continuous functions with compact support on X. For every Radon measure μ, the space $L^1(X,\mu)$ of integrable functions is the completion of $C_c(X)$ with respect to the distance given by the L^1-norm $||f||_{L^1} \mapsto \ell_\mu(|f|)$.*

Haar measures

A topological group is a topological space equipped with a group structure of which the composition and the inverse are both continuous. In particular, the underlying topology is preserved by the left and right translations as well as the inverse. We will denote $l_x(y) = xy$ the left translation by $x \in G$ and $r_x(y) = yx^{-1}$ the right translation.

We define left and right translations on functions, and on functions with compact support, by the formula $(l_x f)(y) = f(x^{-1}y)$ and $(r_x f)(y) = f(yx)$. We define left and right translations on measures by the formula $(l_x\mu)(f) = \mu(l_{x^{-1}}f)$ and $(r_x\mu)(f) = \mu(r_{x^{-1}}f)$. A Radon measure μ on G is said to be left invariant if $l_x\mu = \mu$ for all $x \in G$. We have similar notion of right invariant Radon measures.

Theorem 2.5 (Haar-von Neumann)**.** *There exists a left-invariant Radon measure on every locally compact topological group. This measure is unique up to multiplication by a positive constant.*

Proof. We first prove the *existence* of left-invariant Radon measure. Let $f, g \in C_c^+(G)$ be non-zero non-negative functions. The ratio $\mu(f) : \mu(g)$, μ being the invariant measure we seek to define, must be bounded from above by the sums $\sum_{i=1}^n c_i$ where c_i are positive real numbers such that there exist elements $x_1, \ldots, x_n$ of G satisfying

$$f < \sum_{i=1}^{n} c_i l_{x_i}(g).$$

We define

$$(f : g) = \inf\left\{\sum_{i=1}^{n} c_i \mid f < \sum_{i=1}^{n} c_i l_{x_i}(g)\right\}$$

to be the infimum of those sums.

We have just defined a kind of "ratio" $(f : g)$ for all $f, g \in C_c^+(G)$. The purpose of the quote marks is to remind us that this may not be a genuine ratio $\mu(f) : \mu(g)$ in the sense that there is a triangular inequality

$$(\varphi_1 : \varphi_3) \leq (\varphi_1 : \varphi_2)(\varphi_2 : \varphi_3) \tag{2.2}$$

for all non-zero $\varphi_i \in C_c^+(G)$ but no equality in general.

Let us choose a non-zero positive continuous function with compact support $f_0 \in C_c^+(G)$. We will prove that there exists a left-invariant Radon measure I such that $\mu(f_0) = 1$. We set

$$\nu_\varphi(f) = \frac{(f : \varphi)}{(f_0 : \varphi)} \tag{2.3}$$

We will define $\mu(f)$ as the limit of $\nu_\varphi(f)$ as the support of the non-negative function φ shrinks to an arbitrarily small neighborhood of identity. By its very construction ν_φ

is invariant under left translation i.e. $\nu_\varphi(l_x f) = \nu_\varphi(f)$ for all $x \in G$ and $f \in C_c^+(G)$. It can also be easily checked that ν_φ satisfies the sub-additivity inequality

$$\nu_\varphi(f_1 + f_2) \leq \nu_\varphi(f_1) + \nu_\varphi(f_2) \tag{2.4}$$

but no equality in general. We will prove that by constructing an appropriate limit of ν_φ as the support of φ shrinks to an arbitrarily small neighborhood of identity, the inequality (2.4) will becomes an equality.

As the triangular inequality (2.2) implies lower and upper bounds for ν_φ

$$\frac{1}{(f_0 : f)} \leq \nu_\varphi(f) \leq (f : f_0),$$

the sought after number $\mu(f)$ satisfies the same inequality. Let B_f denote the closed interval $[1/(f_0 : f), (f : f_0)]$. According to the Tychonov theorem, the infinite product

$$B = \prod_{f \in C_c^+(G)} B_f$$

is compact. For every $\varphi \in C_c^+(G)$, the function $f \mapsto \nu_\varphi(f)$ determines an element of $\nu_\varphi \in B$. For each neighborhood V of identity in G, let $\mathscr{I}_V$ denote the closure of the set $\{\nu_\varphi \mid \varphi \in C_c^+(V)\}$ in B. For any finite collection of neighborhoods of identity $V_1, \ldots, V_n$, the intersection $\mathscr{I}_{V_1} \cap \cdots \cap \mathscr{I}_{V_n}$ is non-empty because we can find a function φ with support contained in $V_1 \cap \cdots \cap V_n$. Since B is compact, the intersection of $\mathscr{I}_V$ for all neighborhood V of identity is non-empty. Let μ be an element of this intersection. We will prove that μ can be extended a left-invariant Radon measure of G.

Since ν_φ is invariant under left translation i.e. $\nu_\varphi(l_x f) = \nu_\varphi(f)$ for all $x \in G$ and $f \in C_c^+(G)$, μ satisfies the same invariant property. We only need to prove that μ is additive for non-negative functions i.e. $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$ for all $f_1, f_2 \in C_c^+(G)$. The extension of $\mu(f)$ to all function $f \in C_c(G)$ is then guaranteed because every continuous function with compact support can be written as the difference of two non-negative continuous functions with compact support. The existence of Haar measure would now derive from the following lemma.

Lemma 2.6. *If $\mu \in \bigcap_V \mathscr{I}_V$ with V running over all neighborhoods of identity in G, then $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$ for all $f_1, f_2 \in C_c^+(G)$.*

Proof. For every non-zero non-negative function φ, ν_φ satisfies the sub-additivity inequality

$$\nu_\varphi(f_1 + f_2) \leq \nu_\varphi(f_1) + \nu_\varphi(f_2)$$

according to its very definition. Since μ lies in the closure of the ν_φ, it satisfies the same inequality.

We will prove now the opposite inequality

$$\mu(f_1) + \mu(f_2) \le \mu(f_1 + f_2)$$

for every $f_1, f_2 \in C_c^+(G)$. Let us choose an auxiliary $f' \in C_c^+(G)$ that takes value 1 on the union of supports of f_1 and f_2. We will prove that the inequality

$$\mu(f_1) + \mu(f_2) \le (1 + 2\epsilon)(\mu(f_1 + f_2) + \delta\mu(f')) \tag{2.5}$$

holds for all $\delta, \epsilon > 0$.

Let $f = f_1 + f_2 + \delta f'$ for some positive real number δ. Set $h_i(x) = \frac{f_i(x)}{f(x)}$ for x in the support of f_i and 0 otherwise. The non-vanishing of f' on the support of f_i implies that $h_i \in C_c^+(G)$ for $i \in \{1, 2\}$. We have $f_i = f h_i$ and $h_1 + h_2 < 1$.

The main ingredient that come into the proof of (2.5) is the *uniform continuity* of continuous compactly supported functions. The notion of uniform continuity depends upon group action, the right translation of G on itself in the present case. The functions h_i are right equicontinuous in the sense that for every $\epsilon > 0$ there exists a neighborhood V of identity such that for every $y \in V$ and $x \in G$, $|h_i(xy) - h_i(x)| < \epsilon$ for $i \in \{1, 2\}$.

Let $\varphi \in C_c(V)$ and choose $c^-, \dots, c_n$ and $x_1, \dots, x_n$ such that

$$f < \sum_{j=1}^n c_j l_{x_j} \varphi.$$

The inequality $|h_1(xy) - h_1(x)| < \epsilon$ satisfied by $y \in V$ and $x \in G$ implies

$$f_i(y) = f(y) h_i(y) < \sum_{j=1}^n c_j (h_i(x_j) + \epsilon) l_{x_j} \varphi$$

hence

$$(f_i : \varphi) < \sum_{j=1}^n c_j (h_i(x_j) + \epsilon).$$

It follows that

$$(f_i : \varphi) + (f_2 : \varphi) < \sum_{j=1}^n c_j (1 + 2\epsilon).$$

Since $(f : \varphi)$ is defined as the infimum of numbers $\sum_{i=1}^{n} c_i$ obtained as above, we derives the inequality

$$(f_1 : \varphi) + (f_2 : \varphi) < (1 + 2\epsilon)(f : \varphi).$$

The inequality

$$\nu_\varphi(f_1) + \nu_\varphi(f_2) < (1 + 2\epsilon)(\nu_\varphi(f_1 + f_2) + \delta\, \nu_\varphi(f'))$$

is thus satisfied for all non-zero non-negative function $\varphi \in C_c^+(V)$. Since I belong to the closure of $\mathscr{J}_V$, this implies the inequality (2.5). $\square$

We now turn to the proof of the *uniqueness* of invariant measure. Let μ and ν be two left-invariant Haar measures. For every positive, continuous compactly supported function $f, g \in C_c^+(G)$ we will prove that the difference between the ratios $\nu(f)/\mu(g)$ and $\nu(f)/\mu(f)$ is arbitrarily small using essesntially the equicontiuity of f and g their property of being left-invariant.

Consider a non-negative function $f \in C_c^+(G)$ supported on a compact set C and an auxiliary function $f' \in C_c^+(G)$ that takes value 1 on an open subset U containing C. For every $\epsilon > 0$, there is a symmetric neighborhood V of the identity such that $|f(x) - f(xy)| < \epsilon$ for all $x \in C$ and $y \in V$. We also require that $CV \subset U$. It follows from these assumptions that

$$|f(x) - f(xy)| \leq \epsilon f'(x) \tag{2.6}$$

for all $y \in V$. If x or xy lies in C, the inequality $|f(x) - f(xy)| < \epsilon$ is satisfied as we can replace y by y^{-1}. In this case $x \in C \cup CV \subset U$ so that $f'(x) = 1$. If neither x nor xy lies in C, the left hand side of (2.6) the above inequality vanishes while the right hand side is greater or equal to zero.

Let $h \in C_c^+(V)$ be any non-zero non-negative function supported in V. We also assume h symmetric $h(x) = h(x^{-1})$. We will prove that

$$\left| \frac{\nu(f)}{\mu(f)} - \frac{\nu(h)}{\mu(h)} \right| < \epsilon \frac{\nu(f')}{\mu(f)}. \tag{2.7}$$

We will introduce a temporary notation $\mu(h) = \mu_x h(x)$ where x is a silent variable. Thus on one hand, we write

$$\mu(h)\nu(f) = \mu_y\, \nu_x h(y) f(x)$$

and on the other hand, using Fubini theorem and the left invariance of I and J, we can rewrite $\nu(h)\mu(f)$ as

$$\nu(h)\mu(f) = \mu_y \nu_x h(x) f(y) = \mu_y \nu_x h(y^{-1}x) f(y).$$

Since $h(y^{-1}x) = h(x^{-1}y)$, this can also expressed as

$$\nu(h)\mu(f) = \nu_x \mu_y h(x^{-1}y) f(y) = \mu_y \nu_x h(y) f(xy).$$

The difference of $|\mu(h)\nu(f) - \nu(h)\mu(f)|$ can now be bounded

$$|\mu(h)\nu(f) - \nu(h)\mu(f)| = \mu_y \nu_x h(y)|f(x) - f(xy)| \leq \epsilon \mu_y \nu_x h(y) f'(x)$$

after (2.6). Therefore

$$|\mu(h)\nu(f) - \nu(h)\mu(f)| \leq \epsilon \mu(h)\nu(f')$$

The inequality (2.7) follows by getting multiplied on both sides with $\mu(f)\mu(h)$.

Let $g \in C_c^+(G)$ and if g' is an auxiliary function for g as f' for f. Then we have also an inequality

$$\left| \frac{\nu(g)}{\mu(g)} - \frac{\nu(h)}{\mu(h)} \right| < \epsilon \frac{\nu(g')}{\mu(g)}$$

for every symmetric non-negative function h supported in a small enough neighborhood V of the identity. Combining with (2.7), we get

$$\left| \frac{\nu(f)}{\mu(f)} - \frac{\nu(g)}{\mu(g)} \right| < \epsilon \left(\frac{\nu(f')}{\mu(f)} + \frac{\nu(g')}{\mu(g)} \right)$$

for every $\epsilon > 0$. We remember that only the support of h depends on the choice of ϵ, in letting $\epsilon \to 0$ in the above inequality we infer

$$\frac{\nu(f)}{\mu(f)} = \frac{\nu(g)}{\mu(g)}$$

for all $f, g \in C_c^+(G)$. It follows that $J = cI$ for some positive constant c. □

Fubini theorem

Theorem 2.7. *Let $(X,\mu), (Y,\nu)$ be measured spaces. The product $X \times Y$ is equipped with the measure $\mu \times \nu$. For every integrable function $f \in L^1(X \times Y, \mu \times \nu)$, the function $f_x : y \mapsto f(x, y)$ is integrable for almost all y, and the function $x \mapsto \int_Y f_x(y)\nu(y)$ is in $L^1(X,\mu)$. Moreover, we have the formula*

$$\int_{X \times Y} f(x, y)(\mu \times \nu)(x, y) = \int_X \left(\int_Y f_x(y)\nu(y) \right) \mu(x).$$

Convolution product

Proposition 2.8. *Let G be a locally compact topological group and μ a Haar measure on G. Let $f, g \in L^1(G, \mu)$ be integrable functions on G. Then for almost all x, the function $y \mapsto f(y)g(y^{-1}x)$ is integrable and the integral*

$$(f \star g)(x) = \int_G f(y)g(y^{-1}x)\mu(y)$$

defines an integrable function on (G, μ).

Fourier transform

Let G be a locally compact abelian group. We will call *character* of G any continuous homomorphism $\chi : G \to \mathbb{C}^\times$, and *unitary characters* those characters whose image is contained in the unit circle $\mathbb{C}^1$. We will denote $\Lambda(G)$ the group of all unitary characters of G. The group $\Lambda(G)$ equipped with the compact open topology to be discussed later is called the Pontryagin dual of G.

Let us fix an invariant measure μ on G. For every integrable function $\varphi \in L^1(G)$ with respect to Haar measure of G, we define its *Fourier transform* to be the function of variable $\chi \in \Lambda(G)$ defined by the integral

$$\hat{\varphi}(\chi) = \int_G \varphi(x)\chi^{-1}(x)\mu(x). \tag{2.8}$$

We note that as $|\chi^{-1}(x)| = 1$ for all $x \in G$, the function $x \in \varphi(x)\chi^{-1}(x)$ also belongs to $L^1(G)$ and the above integral converges absolutely. We also note that the inequality

$$|\phi(\chi)| \leq ||\phi||_{L^1} \tag{2.9}$$

holds for all $\chi \in \Lambda(G)$.

Proposition 2.9. *For every $x \in G$, and $f \in \mathrm{L}^1(G, \mu)$, we have*

$$\widehat{l_x f}(\chi) = \chi(x)^{-1}\hat{f}(\chi).$$

For all $f, g \in L^1(G, \mu(x))$, the Fourier transform of the convolution product of f and g is equal to the point wise multiplication of the Fourier transforms of f and g. In other words, the formula

$$\widehat{f \star g}(\chi) = \hat{f}(\chi)\hat{g}(\chi)$$

holds for all $\chi \in \Lambda(G)$.

Compact open topology and Ascoli's theorem

Before discussing the topology of the Pontryagin dual $\Lambda(G)$, we will recall some backgrounds on compact open topology on the space of maps.

Let X, Y be topological spaces, Y^X the space of all functions from X to Y. If E and H are subsets of X and Y respectively, the class of functions $f \in Y^X$ such that $f(E) \subset H$ will be denoted $(E : H)$. The *compact open topology* of Y^X is the topology generated by subclasses of the form $(K : G)$ where K is compact and G is open. A net of functions $f_\alpha : X \to Y$ converges to a function $f : X \to Y$ in the compact open topology if whenever $f(K) \subset G$ for a compact subset K of X and an open subset G of Y, then there exists α such that for all $\beta \geq \alpha$, the relation $f_\beta(K) \subset G$ also holds.

As a comparison, the topology generated by classes of the type $(x : G)$, where G is open will be referred to as the *punctual topology*. A net of functions $f_\alpha : X \to Y$ converges to a function $f : X \to Y$ in the punctual topology if whenever $f(x) \in G$ for an open subset G of Y, then there exists α such that for all $\beta \geq \alpha$, the relation $f_\beta(x) \in G$ also holds. The punctual topology on Y^X coincide with the product topology as in the Tychonov theorem.

For any subset $\mathscr{E}$ of Y^X, we call compact open topology of C the restriction of compact open topology of Y^X. For instance, for every topological space X, the space $C(X)$ of continuous functions is equipped with compact open topology. In this topology, a net of functions $f_\alpha \in C(G)$ converges to a function f if and only if f_α converge to f uniformly on compact subsets.

Following Kelley, we will say that subset $\mathscr{E}$ of Y^X is *evenly continuous* at a point $x \in X$ if for every $y \in Y$ and neighborhood V of y, there exist neighborhood U and W of x and y respectively such that for every $f \in \mathscr{E}$, $f(x) \in W$ implies $f(U) \subset V$. The following statement is a generalization of Ascoli's theorem due to J.D. Weston.

Theorem 2.10. *Let X, Y be topological spaces. Let Y^X be the class of arbitrary functions from X to Y equipped with the compact open topology. Then a closed set $\mathscr{F} \subset Y^X$ is compact if and only if, for every $x \in X$, the closure of $\mathscr{F}(x)$ is compact and $\mathscr{F}$ is evenly continuous.*

The assumption of even continuity will be used in the following:

Lemma 2.11. *Let $\mathscr{E} \subset Y^X$ be evenly continuous at every point $x \in X$. Then $\mathscr{E}$ has the same closure with respect to the punctual topology or the compact open topology. Moreover, these topology have the same restriction to $\mathscr{E}$.*

Proof. Since subsets of evenly continuous set of functions are also evenly continuous, the second assertion derives from the first. Let f be an element in the punctual closure of $\mathscr{E}$. We will prove that f belongs to the closure of $\mathscr{E}$ with respect to the compact open topology. Let $K_1, \dots, K_n$ be compact subsets of X, and let $G_1, \dots, G_n$ be open subsets in Y such that $f(K_i) \subset G_i$. What we need to prove now is that there exists $g \in \mathscr{E}$ satisfying the same relations. For $x \in K_i$ with $i \in \{1, \dots, n\}$ there exist a neighborhood U_x and W_x of x and $f(x)$ respectively such that for $g \in \mathscr{E}$ if $g(x) \in W_x$ then $g(U_x) \subset G_i$. Since K_i is compacts, there exist finitely many points $x_{i,1}, \dots, x_{i,m_i}$ such that $K_i \subset \bigcup_{j=1}^{m_i} U_{x_{i,j}}$. Since f lies in the punctual closure of $\mathscr{E}$, there exists $g \in \mathscr{E}$ such that $g(x_{i,j}) \in W_{x_{i,j}}$ for all i, j. It follows that $g(U_{x_{i,j}}) \subset G_i$ and hence $g(K_i) \subset G_i$. □

Proof of Ascoli's theorem. Let $\mathscr{F}$ be a closed subset of Y^X with respect to the compact open topology, and suppose that, for each point $x \in X$, the closure of $\mathscr{F}(x)$ is compact and $\mathscr{F}$ is evenly continuous. Let Y_x denote the closure of $\mathscr{F}(x)$. The punctual topology of Y^X restricted to $\prod_{x\in X} Y_x$ is the product topology with respect to which $\prod_{x\in X} Y_x$ is compact according to the Tychonov theorem. Now $\mathscr{F}$ is a closed subset with respect to the compact convergence topology, it is also punctually closed since it is even continuous. As a punctually closed subset of $\prod_{x\in X} Y_x$, it has to be compact. Applying the lemma again, we see that $\mathscr{F}$ is compact with respect to the compact open topology. □

Pontryagin dual

Let G be a locally compact topological group. We equip the group $\Lambda(G)$ of unitary characters of G with the compact open topology. We recall that a net of functions $f_\alpha \in C(G)$ converges to a function f in the compact open topology if and only if f_α converge to f uniformly on compact subsets. In other words, the compact open topology on the space $C(G)$ of continuous functions on a topological space X is generated by subsets

$$(K : U) = \{f \in C(G), f(K) \subset U\}$$

for K compact subset of G and U open subset of $\mathbb{C}^1$.

Lemma 2.12. *For every $\varphi \in L^1(G)$, the Fourier transform $\chi \mapsto \hat{\varphi}(\chi)$ is a continuous function on $\Lambda(G)$.*

Proof. For every $\varphi \in L^1(G)$, for every $\epsilon > 0$, there exists a compact subset K of G such that $\int_{G-K} |\varphi(x)|\mu(x) < \epsilon$. Let $(K : \epsilon)$ be the open neighborhood of the trivial character χ_0 on $\Lambda(G)$ consisting of all unitary characters $\chi : G \to \mathbb{C}^1$ such that $|\chi^{-1}(x) - 1| < \epsilon$ for all $x \in K$, then we have

$$|\hat{\varphi}(\chi) - \hat{\varphi}(\chi_0)| \leq (\mathrm{vol}(K, \mu) + 1)\epsilon.$$

This proves that $\hat{\varphi}$ is continuous at χ_0. For every $\chi \in \Lambda(G)$, by replacing φ with the function $x \mapsto \varphi(x)\chi^{-1}(x)$, the above argument then implies that $\mathrm{F}\varphi$ is continuous at χ. $\square$

Theorem 2.13 (Riemann-Lebesgue)**.** *Moreover for every $\varphi \in L^1(G)$, the Fourier transform $\hat{\varphi} \in C(\Lambda(G))$ can be extended to a continuous function on $\Lambda(G) \cup \{\infty\}$, the one-point compactification of $\Lambda(G)$, by setting $\hat{\varphi}(\infty) = 0$.*

We recall that open neighborhoods of ∞ in the one-point compactification of $\Lambda(G) \cup \{\infty\}$ are subsets of the form $U \cup \{\infty\}$ where U is the complement of compact closed subset of $\Lambda(G)$. We derive Riemann-Lebesgue's theorem from the following lemma.

Lemma 2.14. *Let $\varphi \in L^1(G)$ be an integrable function on G and let $\hat{\varphi} \in C(\Lambda(G))$ denote its Fourier transform. For every $a > 0$, the set K_a of $\chi \in \Lambda(G)$ such that $|\hat{\varphi}(\chi)| \geq a$ is a compact subset of $\Lambda(G)$.*

Proof. The Pontryagin dual $\Lambda(G)$, and its subset K_a, will be considered as a closed subsets of the space $\mathrm{Map}(G, \mathbb{C})$ of all complex valued functions on G equipped with the compact open topology. According to Weston's generalization of the Ascoli theorem, to be recalled in the appendix, in order to prove that K_a is compact, it is sufficient to prove that K_a, as a subset of $\mathscr{C}(G)$ is evenly continuous. It is enough to prove that K_a is evenly continuous at the identity element of G i.e. for every $\epsilon > 0$, there exists a neighborhood U of identity in G such that for all $\chi \in K_a$, for all $x \in U$, we have $|\chi(x) - 1| < \epsilon$.

First we claim that for all $\phi \in L^1(G)$, $x \in G$ and $\chi \in \Lambda(G)$, the inequality

$$|(\chi(x) - 1)\hat{\varphi}(\chi)| \leq \|l_{x^{-1}}(\varphi) - \varphi\|_{L^1} \tag{2.10}$$

holds. This inequality derives from the equality

$$(\chi(x) - 1)\hat{\varphi}(\chi) = (\widehat{l_{x^{-1}}(\varphi) - \varphi})(\chi)$$

that follows from the fact that the Fourier transform of $l_{x^{-1}}\varphi$ is the function $\chi \mapsto \chi(x)\hat{\varphi}(\chi)$. Thus for $\chi \in K_a$, the estimate

$$|\chi(x) - 1| \leq a^{-1} \|l_{x^{-1}}(\varphi) - \varphi\|_{L^1}$$

holds for all $x \in G$ by (2.9). Thus the Riemann-Hilbert lemma derives from the continuity of the translation representation of G on $L^1(G)$, as stated in the following lemma. $\square$

Lemma 2.15. *For every $\varphi \in L^1(G)$ and for all $\epsilon > 0$ there exists a neighborhood U of identity in G such that for every $x \in U$, we have $\|l_x(\varphi) - \varphi\|_{L^1} < \epsilon$.*

Proof. Assume that φ is a continuous function with compact support K. For every y in the support of φ, let U_y be a symmetric neighborhood of identity such that $|\varphi(x^{-1}y) - \varphi(y)| < \epsilon'$ for all $x \in U_y$. Since K is compact, it can be covered by finitely many open subsets of the form $U_y y$. Let U be the intersection of those finitely many U_y then for every $x \in U$, and for every $y \in G$, we have $|\varphi(x^{-1}y) - \varphi(y)| < \epsilon'$. Let adjust ϵ' to take the volume of K into account, this will imply $\|l_x(\varphi) - \varphi\|_{L^1} < \epsilon$. For every $\varphi \in L^1(G)$, there exists $\varphi_0 \in C_c(G)$ such that $\|\varphi - \varphi_0\|_{L^1} < \epsilon$ so that if the statement is true for all compactly supported continuous functions, it is also true for all integrable functions. □

Theorem 2.16. *For every locally compact abelian group G, $\Lambda(G)$ is also a locally compact abelian group.*

Proof. We now derive the local compactness of $\Lambda(G)$ from Lemma 2.14. We claim that the identity element in $\Lambda(G)$ has a compact neighborhood. Let K be a compact neighborhood of identity in G and let denote 1_K its characteristic function. Since K contains a non-empty open subset, its volume is a positive number. For a small positive number $\epsilon > 0$, we consider the open subset $(K : \epsilon)$ of $\Lambda(G)$ consisting of all unitary characters $\chi : G \to \mathbb{C}^1$ such that for all $x \in K$, $|\chi(x) - 1| < \epsilon$. It is enough to prove that $(K : \epsilon)$ is contained in a compact subset of $\Lambda(G)$. According to the Riemann-Lebesgue's lemma, it is enough to prove that $\widehat{1_K}(\chi^{-1})$ is bounded from below for $\chi \in (K : \epsilon)$. Indeed, we have the inequality

$$|\widehat{1_K}(\chi^{-1})| \geq (1-\epsilon) \int_G 1_K(x)\mu(x)$$

by developing $\widehat{1_K}(\chi^{-1})$ in the form

$$\widehat{1_K}(\chi^{-1}) = \int_G 1_K(x)\mu(x) - \int_G 1_K(x)(1-\chi(x))\mu(x)$$

and using the assumption $|1 - \chi(x)| < \epsilon$. □

Proposition 2.17. *If G is a compact group then $\Lambda(G)$ is a discrete group. If G is a discrete group then $\Lambda(G)$ is a compact group.*

Proof. Assume that G is a compact group. Let χ_0 denote the trivial character $G \to \mathbb{C}^1$. In order to prove that $\Lambda(G)$ is a discrete group, it is sufficient to prove that its unit element $1_{\Lambda(G)}$ is an isolated point. First we claim that there is a neighborhood D of $1 \in \mathbb{C}^1$ that contains no non trivial subgroup of $\mathbb{C}^1$. We can take D to be the set of norm one complex number z of argument $\arg(z) \in (-\pi/2, \pi/2)$, we take convention that argument of a complex number is in the interval $(-\pi, \pi]$. Let z be any element of D and n the greatest integer such that $|2^n \arg(z)| < \pi/2$. Then $2^{n+1}\arg(z) \notin (-\pi/2, \pi/2)$, thus D does not contain the subgroup generated by z.

The neighborhood $(G : D)$ of χ_0 in $\Lambda(G)$ consists of all unitary characters $\chi : G \to \mathbb{C}^1$ with image contained in D. Since D contains no nontrivial subgroups, this neighborhood is reduced to a singleton, therefore χ_0 is an isolated point in $\Lambda(G)$. Since $\Lambda(G)$ is a topological group with unit element isolated, it is a discrete group.

The set of unitary characters $\chi : G \to \mathbb{C}^1$ can be identified with a subset of the space $\mathrm{Map}(G, \mathbb{C}^1)$ of all maps $G \to \mathbb{C}^1$, equipped with the product topology. Since G is discrete, this subset is determined by the system of equations $\chi(xy) = \chi(x)\chi(y)$ on the unknown $\chi \in \mathrm{Map}(G, \mathbb{C}^1)$ and thus $\Lambda(G)$ is a closed subset of $\mathrm{Map}(G, \mathbb{C}^1)$. The discreteness of G also implies that the compact open topology on $\Lambda(G)$ coincide with the induced topology as subset of $\mathrm{Map}(G, \mathbb{C}^1)$. According to the Tychonov theorem, $\mathrm{Map}(G, \mathbb{C}^1)$ is a compact set and it follows the $\Lambda(G)$ is a compact group. □

Fourier adjunction formula.

Let G be a locally compact abelian group, $\Lambda(G)$ its Pontryagin dual, and $\Lambda(\Lambda(G))$ its bidual. We have a homomorphism $\mathrm{e} : G \to \Lambda(\Lambda(G))$ mapping $x \in G$ on the character $\chi \mapsto \chi(x)$ of $\Lambda(G)$.

Proposition 2.18. *Let μ and ν be Haar measures on G and $\Lambda(G)$. If $f \in L^1(G, \mu)$ and $g \in L^1(G, \nu)$ we have*

$$\int_G f(x)\hat{g}(\mathrm{e}(x))\mu(x) = \int_{\hat{G}} \hat{f}(\chi)g(\chi)\nu(\chi).$$

Proof. The function $F : G \times \Lambda(G) \to \mathbb{C}$ given by

$$F(x, \chi) = f(x)g(\chi)\chi(x)^{-1}$$

is integrable on $G \times \Lambda(G)$ with respect to the product measure $\mu \times \nu$. We obtain the Fourier adjunction formula is an application of the Fubini theorem to e by integrating it first on G then on $\Lambda(G)$ and vice versa. □

Dirac sequences

Let G be a locally compact abelian group and μ a Haar measure. A Dirac sequence is a net $(\varphi_j | j \in J)$ of non-negative functions $\varphi_j \in C_c(G)$ such that

- for all $j \in J$, $\int_A \varphi_j(x)\mu(x) = 1$;
- the support of ϕ_j shrink to the identity as $j \to \infty$ i.e. for every neighborhood U of the identity of G, there exists $j_0 \in J$ such that the support of ϕ_j is contained in U for all $j \geq j_0$;
- $\phi_j(x) = \phi_j(x^{-1})$.

Dirac sequences exist by the Urysohn lemma.

Lemma 2.19. *Let G be a locally compact abelian group with Haar measure $\mu(x)$. Let $(\varphi_j | j \in J)$ be a Dirac sequence. For every $f \in L^1(G)$, $\varphi_j \star f$ and $f \star \varphi_j$ converge to f in $L^1(G)$. For every continuous function $f \in C(G)$, $\varphi_j \star f$ and $f \star \varphi_j$ converge to f punctually.*

Lemma 2.20. *Let G be a locally compact abelian group with Haar measure $\mu(x)$. Let $(\varphi_j | j \in J)$ be a Dirac sequence. Then the Fourier transforms $(\hat{\varphi}_j | j \in J)$ of $(\varphi_j | j \in J)$ converge to the function 1 on $\Lambda(G)$ uniformly on compacta.*

Fourier inversion theorem

Theorem 2.21. *Let G be a locally compact abelian group and $\Lambda(G)$ its Pontryagin dual. For every Haar measure μ on G there exists a unique Haar measure on $\Lambda(G)$, the dual Haar measure, satisfying the property that for every $f \in L^1(G, \mu)$ such that $\hat{f} \in L^1(\Lambda(G), \nu)$, we have*

$$f(0) = \int_{\Lambda(G)} \hat{f}(\chi)\nu(\chi).$$

Pontryagin biduality theorem

The cornerstone of the theory of locally compact abelian groups is the Pontryagin biduality theorem. It asserts that for every locally compact abelian group G, the homomorphism $G \to \Lambda(\Lambda(G))$ mapping $x \in G$ on the character $\chi \mapsto \chi(x)$ of $\Lambda(G)$ is an isomorphism.

The theorem is already non-trivial for $G = \mathbb{Z}$. Its Pontryagin dual of $\mathbb{Z}$ obviously is $\mathrm{Hom}(\mathbb{Z}, \mathbb{C}^1) = \mathbb{C}^1$. The Pontryagin-Van Kampen theorem asserts in this case that all unitary character $\nu : \mathbb{C}^1 \to \mathbb{C}^1$ are of the form $\nu_n(x) = x^n$ for some $n \in \mathbb{Z}$. This is a consequence of the Plancherel theorem for Fourier series. Assume that ν is not one of the $\nu_n(x) = x^n$ with $n \in \mathbb{Z}$ then the scalar product $\langle \nu, \nu_n \rangle = 0$ in $L^2(\mathbb{C}^1)$. This contradicts with the fact that the map $L^2(\mathbb{C}^1) \to L^2(\mathbb{Z})$ given by $f \mapsto \langle f, \nu_n \rangle$ is an isometry as asserted by the Plancherel theorem for Fourier series.

Theorem 2.22 (Pontryagin). *The homomorphism* $\mathrm{e} : G \to \Lambda(\Lambda(G))$*, which maps* $x \in G$ *on the unitary character* $\chi \mapsto \chi(x)$ *of* $\Lambda(G)$*, is a homeomorphism.*

Subgroups and factor groups

Proposition 2.23. *Let G be a locally compact abelian group and H a closed subgroup of G. Let $H^\perp$ denote the orthogonal of H in $\Lambda(G)$ which is the closed subgroup of $\Lambda(G)$ consisting of unitary characters $\chi : G \to \mathbb{C}^1$ whose restriction to H is trivial. Then the orthogonal of $H^\perp$ in $G = \Lambda(\Lambda(G))$ is H. Moreover, there are canonical isomorphisms between $\Lambda(G)/H^\perp$ and $\Lambda(H)$ as well as $\Lambda(G/H)$ and $H^\perp$.*

Proposition 2.24. *Let G be a locally compact abelian group, and H a closed subgroup of G. For $f \in L^1(G)$, we define $f^H \in L^1(G/H)$ by $f^H(x) = \int_H f(xh)\mathrm{d}h$. Then the Fourier transform of f^H is the restriction of $\hat{f}$ to $H^\perp = \Lambda(G/H)$. If moreover $\hat{f}|_{H^\perp} \in L^1(H^\perp)$ then the equality*

$$\int_H f(xh)\mathrm{d}h = \int_{H^\perp} \hat{f}(\chi)\chi(x)\mathrm{d}\chi \tag{2.11}$$

holds for almost all x. If moreover f^H is everywhere defined and continuous, then the above equality holds for all $x \in G$.

3 Fourier transform on additive groups

Additive characters

A character of a topological group G is a continuous homomorphism $\mathrm{e} : G \to \mathbb{C}^\times$. We say that e is a unitary character if its image is contained in the subgroup $\mathbb{C}^1$ of complex numbers of norm one. We will say e is a positive character of its image lies in the subgroup $\mathbb{R}_+^\times$ of positive real numbers. Every character $\mathrm{e} : G \to \mathbb{C}^\times$ can be written uniquely as $\mathrm{e} = \mathrm{e}_u \mathrm{e}_+$ where e_u is a unitary character and e_+ is a positive character just as a complex number can be written uniquely as product of a complex number of module one and a positive real number.

We will set out to study and determine all characters of additive groups of local fields. The discussion will be similar for unitary characters but different for non-unitary characters in archimedean and non-archimedean cases. The reason is that archimedean local fields have no non-trivial compact subgroups as opposed to non-archimedean local fields which are union of its compact subgroups. We will also discuss characters of groups of adèles of global fields. The determination of characters and unitary characters of additive groups relies very much on the following elementary fact on complex numbers.

Lemma 3.1. *There exists an open neighborhood U of 1 in $\mathbb{C}^\times$ which does not contain any non trivial subgroup of $\mathbb{C}^\times$.*

Proof. □

First, we will construct a preferred choice of unitary characters for local fields and groups of adèles. This construction only depends on the choice of a complex number i satisfying $i^2 = -1$. We fix a choice of i. This choice gives rise to an isomorphism $\mathrm{e}_1 : \mathbb{R}/\mathbb{Z} \to \mathbb{C}^1$ given by $x \mapsto \exp(2i\pi x)$. Using the isomorphism (1.11), it also induces an isomorphism $\mathrm{e}_1 : \mathbb{A}/(\mathbb{Q}+\hat{\mathbb{Z}}) \to \mathbb{C}^1$ and therefore adelic additive character $\mathrm{e}_1^{\mathbb{Q}} : \mathbb{A} \to \mathbb{C}^1$ as well as local additive character $\mathrm{e}_1^F : F \to \mathbb{C}^1$ for all completions F of $\mathbb{Q}$. For $F = \mathbb{R}$, we recover again the formula $\mathrm{e}_1^{\mathbb{R}}(x) = \exp(2i\pi x)$. For $F = \mathbb{Q}_p$, $\mathrm{e}_1^{\mathbb{Q}_p} : \mathbb{Q}_p \to \mathbb{C}^1$ is the unique character satisfying the formula $\mathrm{e}_1^{\mathbb{Q}_p}(a+n/p^r) = \exp(-2i\pi n/p^r)$ for all p-adic integers $a \in \mathbb{Z}_p$, integers $n \in \mathbb{Z}$ and non-negative integers r.

For every number field k, we have a trace map $\mathrm{tr}_{k/\mathbb{Q}} : \mathbb{A}_k \to \mathbb{A}$ which gives rise to a character $\mathrm{e}_1^k : \mathbb{A}_k \to \mathbb{C}^1$ defined by $\mathrm{e}_1^k(x) = \mathrm{e}_1^{\mathbb{Q}} \circ \mathrm{tr}_{k/\mathbb{Q}}(x)$. The character $\mathrm{e}_1^k : \mathbb{A}_k \to \mathbb{C}^1$ is trivial on the discrete subgroup k. For every place u of k, the restriction e_1^k to $F = k_u$ gives rise to a character $\mathrm{e}_1^F : F \to \mathbb{C}^1$. If u is over a place v of $\mathbb{Q}$ then we have the formula $\mathrm{e}_1^{k_u}(x) = \mathrm{e}_1^v(\mathrm{tr}_{k_u/\mathbb{Q}_v}(x))$ which shows that the character $\mathrm{e}_1^F : F \to \mathbb{C}^1$ depends on the local field F alone and not on its realization as a completion of a global field.

Proposition 3.2. *Every positive character $\mathrm{e}_+ : \mathbb{R} \to \mathbb{R}_+^\times$ can be written uniquely in the form $\mathrm{e}_+(x) = \exp(\lambda x)$ for some $\lambda \in \mathbb{R}$. Each unitary character $\mathrm{e}_u : \mathbb{R} \to \mathbb{C}^1$ can be written uniquely in the form $\mathrm{e}_u(x) = \mathrm{e}_1^{\mathbb{R}}(\xi x)$ for some $\xi \in \mathbb{R}$. In other words, both spaces of positive characters and unitary characters of $\mathbb{R}$ are one-dimensional $\mathbb{R}$-vector spaces.*

Proof. As the natural logarithm $\log : \mathbb{R}_+^\times \to \mathbb{R}$ is an isomorphism of topological groups, the first assertion is equivalent to the following lemma.

Lemma 3.3. *Every continuous homomorphism $l : \mathbb{R} \to \mathbb{R}$ can be written uniquely in the form $l(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.*

Proof. Let $l : \mathbb{R} \to \mathbb{R}$ be a homomorphism of topological groups. If we set $\lambda = l(1)$ then the identity $l(x) = \lambda x$ holds for all $x \in \mathbb{Z}$ using the compatibility with group law, and then for all $x \in \mathbb{Q}$ for the multiplication by a nonzero number is injective. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the identity $l(x) = \lambda x$ holds for all $x \in \mathbb{R}$. □

In order to determine unitary characters of $\mathbb{R}$, we will need another lemma.

Lemma 3.4. *Every homomorphism $\nu : \mathbb{C}^1 \to \mathbb{C}^1$ can be uniquely written in the form $\nu(x) = x^n$ for some integer $n \in \mathbb{Z}$.*

Proof. There are more than one way to prove this statement. On the one hand, we can use the Plancherel theorem for Fourier series. Assume that ν is not one of the $\nu_n(x) = x^n$ with $n \in \mathbb{Z}$ then the scalar product $\langle \nu, \nu_n \rangle = 0$ in $L^2(\mathbb{C}^1)$. This contradicts with the fact that the map $L^2(\mathbb{C}^1) \to L^2(\mathbb{Z})$ given by $f \mapsto \langle f, \nu_n \rangle$ is an isometry as asserted by the Plancherel theorem.

On the other hand, one can also argue on the ground of the knowledge of the universal covering of $\mathbb{C}^1$. Since the map $\mathbb{R} \to \mathbb{C}^1$ given by $x \mapsto \exp(2i\pi x)$ is the universal covering of $\mathbb{C}^1$, the continuous homomorphism $nu : \mathbb{C}^1 \to \mathbb{C}^1$ induces a continuous homomorphism $\tilde{\nu} : \mathbb{R} \to \mathbb{R}$ of universal coverings mapping $\mathbb{Z}$ into itself. By Lemma 3.3, $\tilde{\nu}$ is of the form $\tilde{\nu}(x) = nx$ for some $n \in \mathbb{R}$. The assumption $x \mapsto nx$ preserves $\mathbb{Z}$ implies that $n \in \mathbb{Z}$. It follows that $\nu = \nu_n$. □

We are now in position to complete the proof of Proposition 3.2. Let $\mathrm{e}_u : \mathbb{R} \to \mathbb{C}^1$ be a nontrivial homomorphism of topological groups. Its kernel A is then necessarily a discrete subgroup of $\mathbb{R}$. As A is a discrete subgroup of $\mathbb{R}$, it has a unique positive generator $a \in \mathbb{R}_+$. Now $x \mapsto \mathrm{e}_u(a^{-1}x)$ induces an injective continuous homomorphism $\nu : \mathbb{R}/\mathbb{Z} \to \mathbb{C}^1$. We will identify $\mathbb{R}/\mathbb{Z} \simeq \mathbb{C}^1$ using $x \mapsto \exp(2i\pi x)$. After Lemma 3.4, ν is of the form $\nu(x) = x^n$ for some $n \in \mathbb{Z}$, and it is injective if and only if $n \in \{\pm 1\}$. Thus there exists a unique $\xi \in \mathbb{R}^\times$, in fact $\xi \in \{\pm a\}$, such that $\mathrm{e}_u(x) = \exp(2i\pi\xi x)$. □

Proposition 3.5. *If F be a non-archimedean local field, then all characters $\mathrm{e} : F \to \mathbb{C}^\times$ are unitary. The group of characters of F is equipped with a structure of F-vector space by defining scalar multiplication of $\mathrm{e} : F \to \mathbb{C}^\times$ by $a \in F$ to be $x \mapsto \mathrm{e}(ax)$. As F-vector space, it has dimension one.*

Proof. The module of an additive character $\mathrm{e} : F \to \mathbb{C}^\times$ is given by a positive character $\mathrm{e}_+ : F \to \mathbb{R}_+^\times$. Since F is the union of its compact subgroups, and $\mathbb{R}_+^\times$, being isomorphic to $\mathbb{R}$, has no nontrivial compact subgroup, e_+ has to be trivial. In other words, e is unitary.

We also claim that for every character $\mathrm{e} : F \to \mathbb{C}^\times$, there exists an open compact $\mathcal{O}_F$-submodule C of F which is contained in the kernel of e. Indeed, we can construct an open neighborhood U of $1 \in \mathbb{C}^\times$ such that for all non-triial subgroup A of $\mathbb{C}^\times$, we have $A \not\subset U$. Since e is continuous, $\mathrm{e}^{-1}(U)$ is an open neighborhood of 0 in F. Since F is totally disconnected, $\mathrm{e}^{-1}(U)$ contains an open compact $\mathcal{O}_F$-submodule C of F. Since e is a character, $\mathrm{e}(C)$ is a subgroup of $\mathbb{C}^\times$ contained in U. It follows that $\mathrm{e}(C) = \{1\}$. For every non-trivial character $\mathrm{e} : F \to \mathbb{C}^\times$, we will denote $C(\mathrm{e})$ the largest compact open $\mathcal{O}_F$-submodule contained in the kernel of e, and we will call it the conductor of e. We will denote $C_1 = C(\mathrm{e}_1^F)$ the conductor of our preferred character e_1^F. If $x \in F^\times$, and $\mathrm{e}_x(y) = \mathrm{e}_1(xy)$ then $C(\mathrm{e}_x) = x^{-1}C_1$.

Let $\mathrm{e}_1 : F \to \mathbb{C}^1$ be a non-trivial additive character. We may take, for instance, $\mathrm{e}_1 = \mathrm{e}_1^F$ our preferred additive character. We will prove that every additive character $\mathrm{e} : F \to \mathbb{C}^1$ can be written uniquely in the form $\mathrm{e} = \mathrm{e}_x$ for some $x \in F$.

It is enough to prove that every character $\mathrm{e} : F \to \mathbb{C}^\times$, trivial on C_1 can be written uniquely of the form $\mathrm{e}(x) = \mathrm{e}_1^F(\xi x)$ for some $\xi \in \mathcal{O}_F$. If we denote ϖ_F an uniformizing parameter of F, then it is enough to prove that every character $\varpi_F^{-n}C_1/C_1 \to \mathbb{C}^1$ can be written uniquely of the form $x \mapsto \mathrm{e}_1^F(\xi x)$ with $\xi \in \mathcal{O}_F/\varpi^n\mathcal{O}_F$. This is equivalent to saying that the induced homomorphism from $\mathcal{O}_F/\varpi^n\mathcal{O}_F$ to the dual group of $\varpi_F^{-n}C_1/C_1$ is an isomorphism. Since these groups have the same cardinal, it is enough to prove that it is injective. This follows from the definition of conductor. $\square$

Let k be a global field and $\mathbb{A} = \mathbb{A}_k$ its ring of adèles. Let $\mathrm{e}_1^{\mathbb{A}} : \mathbb{A} \to \mathbb{C}^1$ be the (preferred) unitary character defined by

$$\mathrm{e}_1^{\mathbb{A}}(x) = \prod_{v \in |k|} \mathrm{e}_1^{k_v}(x_v) \tag{3.1}$$

where $\mathrm{e}_1^{k_v} : k_v \to \mathbb{C}^1$ is our preferred additive character of k_v. The above formula makes sense because $\mathrm{e}_1^{k_v}$ is trivial on $\mathcal{O}_v$ for almost all v and $x_v \in \mathcal{O}_v$ for almost all v.

Proposition 3.6. *The unitary character $\mathrm{e}_1^{\mathbb{A}} : \mathbb{A} \to \mathbb{C}^1$ gives rise to an homomorphism $\mathbb{A} \to \Lambda(\mathbb{A})$ mapping $x \in \mathbb{A}$ on the character $\mathrm{e} = \mathrm{e}_x$ where $\mathrm{e}_x(y) = \mathrm{e}_1^{\mathbb{A}}(xy)$. This morphism is an isomorphism and gives rise to an identification between the group of adèles $\mathbb{A}$ and its Pontryagin dual. Through this isomorphism, the discrete subgroup k is identified with its own orthogonal, and therefore the Pontryagin dual of k is identified with $\mathbb{A}/k$.*

Proof. The character $\mathrm{e} : \mathbb{A} \to \mathbb{C}^1$ induces at every place an additive character $\mathrm{e}_v : k_v \to \mathbb{C}^1$. We know that there exists a unique $x_v \in k_v$ such that $\mathrm{e}_v = \mathrm{e}_{x_v}$ where $\mathrm{e}_{x_v} : k_v \to \mathbb{C}^1$ is given by $x \mapsto \mathrm{e}_1^{k_v}(xx_v)$. It remains to prove that the x_v form an adèle i.e. $x_v \in \mathcal{O}_v$ for almost all v.

It is enough to prove that e_v is trivial on $\mathcal{O}_v$ for almost all v. This assertion follows from the continuity of e. Pick a neighborhood V of 1 in $\mathbb{C}^1$ which doesn't contain any non trivial subgroup of $\mathbb{C}^1$. Since e is continuous, there exist an open neighborhood U of 0 in $\mathbb{A}$ such that $\mathrm{e}(U) \subset V$. We may assume that U is of the form $U = \prod U_v$ where U_v is an open neighborhood of 0 in k_v and $U_v = \mathcal{O}_v$ for almost all v. It follows that for almost all v, $\mathrm{e}_v(\mathcal{O}_v) \subset V$ which implies that e_v is trivial on $\mathcal{O}_v$. □

Self-dual measures

We recall that for every Haar measure μ on a locally compact abelian group G, there exists a unique dual Haar measure ν on the Pontryagin dual $\Lambda(G)$ such that the the Fourier inversion formula holds

$$\int_{\Lambda(G)} \hat{\phi}(\xi)\nu(\xi) = \phi(0) \tag{3.2}$$

for all $\phi \in L^1(G)$ such that $\hat{\phi} \in L^1(\Lambda(G))$. If we multiply μ by a positive constant a, then we have to multiply $\mathrm{d}\xi$ by its inverse a^{-1} to obtain the dual Haar measure.

For every local field F with preferred additive character $\mathrm{e}_1^F : F \to \mathbb{C}^1$, we have identified F with its Pontryagin dual. Then there exists a unique Haar measure μ on F which is self-dual with respect to the identification of F with its Pontryagin dual.

- If $F = \mathbb{R}$, $\mathrm{e}_1^{\mathbb{R}}(x) = \exp(2i\pi x)$, then the self-dual Haar measure is the usual Lebesgue measure that assigns to the interval $[0,1]$ the measure 1.

- If $F = \mathbb{C}$, for the preferred character $\mathrm{e}_1^{\mathbb{C}}(x+iy) = \mathrm{e}_1^{\mathbb{R}}(2x)$, the self-dual Haar measure is $2\mathrm{d}x\mathrm{d}y$, twice the ordinary measure.

- For $F = \mathbb{Q}_p$ and its preferred additive character, the self-dual Haar measure assigns to $\mathbb{Z}_p$ the measure 1.

- More generally for F is a p-adic local field, and its preferred additive character $\mathrm{e}_1^F(x) = \mathrm{e}_{\mathbb{Q}_p}^1(\mathrm{Tr}_{F/\mathbb{Q}_p}(x))$, the self-dual Haar measure assigns to $\mathcal{O}_F$ the measure $|N_{F/\mathbb{Q}_p}(\mathcal{O}_F^{\perp})|^{-1/2}$ where $\mathcal{O}_F^{\perp}$ is the fractional ideal of F consisting of elements $x \in F$ such that $\mathrm{tr}_{F/\mathbb{Q}_p}(x\mathcal{O}_F) \subset \mathbb{Z}_p$. Indeed, if $b = 1_{\mathcal{O}_F}$ is the characteristic function of $\mathcal{O}_F$, its Fourier transform is $\hat{b} = \mathrm{vol}(\mathcal{O}_F)1_{\mathcal{O}_F^\perp}$. The equation (3.2) hold if and only if

$$\mathrm{vol}(\mathcal{O}_F)\mathrm{vol}(\mathcal{O}_F^\perp) = 1.$$

As $\mathrm{vol}(\mathcal{O}_F^\perp) = |N_{F/\mathbb{Q}_p}(\mathcal{O}_F^\perp)|\mathrm{vol}(\mathcal{O}_F)$, the above equation holds if and only if $\mathrm{vol}(\mathcal{O}_F) = |N_{F/\mathbb{Q}_p}(\mathcal{O}_F^\perp)|^{-1/2}$. We notice that $\mathrm{vol}(\mathcal{O}_F) = 1$ is and only if F is an unramified extension of $\mathbb{Q}_p$.

For a global field k with ring of adèles $\mathbb{A} = \mathbb{A}_k$, the preferred additive character $\mathrm{e}_1^{\mathbb{A}} : \mathbb{A} \to \mathbb{C}^1$ gives rise to an identification of $\mathbb{A}$ with its Pontryagin dual $\Lambda(\mathbb{A})$. The self-dual Haar measure $\mathrm{d}x$ on $\mathbb{A}$ is the product of Haar measures of local fields k_v for $v \in |k|$ in the sense that if $B_c = \prod_{v\in|k|} B_v$ is the compact subset of $\mathbb{A}$ given by $B_{c,v} = \{x_v \in k_v | |x_v| \leq c_v\}$ where $c = (c_v)$ is a collection, indexed by $v \in |k|$, of positive real numbers c_v with $c_v = 1$ for almost all $v \in |k|$, then

$$\mathrm{vol}(B_c, d_x) = \prod_{v\in|k|} \mathrm{vol}(B_{c,v}, \mathrm{d}x_v)$$

where $\mathrm{d}x_v$ is the self-dual Haar measure of F_v. This infinite product makes an obvious sense for we have $\mathrm{vol}(B_{c,v}, \mathrm{d}x_v) = 1$ for almost all v as k_v is unramified for almost all $v \in |k|$.

We have yet another characterization of the self-dual Haar measure by the volume to be given to the compact quotient $\mathbb{A}/k$. This characterization implies that the self-dual Haar measure on $\mathbb{A}$ is absolutely canonical in the sense that it is independent of the choice of additive characters.

Proposition 3.7. *For every invariant measure* $\mathrm{d}x$ *on* $\mathbb{A}$*, there exists a unique invariant measure* $\mathrm{d}y$ *on the quotient* $\mathbb{A}/k$ *of* $\mathbb{A}$ *by the discrete cocompact subgroup* k *such that for every continuous function with compact support* $f \in C_c(\mathbb{A})$*, and* ϕ *the function on* $\mathbb{A}/k$ *defined by* $\phi(x) = \sum_{\alpha\in k} f(x+\alpha)$*, we have* $\int_{\mathbb{A}} f(x)\mathrm{d}x = \int_{\mathbb{A}/k} \phi(y)\mathrm{d}y$.

If $\mathrm{d}x$ *is the self-dual Haar measure on* $\mathbb{A}$ *with respect to an additive unitary character* $\mathrm{e} : \mathbb{A} \to \mathbb{C}^1$ *for which the discrete cocompact subgroup* k *is its own orthogonal, then* $\mathrm{vol}(\mathbb{A}/k, \mathrm{d}y) = 1$ *for the induced invariant measure* $\mathrm{d}y$ *on the compact quotient* $\mathbb{A}/k$.

Proof. Let $f \in L^1(\mathbb{A})$ be an integral function whose Fourier transform $\hat{f} \in L^1(\mathbb{A})$ is also integrable. In this case, both f and $\hat{f}$ can be represented by continuous function. If $\mathrm{d}x$ is the self-dual Haar measure on $\mathbb{A}$ then the Fourier inversion formula (3.2) holds. The infinite $\phi(x) = \sum_{\alpha\in k} f(x+\alpha)$ defines an integrable function on $\mathbb{A}/k$ whose Fourier transform is the restriction of $\hat{f}$ to k.

Assume that the series $\sum_{\alpha\in k}\hat{f}(x+\alpha)$ is absolutely convergent i.e. the function $\hat{\phi}\in L^1(k)$, then ϕ can be represented by a continuous function on $\mathbb{A}$ and we have the Fourier inversion formula

$$\phi(0)=\mathrm{vol}(\mathbb{A}/k)\sum_{\alpha\in k}\hat{f}(\alpha).$$

Since ϕ is continuous, the formula

$$\phi(x)=\sum_{\alpha\in k}f(x+\alpha)$$

holds for all x and we have

$$\sum_{\alpha\in k}f(\alpha)=\mathrm{vol}(\mathbb{A}/k)\sum_{\alpha\in k}\hat{f}(\alpha).$$

By exchanging the role of f and $\hat{f}$, we derive $\mathrm{vol}(\mathbb{A}/k)=1$. □

The above argument also implies that the Poisson summation formula

$$\sum_{\alpha\in k}f(\alpha)=\sum_{\alpha\in k}\hat{f}(\alpha). \tag{3.3}$$

holds for all functions integrable function f on $\mathbb{A}$, whose Fourier transform $\hat{f}$ is also integrable, under the assumption that both Poisson sums are absolutely convergent.

Schwartz functions

Let V be a $\mathbb{R}$-vector space. A measurable function $\phi : V\to\mathbb{C}$ is said to be of rapid decay if for all polynomial function $P : V\to\mathbb{C}$, the function $|P(x)\phi(x)|$ is bounded. We notice that a measurable function ϕ of rapid decay is automatically integrable since there exists a polynomial function $P : V\to\mathbb{R}$ such that $|P|^{-1}$ is integrable, for instance $P(x)=x_1^2+\cdots+x_n^2$ where $x_1,\dots,x_n$ is a set of coordinates of V. The Fourier transform $\hat{\phi}$ of a measurable function ϕ with rapid decay makes sense as a continuous function on the dual vector space V^*.

A measurable function $\phi : V\to\mathbb{C}$ is a Schwartz function if both ϕ and $\hat{\phi}$ are of rapid decay. We denote by $\mathscr{S}(\mathbb{R})$ the space of all Schwartz functions. This space is non empty for the function $e^{-(x_1^2+\cdots+x_n^2)}$ is of rapid decay and equals its own Fourier transform up to normalization. For $V=\mathbb{R}$, the Gaussian function $b_\infty(x)=\exp(-\pi x^2)$ is of rapid decay and stable under the Fourier transform and therefore is an element of $\mathscr{S}(\mathbb{R})$. We will call it the basic element of $\mathscr{S}(\mathbb{R})$.

Proposition 3.8. *A measurable function $\phi : V \to \mathbb{C}$ is a Schwartz function if and only if ϕ is a smooth function such that ϕ and all its partial derivatives are of rapid decay.*

Proof. For simplicity, we assume $V = \mathbb{R}$. Let ϕ be a function such that both ϕ and $\hat{\phi}$ are of rapid decay. It follows that they are both integrable, continuous and the Fourier inversion formula

$$\phi(x) = \int_{\mathbb{R}} \hat{\phi}(y) \mathrm{e}_1^{\mathbb{R}}(xy)\mu(y)$$

is holds for all $x \in \mathbb{R}$. Since $\hat{\phi}(y)$ is of rapid decay, one can differentiate under the integral sign. This implies that ϕ is differentiable and ϕ' is integrable. It also implies that ϕ' is, up to a constant, the Fourier transform of the function $y \mapsto y\hat{\phi}(y)$ which is also a Schwartz function, therefore also differentiable in its turn, and ϕ'' is integrable. By induction, we know that ϕ is differentiable at any order i.e. is a smooth function, and all its derivatives are integrable. By inverting the role of ϕ and $\hat{\phi}$, the argument shows that $\hat{\phi}$ is a smooth function and all its derivatives are integrable. Using integration by part, this implies that all derivatives of ϕ and $\hat{\phi}$ are of rapid decay.

Now we assume that ϕ is a smooth function such that ϕ an all its derivatives are of rapid decay. It follows that ϕ is integrable, and therefore $\hat{\phi}$ is well defined as a continuous function. Integration part again shows that $\hat{\phi}$ is of rapid decay. □

Let V be a totally disconnected locally compact topological group which is isomorphic to its Pontryagin dual. For instance, V can be a finite dimensional vector space over a non-archimedean local field, or the ring of finite adèles over a number field. A continuous function ϕ of compact support in V is said to be a Schwartz-Bruhat function if both $\hat{\phi}$ is also of compact support. We denote by $\mathscr{S}(V)$ the space of all Schwartz-BRuhat functions on V. If $V = \mathbb{Q}_p$, the characteristic function $b_p = 1_{\mathbb{Z}_p}$ is a Schwartz function for it is a compactly supported function equal to its own Fourier transform.

Proposition 3.9. *Let V be a totally disconnected locally compact topological group which is isomorphic to its Pontryagin dual. A continuous function ϕ on V is a Schwartz-Bruhat function of and only if it is of compact support and invariant under a compact open subgroup.*

Proof. Let $(V_\alpha, \alpha \in A)$ be compact open subgroups of V forming a system of neighborhoods of 0, stable under finite intersection. We have in particular $\bigcap_{\alpha \in A} V_\alpha = \{0\}$. Their orthogonal subgroups $(V_\alpha^\perp, \alpha)$ form a family of compact open subgroups, stable under finite sum, and we have $\bigcup_{\alpha\in A} V_\alpha^\perp = V$. For every compact subset K of V, there exists $\alpha \in A$ such that $K \subset V_\alpha^\perp$.

Let ϕ be a continuous function on V such that both ϕ and $\hat{\phi}$ are of compact support. The Fourier inversion formula gives

$$\phi(x) = \int_V \hat{\phi}(y)\mathrm{e}(x, y)\mu(y).$$

Since $\hat{\phi}$ is of compact support, there exists $\alpha \in A$ such that $\mathrm{supp}(\hat{\phi}) \subset V_\alpha^\perp$. We observe that the restriction of the function $\phi(xy)$ to $y \in V_\alpha^\perp$ depends only on x modulo A_α. It follows that ϕ is an V_α-invariant function.

Conversely, if ϕ is a V_α-invariant then its Fourier transform $\hat{\phi}$ is supported in $V_\alpha^\perp$. □

Let $\mathbb{A} = \mathbb{A}_k$ be the ring of adèles of a number field k. We define

$$\mathscr{S}(\mathbb{A}) = \mathscr{S}(\mathbb{A}_{\mathrm{fin}}) \otimes \mathscr{S}(k_{\mathbb{R}}) \tag{3.4}$$

where $\mathscr{S}(\mathbb{A}_{\mathrm{fin}})$ is the space of Schwartz-Bruhat functions on the totally disconnected locally compact abelean group $\mathbb{A}_{\mathrm{fin}}$, and $\mathscr{S}(k_{\mathbb{R}})$ is the space of Schwartz functions on the real vector space $k_{\mathbb{R}}$. An element of $\mathscr{S}(\mathbb{A})$ is called a Schwartz function on $\mathbb{A}$.

Proposition 3.10. *A Schwartz function on A is a finite linear combination of function of the form $\bigotimes_{v\in|k|}\phi_v$ where ϕ_∞ is a Schwartz function on the real vector space $k_{\mathbb{R}}$, for each finite place v, ϕ_v is a Schwartz-Bruhat function on k_v, and for almost all v, $\phi_v = 1_{\mathscr{O}_v}$.*

Poisson summation formula for Schwartz functions

Theorem 3.11. *For all function $\phi \in \mathscr{S}(\mathbb{A})$, we have*

$$\sum_{a\in k} \phi(a) = \sum_{a\in k} \hat{\phi}(a).$$

Proof. Following the discussion around the formula (3.3), it is enough to prove that for every function $\phi \in \mathscr{S}(\mathbb{A})$, the Poisson sum $\sum_{a\in k}\phi(a)$ is absolutely convergent.

We can assume that $\phi = \phi_{\mathrm{fin}} \otimes \phi_\infty$ where ϕ_{fin} is a Schwartz-Bruhat function on $\mathbb{A}_{\mathrm{fin}}$ and ϕ_∞ is a Schwartz function on the real vector space $k_{\mathbb{R}}$. The support of the finite part ϕ_{fin} is contained in a compact open subgroup K of V and is bounded by a constant $c > 0$ in module. We have

$$\sum_{a \in k} |\phi(a)| \leq c \sum_{a \in K \cap k} |\phi_\infty(a)|$$

where $K \cap k$ is a $\mathbb{Z}$-lattice in the real vector space $k_{\mathbb{R}}$. Now every Schwartz function ϕ can be bounded from above in module by a multiple of the function $(x_1^2 + \cdots + x_n^2)^{-1}$ where $x_1, \ldots, x_n$ is a system of coordinates given by a basis of the lattice. It follows that the sum $\sum_{a \in K \cap k} |\phi_\infty(a)|$ can be bounded from above by a multiple of ... $\square$

Fourier transform of tempered distributions

Let F be a local field. A tempered distribution on F is a continuous linear function $a : \mathscr{S}(F) \to \mathbb{C}$. When F is non-archimedean, the space of Schwartz-Bruhat functions $\mathscr{S}(F)$ is equipped with the discrete topology so that the continuity condition is vacuous. If F is archimedean, we equip $\mathscr{S}(F)$ with the topology defined by the family of semi-norms $\phi \mapsto \sup |P\phi^{(n)}|$ where $P\phi^{(n)}$ is the n-th derivative of ϕ multiplied by a polynomial function P. We denote by $\mathscr{S}'(F)$ the space of all tempered distributions on F.

For every $x \in F$, the linear functional

$$\phi \mapsto \phi(x)$$

is tempered distribution to be denoted by δ_x, the Dirac function at x. For a measurable f (with polynomial growth in the archimedean case), provided that integral the linear functional

$$\phi \mapsto \int_F \phi(x) f(x) \mu(x)$$

converges absolutely for all
$phi \in \mathscr{S}(F)$, defines a tempered distributionto be denoted by $f\,\mathrm{d}x \in$.

The Fourier transform $\phi \mapsto \hat{\phi}$ on $\mathscr{S}(F)$ induces a Fourier transform $a \mapsto \hat{a}$ on the space of tempered distributions by the formula

$$\hat{a}(\phi) = a(\hat{\phi}). \tag{3.5}$$

For instance, the Fourier transform of the Dirac function δ_0 is the constant function 1_F. More generally, the Fourier transform of the Dirac function δ_x, for every $x \in F$, is the function e_x^{-1} given by $\mathrm{e}_x^{-1}(y) = \mathrm{e}_1(xy)^{-1}$.

4 Mellin transform

Module in local fields

If F is a local field, $F^\times$ acts continuously on F by multiplication $(a,x)\mapsto ax$. This induces an action of $F^\times$ on the space $C_c(F)$ of continuous functions with compact support: for $a\in F^\times$ and $\phi\in C_c(F)$, we denote by $\phi_a\in C_c(F)$ the function $\phi_a(x)=\phi(a^{-1}x)$. If μ is a Radon measure on F i.e. a continuous linear functional $C_c(F)\to\mathbb{C}$, we define the Radon measure μ_a by the formula

$$\langle \mu_a,\phi\rangle=\langle \mu,\phi_{a^{-1}}\rangle. \tag{4.1}$$

This definition guarantees that the pairing between $C_c(F)$ and the space of Radon measures is invariant with respect to the diagonal action of $F^\times$.

If μ is a Haar measure μ of F, then μ_a is also a Haar measure for every $a\in F^\times$. There exists a unique positive real number $|a|_F\in\mathbb{R}_+^\times$ such that

$$\mu_a=|a|^{-1}\mu \tag{4.2}$$

We may want to unravel this definition by writing down integral of a test function $\phi\in C_c(F)$ as follows:

$$\begin{aligned}\int_F \phi(x)\mu_a(x) &= \int_F \phi(ax)\mu(x)\\ &= |a|^{-1}\int_F \phi(y)\mu(y).\end{aligned}$$

In the above formula can be thought of as the chain rule for the change of variable $y=ax$

$$\mu(x)=|a|^{-1}\mu(ax)$$

which doesn't have a proper sense in the general notational scheme that we have chosen, but should be thought of as a mnemonic device to manipulate formulas.

To get an intuition on how the group $F^\times$ acts on functions on F, the following example is instructive. Recall that for every $c\in\mathbb{R}_+^\times$, the subset B_c of F consisting of elements $x\in F$ such that $|x|\le c$ is a compact. If 1_{B_c} is the characteristic function of B_c, then for every $a\in F^\times$, we have

$$(1_{B_c})_a=1_{B_{|a|c}}.$$

Indeed, it is clear that $a^{-1}x \in B_c$ if and only if $x \in B_{|a|c}$.

If there are more than on local field F to consider, as in the adelic context, we will write $|a|_F$ instead of F. For $F = \mathbb{R}$, $|a|_{\mathbb{R}}$ is the usual real absolute value of $a \in \mathbb{R}$. For $F = \mathbb{C}$, and $z = x + iy \in \mathbb{C}$ is the usual representation of a complex number, we have $|z|_{\mathbb{C}} = x^2 + y^2$. For $F = \mathbb{Q}_p$, we have $|a|_{\mathbb{Q}_p} = p^{-\mathrm{val}_p(a)}$. More generally, if F is a p-adic field, we have

$$|a|_F = q^{-\mathrm{val}_F(a)}$$

where q is the cardinality of the residue field of F and $\mathrm{val}_F : F^\times \to \mathbb{Z}$ is the valuation function of F.

Let k be a global field and $\mathbb{A} = \mathbb{A}_k$ its ring of adèles. The group of idèles $\mathbb{A}^\times$ acts continuously on $\mathbb{A}$. For every $a \in \mathbb{A}^\times$, we define $|a|_{\mathbb{A}}$ by the formula

$$\mu_a = |a|_{\mathbb{A}}^{-1}\mu \tag{4.3}$$

where μ is a Haar measure on $\mathbb{A}$ and μ_a is defined as in (4.1). For every function $\phi \in C_c(\mathbb{A})$, we have

$$\int_{\mathbb{A}} \phi_a(x)\mu(x) = |a|_A \int_{\mathbb{A}} \phi(x)\mu(x). \tag{4.4}$$

If $a = (a_v, v \in |k|)$ where $a_v \in k_v^\times$ with $a_v \in \mathcal{O}_v^\times$ for almost all v then we have the formula

$$|a|_{\mathbb{A}} = \prod_v |a_v|_{k_v}$$

in which $|a_v|_{k_v} = 1$ for almost all v. We recall the product formula

$$|a|_{\mathbb{A}} = \prod_v |a_v|_{k_v} = 1$$

hold for all $a \in k^\times$. This fact can be checked directly upon the definition of local modules $|a_v|_{k_v}$, it is worth to note that it a measure theoretic interpretation.

The measure theoretic proof of the product formula is based on the existence of a fundamental domain for the action of k on $\mathbb{A}$ by translation. We define a fundamental domain of the action of k on $\mathbb{A}$ as a closed subset B of A such that the projection $B \to \mathbb{A}/k$ is finite and surjective while its restriction the interior B' of B is injective. We also require that the complement of B' in B is of measure 0, which is automatic in any practical construction of fundamental domain. For $k = \mathbb{Q}$, the cube

$$B = \{(x_p, x_\infty) || x_p| \leq 1, |x_\infty| \leq 1/2\}$$

is a fundamental domain for the action of $\mathbb{Q}$ on $\mathbb{A}_{\mathbb{Q}}$ by translation. One can use the fundamental domain for $\mathbb{Q} \subset \mathbb{A}_{\mathbb{Q}}$ to construct a fundamental domain for $k \subset \mathbb{A}_k$ by considering k as a finite dimensional vector space over $\mathbb{Q}$.

Let B be a fundamental domain for the action of k on $\mathbb{A} = \mathbb{A}_k$ by translation. We then have $\mathrm{vol}(B, \mu) = \mathrm{vol}(\bar{A}/k, \bar{\mu})$ where μ is a Haar measure on $\mathbb{A}$ and $\bar{\mu}$ is the induced measure on the compact quotient of $\mathbb{A}$ by the discrete subgroup k. For instance, if μ is the self-dual measure, we have

$$\mathrm{vol}(B, \mu) = \mathrm{vol}(\bar{A}/k, \bar{\mu}) = 1.$$

For every $a \in k^\times$, aB is also a fundamental domain. It follows that $\mathrm{vol}(aB) = \mathrm{vol}(B)$ which implies the product formula $|a|_{\mathbb{A}} = 1$.

Multiplicative Haar measure

Structure of the group of characters

We will study the structure the group of characters $\Omega(F^\times)$ where F is a local field, and $\Omega(\mathbb{A}^\times/k^\times)$ where k is a global field and $\mathbb{A}$ is its ring of adèles. These groups have a distinguished element given by the module character. From now on, we will denote by ω^1 the module character $\omega^1(x) = |x|$. Both in local and global cases, the kernel of the module character is compact.

- If $F = \mathbb{R}$, the kernel of ω^1 is the finite subgroup $\{\pm 1\}$. We have $\mathbb{R}^\times = \{\pm 1\} \times \mathbb{R}_+^\times$. Since $\Omega(\mathbb{R}_+^\times) = \mathbb{C}$, we have

  $$\Omega(\mathbb{R}^\times) = \{\pm 1\} \times \mathbb{C}$$

  where $(+, s)$ corresponds to the character $x \mapsto |x|^s$ of $\mathbb{R}^\times$ and $(-, s)$ the character $x \mapsto \mathrm{sgn}(x)|x|^s$.

- If $F = \mathbb{C}$, the kernel of ω^1 is the unit circle $\mathbb{C}^1$. We have a splitting $\mathbb{C}^\times = \mathbb{C}^1 \times \mathbb{R}_+^\times$ where we use the $\mathbb{R}_+^\times \to \mathbb{C}^\times$ given by $t \mapsto t^{1/2}$ to produce a section of $\omega_{\mathbb{C}}^1$. We have thus

  $$\Omega(\mathbb{C}^\times) = \mathbb{Z} \times \mathbb{C}$$

  where (n, s) correspond to the character

  $$e^{2i\pi\theta} r \mapsto e^{n2i\pi\theta} r^{2s}$$

  where we have written a complex number $z \in \mathbb{C}^\times$ as $z = e^{i\theta} r$ with $\theta \in \mathbb{R}/\mathbb{Z}$ and $r \in \mathbb{R}_+^\times$.

- If F is a non-archimedean local field, the kernel of ω^1 is the compact subgroup $\mathcal{O}_F^\times$ where $\mathcal{O}_F$ is the valuation ring of F. We have a splitting $F^\times = \mathcal{O}_F^\times \times q^{\mathbb{Z}}$ where the section of $\omega_F^1 : F^\times \to q^{\mathbb{Z}}$ is produced upon the choice of an uniformizing parameter ϖ of F. We have

  $$\Omega(F^\times) = \Lambda(\mathcal{O}_F^\times) \times \mathbb{C}/2i\pi \log(q)\mathbb{Z}.$$

  An element $(\lambda, s) \in \Lambda(\mathcal{O}_F^\times) \times \mathbb{C}$ corresponds to the character

  $$\alpha\varpi^n \mapsto \lambda(\alpha)q^{-ns}$$

  where we have written an element $x \in F^\times$ as a product of $\alpha \in \mathcal{O}_F^\times$ with a power of the uniformizing parameter ϖ. Since $\mathcal{O}_F^\times$ is a profinite group, $\Lambda(\mathcal{O}_F^\times)$ is a union of finite groups.

- If $\mathbb{A}$ is the ring of adèles of a number field, the kernel of $\omega_{\mathbb{A}}^1 : \mathbb{A}^\times/k^\times \to \mathbb{R}_+^\times$ is the compact group $\mathbb{A}^\times/k^\times$ by Theorem 1.28. We have a splitting $\mathbb{A}^\times/k^\times = \mathbb{A}^1/k^\times \times \mathbb{R}_+^\times$ where a section of $\omega_{\mathbb{A}}^1 : \mathbb{A}^\times/k^\times \to \mathbb{R}_+^\times$ can be produced upon the choice of an archimedean place of k.

We will now discuss all these cases in a uniform manner. We will restrict our attention to locally compact abelian groups G equipped with a module character $\omega^1 : G \to \mathbb{R}_+^\times$ whose kernel G^1 is compact. The image H of the module character can be either $\mathbb{R}_+^\times$ or a discrete subgroup of $\mathbb{R}$ which must be of the form $q^{\mathbb{Z}}$ for some $q \in \mathbb{R}_+^\times$ with $q > 1$. We have an exact sequence

$$0 \to G^1 \to G \to H \to 0.$$

For instant, if $G = \mathbb{R}^\times$ then $G^1 = \{\pm 1\}$ and $H = \mathbb{R}_+^\times$. If $G = \mathbb{C}^\times$ then $G^1 = \mathbb{C}^1$ and $H = \mathbb{R}^+$. If $G = \mathbb{Q}_p^\times$, the module character $\mathbb{Q}_p^\times \to \mathbb{R}_+^\times$ has image $H = p^{\mathbb{Z}}$ and kernel $G^1 = \mathbb{Z}_p^\times$. If $G = \mathbb{A}_K^\times/K^\times$ is the group of idèles classes in number field L, the module character is surjective and its kernel $G^1 = \mathbb{A}^1/K^\times$, the group of idèle classes of norm 1.

We call character of G every continuous homomorphism $\chi : G \to \mathbb{C}^\times$. Let $\Omega(G)$ denote the group of characters of G equipped with the compact open topology. It contains a distinguished non-trivial element $||_G \in \Omega(G)$ representing the module character of G. The restriction of $\chi \in \Omega(G)$ to the compact subgroup G^1 has image contained in $\mathbb{C}^1$, in other words $\chi|_{G^1}$ is a unitary character. Recall that the Pontryagin dual of a compact group is a discrete group, in particular $\Lambda(G^1)$ is discrete. We have an exact sequence

$$0 \to \Omega(H) \to \Omega(G) \to \Lambda(G^1) \to 0$$

where $\Omega(H)$ is the group of characters of H. For each complex number s, we will denote $\omega^s : G \to \mathbb{C}^\times$ the character defined by

$$\omega^s(x) = |x|^s. \tag{4.5}$$

This character is trivial on G^1 and and one can check that every character of G, trivial on G^1 is of this form. The homomorphism $\mathbb{C} \to \Omega(H)$ defined by $s \mapsto \omega^s$ is surjective. It is an isomorphism if $H = \mathbb{R}_+^\times$. If H is a discrete group generated by an some positive real number $q > 1$, then $\omega^s = 1$ if and only if $s \in \frac{2i\pi}{\log(q)}\mathbb{Z}$, thus

$$\Omega(H) = \mathbb{C} \Big/ \frac{2i\pi}{\log(q)}\mathbb{Z}.$$

In both cases, in addition to its group structure, $\Omega(H)$ is naturally equipped with a structure of one-dimensional complex variety. The space $\Omega(G)$ being a disjoint union of copies of $\Omega(H)$ can also be naturally equipped with a structure of one-dimensional complex variety.

The exponent $\Re(\chi)$ of a character $\chi \in \Omega(G)$ is a as follows. For every $\chi \in \Omega(G)$, its absolute value $|\chi|(g) = |\chi(g)|$ is a character of real positive values. Since the only compact subgroup of $\mathbb{R}_+^\times$ is the trivial group, the restriction of $|\chi|$ to the compact subgroup G^1 is trivial. It follows that the restriction of $|\chi|$ to the compact subgroup G^1 is trivial. It follows that there exists a unique $\sigma \in \mathbb{R}$ such that $|\chi(g)| = |g|^\sigma$ for all $g \in G$ and we set the exponent $\Re(\chi)$ of χ to be σ. A character $\chi \in \Omega(G)$ is unitary if and only if $\Re(\chi) = 0$. For $\chi = \omega^s$, we have $\Re(\chi) = \Re(s)$ where $\Re(s)$ is the real party of the complex number s.

Lemma 4.1. *Let $\chi : \mathbb{A}^\times/k^\times \to \mathbb{C}^\times$ be a character of the group of idèle classes. For every place v of k, we denote χ_v the induced character on $F_v^\times$ obtained by composing χ with the homomorphism $k_v^\times \to \mathbb{A}^\times/k^\times$. Then we have $\Re(\chi) = \Re(\chi_v)$.*

Proof. If $\sigma = \Re(\chi)$ then $\chi\omega_\sigma : \mathbb{A}^1/k^\times \to \mathbb{C}^1$ is unitary, where ω_σ is defined by formula (4.5). By restricting to $k_v^\times$, the character $\chi_v\omega_\sigma : k_v^\times \to \mathbb{C}^\times$ is unitary, and therefore $\Re(\chi_v) = \sigma$. $\square$

Mellin transform on $\mathbb{R}_+^\times$

The group of characters of $\mathbb{R}_+^\times$ is $\Omega(\mathbb{R}_+^\times) = \mathbb{C}$ for every $s \in \mathbb{C}$ corresponds to the character $x \mapsto \omega^s(x) = x^s$. We will study the Mellin transform

$$\mathscr{M}\phi(s) = \int_{\mathbb{R}_+^\times} \phi(x) x^s \mu^\times(x)$$

for test smooth functions ϕ on $\mathbb{R}_+^\infty$ satisfying decay conditions at 0 and ∞.

Let $\mathscr{S}(\mathbb{R}_+^\times)$ denote the space of functions $\phi : \mathbb{R}_+ \to \mathbb{C}$ which have rapid decay both at 0 and ∞ i.e.the functions $|x^m\phi(x)|$ are bounded for all $m \in \mathbb{Z}$ and whose derivatives $\phi^{(n)}(x)$ also have rapid decay at both 0 and ∞. Such a function is called a Schwartz function on $\mathbb{R}_+^\times$.

Theorem 4.2. *Let $\mathscr{S}(\mathbb{R}_+^\times)$ the space of Schwartz functions on $\mathbb{R}_+^\times$. Then for every $s \in \mathbb{C}$, the integral*

$$\mathscr{M}\phi(s) = \int_{\mathbb{R}_+^\times} \phi(x)x^s\mu^\times(x)$$

with $\mu^\times = \omega^{-1}\mu$ being the normalized Haar measure on $\mathbb{R}_+^\times$, is absolutely convergent. It defines a holomorphic function of variable $s \in \mathbb{C}$.

Let us denote $\mathscr{Z}(\mathbb{C})$ the space of holomorphic functions $g(s)$ on $\mathbb{C}$ which are bounded on every finite vertical trip $\sigma_1 \leq z \leq \sigma_2$, and so are functions of the form $P(s)g(s)$ where $P(s)$ is a polynomial of variable s. Then for every $\phi \in \mathscr{S}(\mathbb{R}_+^\times)$, the Mellin transform $\mathscr{M}\phi$ lies in $\mathscr{Z}(\mathbb{C})$.

For every $g \in \mathscr{Z}(\mathbb{C})$, the integral along any vertical axe

$$\phi(x) = \frac{1}{2i\pi}\int_{\sigma+i\mathbb{R}} g(s)x^{-s}\mathrm{d}s$$

determines a function $\phi = \mathscr{N}g \in \mathscr{S}(\mathbb{R}_+^\times)$.

The maps $\mathscr{M} : \mathscr{S}(\mathbb{R}_+^\times) \to \mathscr{Z}(\mathbb{C})$ and $\mathscr{N} : \mathscr{Z}(\mathbb{C}) \to \mathscr{S}(\mathbb{R}_+^\times)$ are inverse of each other.

Proof. The proof of the theorem is naturally divided into two parts of which the first deals with the Mellin transform $\phi \mapsto g$ and the second with the inverse Mellin transform $g \mapsto \phi$.

Mellin transform $\phi \mapsto g$. Let us break the Mellin integral into two parts $x < 1$ and $x > 1$:

$$\int_{\mathbb{R}_+^\times} \phi(x)x^s\mu^\times(x) = \int_{x<1} \phi(x)x^s\mu^\times(x) + \int_{x>1} \phi(x)x^s\mu^\times(x).$$

The decay conditions imply that there exists constant c_1, c_2 such that $|\phi(x)|x^s \leq c_1x^2$ and $|\phi(x)|x^s \leq c_1x^{-2}$. The first bound implies that the integral $\int_{x<1}$ in the above equality is absolutely convergent, the second bound implies that the integral $\int_{x>1}$ is absolutely convergent. Thus the Mellin integral $g(s) = \mathscr{M}\phi(s)$ makes sense for all $s \in \mathbb{C}$.

As a function of complex variable s, x^s is holomorphic whose complex derivative is $\log(x)x^s$. The same argument as above implies that the integral

$$\int_{\mathbb{R}_+^\times} \phi(x)\log(x)x^s\mu^\times(x)$$

is absolutely convergent. It follows that $g = \mathscr{M}\phi$ has complex derivative everywhere i.e. g is a holomorphic function.

Let us consider the restriction of g to a vertical axis $\sigma + it$ for a given $\sigma \in \mathbb{R}$. We have

$$g(\sigma + it) = \int_0^\infty \phi(x)x^\sigma x^{it}\mu^\times x$$

We have shown that for every $\sigma \in \mathbb{R}$, the function $|\phi(x)x^\sigma|$ is integrable with respect to the measure $\mu^\times(x)$. The argument shows in fact that the L^1-norm of $|\phi(x)x^\sigma|$ is bounded as σ varies in a finite interval $[\sigma_1, \sigma_2]$. It follows that $g(\sigma + it)$ is bounded on the vertical trip $\sigma_1 \leq \Re(s) \leq \sigma_2$.

In order to show that $\mathscr{M}\phi \in \mathscr{Z}(\mathbb{C})$, it remains to show that for every polynomial function $P(s)$, the function $P(s)\mathscr{M}\phi(s)$ is also bounded on every vertical trips in the above sense. To prove this, we use the differential operator $x\partial_x$ which is invariant with respect to the action of $\mathbb{R}_+^\times$ on itself by translation. Using integration by parts, we get the formula

$$\begin{aligned}
\mathscr{M}(x\partial_x\phi)(s) &= \int_0^\infty \partial_x\phi(x)x^s\mu(x) \\
&= -s\int_0^\infty \partial_x\phi(x)x^{s-1}\mu(x) \\
&= -s\mathscr{M}\phi(s)
\end{aligned}$$

which shows that the function $s\mathscr{M}\phi(s)$ in bounded on every vertical trip. By apply the differential operator $x\partial_x$ as many times as needed, we prove that this statement remains true when we replace $s\mathscr{M}\phi(s)$ by $P(s)\mathscr{M}\phi(s)$ for any polynomial $P(s)$.

Inverse Mellin transform $g \mapsto \phi$. We prove that if $g \in \mathscr{Z}(\mathbb{C})$ is a holomorphic function such that for every polynomial $P(s)$, the function $P(s)g(s)$ is bounded on every vertical trip, then the integral

$$\phi(x) = \frac{1}{2i\pi}\int_{-\infty}^{+\infty} g(\sigma+it)x^{-(\sigma+it)}\mu(t) \tag{4.6}$$

$$= \frac{1}{2i\pi}\int_{\sigma+i\mathbb{R}} g(s)x^{-s}\mathrm{d}s. \tag{4.7}$$

over any vertical axis, defines a function $\phi = \mathscr{N}g \in \mathscr{S}(\mathbb{R}_+^\times)$.

First, the growth condition on vertical trips implies that the function $g(\sigma+it)$ of variable $t \in \mathbb{R}$ is integrable so that the integral (4.8) makes sense. In vertu of contour integration as in (4.8), the growth condition on vertical trips also assures that we obtain the same value $\phi(x)$ no matter what vertical axis we choose. For a given σ, it follows that $\phi(x)x^\sigma$ is a continuous function of variable x bounded by 2π the L^1-norm of $g(\sigma+it)$. As $\phi(x)x^\sigma$ is bounded for every σ, the function ϕ has rapid decay as $x \to \infty$ and $x \to 0$.

The growth condition on vertical trips implies that the function $g(\sigma+it)\partial_x x^{-(\sigma+it)}$ of variable t is also integrable. It follows that $\partial_x\phi$ exists and it is equal to the integral of $g(\sigma+it)\partial_x x^{-(\sigma+it)}$ against the additive measure $(2i\pi)^{-1}\mu(t)$. Again by letting $\sigma \to \infty$, we prove that $\partial_x\phi$ has rapid decay. By repeating this argument as many times as necessary, we prove that ϕ is a smooth function of variable $x \in \mathbb{R}_+$ and all its derivatives have rapid decay as $x \to \infty$ and $x \to 0$.

$\mathscr{M}$ and $\mathscr{N}$ are inverse of each other. This statement is nothing but the Fourier inversion formula, after a change of variables. Let us identify $\mathbb{R} \to \mathbb{R}_+^\times$ with help of the exponential function $y \mapsto x = e^y$. Via this identification, the normalized multiplicative measure $\mu^\times$ on $\mathbb{R}_+^\times$ corresponds to the additive measure μ on $\mathbb{R}$. For every $\phi \in \mathscr{S}(\mathbb{R}_+^\times)$, for every σ, the function $\phi(x)x^\sigma = \phi(e^y)e^{y\sigma}$ of variable $y \in \mathbb{R}$ is integrable with respect to the additive measure μ, and its Fourier transform:

$$\mathscr{M}\phi(\sigma+it) = \int_{\mathbb{R}} \phi(e^y)e^{\sigma y}e^{ity}\mu(y)$$

is just the restriction of $\mathscr{M}\phi$ to the vertical axis $\sigma+it$.

Since for every polynomial $P(s)$ the function $P(s)\mathscr{M}\phi(s)$ is bounded on every vertical trips, the function $\mathscr{M}\phi(\sigma+it)$ over variable $t \in \mathbb{R}$ is integrable. We can thus apply the Fourier inversion formula

$$\phi(x)x^\sigma = \phi(e^y)e^{y\sigma} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \mathscr{M}\phi(\sigma+it)e^{-iyt}\mu(t).$$

This shows that $\mathscr{N}\mathscr{M}\phi = \phi$. The equality $\mathscr{M}\mathscr{N}g = g$ can be proven similarly using the fact that the function $\phi(e^y)e^{\sigma y}$ is integrable for all $\sigma \in \mathbb{R}$. $\square$

Let $\mathscr{S}_+(\mathbb{R}_+^\times)$ denote the space of functions $\phi : \mathbb{R}_+ \to \mathbb{C}$ which has a Taylor expansion at 0 and behave as a Schwartz function at $+\infty$ i.e. $|x^m\phi^{(n)}(x)|$ is bounded for all non-negative integers m, n.

Theorem 4.3. *Let $\mathscr{S}_+(\mathbb{R}_+^\times)$ denote the space of functions $\phi : \mathbb{R}_+ \to \mathbb{C}$ which has a Taylor expansion at 0 and behave as a Schwartz function at $+\infty$ i.e. $|x^m\phi^{(n)}(x)|$ is bounded for all non-negative integers m, n. For every Schwartz function $\phi \in \mathscr{S}_+(\mathbb{R}_+^\times)$, the integral*

$$\mathscr{M}\phi(s) = \int_{\mathbb{R}_+^\times} \phi(x)x^s\mu^\times(x)$$

with $\mu^\times = \omega^{-1}\mu$ being the normalized Haar measure on $\mathbb{R}_+^\times$, is absolutely convergent and defines a holomorphic function on $\mathscr{M}\phi$ on the half-plane $\Re(s) > 0$. The function $\mathscr{M}\phi$ can be extended as a meromorphic functions on $\mathbb{C}$ with at most simple poles at non-positive integers.

Let us denote $\mathscr{Z}_+(\mathbb{C})$ the space of meromorphic functions g on $\mathbb{C}$ with at most simple poles at non-positive integers such that after removing a small disc around each non-positive integers, g is bounded on every finite vertical trip $\sigma_1 \leq z \leq \sigma_2$, and so are functions of the form Pg where P are polynomials of variable s. Then $\phi \mapsto \mathscr{M}\phi$ defines a linear map from $\mathscr{S}_+(\mathbb{R})$ to $\mathscr{Z}_+(\mathbb{C})$.

For every $g \in \mathscr{Z}_+(\mathbb{C})$, the integral along any vertical axe $\Re(s) = \sigma > 0$

$$\phi(x) = \frac{1}{2i\pi}\int_{\sigma+i\mathbb{R}} g(s)x^{-s}\mathrm{d}s$$

defines a function $\phi = \mathscr{N}g \in \mathscr{S}_+(\mathbb{R}_+^\times)$.

The maps $\mathscr{M} : \mathscr{S}_+(\mathbb{R}_+^\times) \to \mathscr{Z}_+(\mathbb{C})$ and $\mathscr{N} : \mathscr{Z}_+(\mathbb{C}) \to \mathscr{S}_+(\mathbb{R}_+^\times)$ are inverse of each other. The Taylor coefficients $a_0, a_1, \ldots$ in the Taylor development of ϕ at 0

$$a_0 + a_1x + a_2x^2 + \cdots$$

are the residues $b_0, b_1, \ldots$ of g at $0, -1, -2, \ldots$.

Proof. The proof of the theorem is naturally divided into two parts of which the first deals with the Mellin transform $\phi \mapsto g$ and the second with the inverse Mellin transform $g \mapsto \phi$.

Mellin transform $\phi \mapsto g$. Let us break the Mellin integral into two parts $x \le 1$ and $x > 1$:

$$\int_{\mathbb{R}_+^\times} \phi(x)x^s\mu^\times(x) = \int_{x\le 1} \phi(x)x^s\mu^\times(x) + \int_{x>1} \phi(x)x^s\mu^\times(x).$$

Since ϕ has rapid decay as $x \to \infty$, the integral $\int_{x>1}$ is absolutely convergent for every $s \in \mathbb{C}$. Since ϕ is continuous at 0, its is bounded on the interval $[0,1]$. The integral $\int_{x\le 1}$ is now absolutely convergent for $\Re(s) > 0$ as the integral

$$\int_0^1 x^s\mu^\times(x)$$

does. It follows that for $\Re(s) > 0$, the function $\phi(x)x^s \in L^1(\mathbb{R}_+^\times, \mu^\times)$. In particular, $\mathscr{M}\phi(s)$ is an absolutely convergent integral for s with positive real part.

As a function of complex variable s, x^s is holomorphic whose complex derivative is $\log(x)x^s$. We observe that the integral

$$\int_{\mathbb{R}_+^\times} \phi(x)\log(x)x^s\mu^\times(x) = \int_{x\le 1} \phi(x)\log(x)x^s\mu^\times(x) + \int_{x>1} \phi(x)\log(x)x^s\mu^\times(x)$$

is absolutely convergent for $\Re(s) > 1$ for similar reason. On one hand, since the function $\phi(x)\log(x)$ has rapid decay as $x \to \infty$, the integral $\int_{x>1}$ is absolutely convergent for every $s \in \mathbb{C}$. The integral $\int_{\le 1} \phi(x)\log(x)x^s\mu^\times(x)$ is absolutely convergent for $\Re(s) > 1$ as the integral

$$\int_0^1 \log(x)x^s\mu^\times(x)$$

does. It follows that for $\Re(s) > 1$, $\mathscr{M}\phi(s)$ is holomorphic and its complex derivative is

$$\int_{\mathbb{R}_+^\times} \phi(x)\log(x)x^s\mu^\times(x).$$

We now consider the derivative $\partial_x\phi$ of ϕ with respect to the variable x. It is clear that $\partial_x\phi \in \mathscr{S}_+(\mathbb{R}_+^\times)$. We now rewrite the integral $\mathscr{M}(\partial_x\phi)(s)$ with respect to the additive measure and apply the formula of integration by parts which show that the following formulas are valid for $\Re(s) > 1$:

$$\begin{aligned}
\mathscr{M}(\partial_x \phi)(s) &= \int_0^\infty \partial_x \phi(x) x^{s-1} \mu(x) \\
&= -(s-1) \int_0^\infty \phi(x) x^{s-2} \mu(x) \\
&= -(s-1) \mathscr{M}\phi(s-1).
\end{aligned}$$

Since the function $\mathscr{M}(\partial_x \phi)(s)$ is known to be holomorphic for $\Re(s) > 1$, this formula shows that $\mathscr{M}\phi$ is holomorphic for $\Re(s) > 0$. By applying repeatedly, we obtain a meromorphic continuation of $\mathscr{M}\phi$ to $\mathbb{C}$ with at worst simple poles located at non-positive integers.

Let us now prove that $\mathscr{M}\phi(s)$ is bounded on every vertical trip

$$0 < \sigma_1 \leq \Re(s) \leq \sigma_2.$$

The above translation argument shows that its meromorphic continuation enjoys the same property after we remove small discs around non-positive integers. This assertion follows from the fact we have shown that the function $\phi(x)x^s$ belongs to $L^1(\mathbb{R}_+^\times, \mu^\times)$ for $\Re(s) > 0$. Its L^1-norm depends only on $\Re(s)$ and can be bounded uniformly on the interval $[\sigma_1, \sigma_2]$.

In order to show that $\mathscr{M}\phi \in \mathscr{Z}_+(\mathbb{C})$, it remains to show that for every polynomial function $P(s)$, the function $P(s)\mathscr{M}\phi(s)$ is also bounded on every vertical trips in the above sense. To prove this, we use the differential operator $x\partial_x$ which is invariant with respect to the action of $\mathbb{R}_+^\times$ on itself by translation. Using again integration by parts, we get the formula

$$\begin{aligned}
\mathscr{M}(x\partial_x \phi)(s) &= \int_0^\infty \partial_x \phi(x) x^{s} \mu(x) \\
&= -s \int_0^\infty \partial_x \phi(x) x^{s-1} \mu(x) \\
&= -s \mathscr{M}\phi(s)
\end{aligned}$$

which shows that the function $s\mathscr{M}\phi(s)$ in bounded on every vertical trip. By apply the differential operator $x\partial_x$ multiple times, we prove that the statement remains true when we replace $s\mathscr{M}\phi(s)$ by $P(s)\mathscr{M}\phi(s)$ for any polynomial $P(s)$.

Inverse Mellin transform $g \mapsto \phi$. We prove that if $g \in \mathscr{Z}_+(\mathbb{C})$ is a meromorphic function with at most simple poles at non-positive integers such that for every polynomial

$P(s)$, the function $P(s)g(s)$ is bounded on every vertical trip, then the integral

$$\begin{aligned} \phi(x) &= \frac{1}{2i\pi}\int_{-\infty}^{+\infty} g(\sigma+it)x^{-(\sigma+it)}\mu(t) && (4.8) \\ &= \frac{1}{2i\pi}\int_{\sigma+i\mathbb{R}} g(s)x^{-s}\mathrm{d}s. && (4.9) \end{aligned}$$

over any vertical axis of real coordinate $\sigma > 0$, defines a function $\phi \in \mathscr{S}_+(\mathbb{R}_+^\times)$.

First, the growth condition on vertical trips implies that the function $g(\sigma+it)$ of variable $t \in \mathbb{R}$ is integrable so that the integral (4.8) makes sense. It follows that $\phi(x)x^\sigma$ is a continuous function of variable x bounded by 2π the L^1-norm of $g(\sigma+it)$. As $\phi(x)x^\sigma$ is bounded for every $\sigma > 0$, the function ϕ has rapid decay as $x \to \infty$.

The growth condition on vertical trips implies that the function

$$g(\sigma+it)\partial_x x^{-(\sigma+it)}$$

of variable t is also integrable. It follows that $\partial_x\phi$ exists and it is equal to the integral of $g(\sigma+it)\partial_x x^{-(\sigma+it)}$ against the additive measure $(2i\pi)^{-1}\mu(t)$. Again by letting $\sigma \to \infty$, we prove that $\partial_x\phi$ has rapid decay. By repeating this argument as many times as necessary, we prove that ϕ is a smooth function of variable $x \in \mathbb{R}_+$ and all its derivatives have rapid decay as $x \to \infty$.

Let $g \in \mathscr{Z}_+(\mathbb{C})$. We will prove that the function $\phi = \mathscr{N}g : \mathbb{R}_+^\times \to \mathbb{C}$ has a Taylor expansion at 0, we will move the integration line to the left passing on the poles at $0,-1,-2,\ldots$. If $b_0, b_1, \ldots$ are residues of $g(s)$ at $0,-1,\cdots$, then the residues of $g(s)x^{-s}$ at $0,-1,-2,\ldots$ are $b_0, b_1x, b_2x^2,\ldots$. By applying the residues formula, we have

$$\phi(x) = \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} g(s)x^{-s}\mathrm{d}s = \sum_{i=0}^{n} b_i x^i + \frac{1}{2\pi i}\int_{\sigma'+i\mathbb{R}} g(s)x^{-s}\mathrm{d}s$$

for $\sigma > 0$ and $-n > \sigma' > -n-1$. As the function $g(\sigma'+it)$ of variable t is integrable, the above formula implies that

$$\phi(x) = \sum_{i=0}^{n} b_i x^i + O(x^{-\sigma'})$$

for every $\sigma' \in (-n,-n-1)$. It follows that ϕ has a Taylor expansion at 0 and it is equal to the formal series $\sum_{i=0}^{\infty} b_i x^i$.

If the Taylor expansion of $\phi \in \mathscr{S}_+(\mathbb{R}_+^\times)$ is $\sum_{i=0}^\infty a_i t^i$, then by reversing the above argument, we prove that $a_0, a_1, a_2, \ldots$ are the residues of $g = \mathscr{M}\phi$ at $0, -1, -2, \ldots$.

$\mathscr{M}$ and $\mathscr{N}$ are inverse of each other. This statement is nothing but the Fourier inversion formula after a change of variables. Let us identify $\mathbb{R} \to \mathbb{R}_+^\times$ with help of the exponential function $y \mapsto x = e^y$. Via this identification, the normalized multiplicative measure $\mu^\times$ on $\mathbb{R}_+^\times$ corresponds to the measure μ on $\mathbb{R}$. For every $\phi \in \mathscr{S}_+(\mathbb{R}_+^\times)$, for every $\sigma > 0$, the function $\phi(x)x^\sigma = \phi(e^y)e^{y\sigma}$ of variable $y \in \mathbb{R}$ is integrable with respect to the additive measure μ, and the restriction of $\mathscr{M}\phi$ to the vertical axis $\sigma + it$ is its Fourier transform:

$$\mathscr{M}\phi(\sigma + it) = \int_{\mathbb{R}} \phi(e^y)e^{y\sigma}e^{iyt}\mu(y).$$

Since for every polynomial $P(s)$ the function $P(s)\mathscr{M}\phi(s)$ is bounded on every vertical trips, the function $\mathscr{M}\phi(\sigma + it)$ over variable $t \in \mathbb{R}$ is integrable. We can thus apply the Fourier inversion formula

$$\phi(x) = \phi(e^y) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \mathscr{M}\phi(\sigma + it)e^{-y\sigma}e^{-iyt}\mu(t).$$

It follows that $\mathscr{N}\mathscr{M}\phi = \phi$. The other identity $\mathscr{M}\mathscr{N}g = g$ can be proved in the same way by using the Fourier inversion formula. □

Proposition 4.4. *The space of Schwartz functions $\mathscr{S}(\mathbb{R}_+^\times)$ is an algebra with respect to the convolution product:*

$$\phi_1 \star \phi_2(y) = \int_{\mathbb{R}_+^\times} \phi_1(x)\phi_2(x^{-1}y)\mu^\times(x). \tag{4.10}$$

For every $s \in \mathbb{C}$, we have

$$\mathscr{M}(\phi_1 \star \phi_2)(s) = \mathscr{M}\phi_1(s)\mathscr{M}\phi_2(s). \tag{4.11}$$

The space $\mathscr{S}_+(\mathbb{R}_+^\times)$ is a free module over $\mathscr{S}(\mathbb{R}_+^\times)$ with respect to the convolution product. The function e^{-x} is a generator of this module.

Proof. The Mellin transform of e^{-x} is the classical Gamma function:

$$\Gamma(s) = \int_{\mathbb{R}_+^\times} e^{-x}x^s\mu^\times(x). \tag{4.12}$$

The above integral is absolutely convergent for $\Re(s) > 0$ and determine Γ as a holomorphic function on this half-plane. The functional equation

$$\Gamma(s+1) = s\Gamma(s)$$

implies the existence of a meromorphic continuation of Γ to $\mathbb{C}$ with simple poles at non-positive integers.

In order to prove that e^{-x} is a generator of $\mathscr{S}_+(\mathbb{R}_+^\times)$ as module over $\mathscr{S}(\mathbb{R}_+^\times)$ with respect to the convolution product, we only need to prove that $\Gamma(s) \neq 0$ for all $s \in \mathbb{C} - \{0, -1, \dots\}$. This follows from Euler's reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{2i\pi}{e^{i\pi s} - e^{-i\pi s}}.$$

As the right hand side have only simple poles at $s \in \mathbb{Z}$, is non-vanishing in $s \in \mathbb{C} - \mathbb{Z}$, and $\Gamma(s)$ has simple poles at non-positive integers, Γ is non-vanishing everywhere where it is defined. □

Mellin transform over $\mathbb{R}^\times$

Since $\mathbb{R}^\times = \mathbb{R}_+^\times \times \{\pm 1\}$, $\Omega(\mathbb{R}^\times)$ has two connected components. The neutral component $\Omega^+(\mathbb{R}^\times)$ consists of characters of the form $\omega^s(x) = |x|^{-s}$ and the other component $\Omega^-(\mathbb{R}^\times)$ consists of characters of the form $\omega^s \mathrm{sgn}(x) = \mathrm{sgn}(x)|x|^{-s}$ with $s \in \mathbb{C}$.

Proposition 4.5. *For every $\phi \in \mathscr{S}(\mathbb{R})$, the integral*

$$\mathscr{M}\phi(\chi) = \int_{\mathbb{R}^\times} \phi(x)\chi(x)\mu^\times(x)$$

converges absolutely for $\Re(\chi) > 0$ and uniformly on compact sets in this domain.

Let $\mathscr{L}(\Omega(\mathbb{R}^\times))$ denote the space of meromorphic functions g on $\Omega(\mathbb{R}^\times)$ with at most simple poles at ω^{-2n} in $\Omega^+(\mathbb{R}^\times)$ and at $\omega^{-2n-1}\mathrm{sign}$ of the module in $\Omega^-(\mathbb{R}^\times)$, satisfying the boundedness on vertical trips as in Theorem 4.3. Then for every $\phi \in \mathscr{S}(\mathbb{R})$, the integral $\mathscr{M}\phi(\chi)$ well defined for $\Re(\chi) > 0$ has a meromorphic continuation on $\Omega(\mathbb{R}^\times)$ and gives rise to an element $g \in \mathscr{L}(\Omega(\mathbb{R}^\times))$.

For every $g \in \mathscr{L}(\Omega(\mathbb{R}))$, the integral along any vertical axes in $\Omega^+(\mathbb{R}^\times)$ and $\Omega^-(\mathbb{R}^\times)$

$$\phi(x) = \frac{1}{2i\pi}\int_{\Re(\chi)=\sigma} g(s)x^{-s}\mathrm{d}s$$

defines a function $\phi = \mathscr{N} g \in \mathscr{S}(\mathbb{R})$ for every $\sigma > 0$.

The maps $\mathscr{M}$ and $\mathscr{N}$ are inverse of each other.

Proof. The Mellin transform of even functions $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^\times)$ is supported on $\Omega^+(\mathbb{R}^\times)$ whereas the Mellin transform of odd functions is supported on $\Omega^-(\mathbb{R}^\times)$. We will decompose $\mathscr{S}(\mathbb{R})$ as a direct sum

$$\mathscr{S}(\mathbb{R}) = \mathscr{S}_+(\mathbb{R}) \oplus \mathscr{S}_-(\mathbb{R}) \tag{4.13}$$

where $\mathscr{S}_+(\mathbb{R})$ is the space of even Schwartz functions and $\mathscr{S}_-(\mathbb{R})$ is the space of odd Schwartz functions, and consider separately the Mellin transform of even and odd Schwartz functions. What we have to prove is a consequence of Theorem 4.3 applied to even and odd functions separately.

For $\phi \in \mathscr{S}_+(\mathbb{R})$, we have

$$\mathscr{M}\phi(\omega^s) = 2\int_{\mathbb{R}_+^\times} \phi(t)t^{s-1}\mu(t)$$

and $M\phi(\omega^s \mathrm{sgn}) = 0$. Similarly, for $\phi \in \mathscr{S}_-(\mathbb{R})$, we have

$$\mathscr{M}\phi(\omega^s \mathrm{sgn}) = 2\int_{\mathbb{R}_+^\times} \phi(t)t^{s-1}\mu(t)$$

and $\mathscr{M}\phi(\omega^s) = 0$ for all $s \in \mathbb{C}$.

Let us write a function $\phi \in \mathscr{S}(\mathbb{R})$ as $\phi = \phi_+ + \phi_-$ where ϕ_+ is an even Schwartz function and ϕ_- is an odd Schwartz function. We can now apply Theorem 4.3 to ϕ_+ and ϕ_- and deduce meromorphic continuations of $\mathscr{M}\phi_+$ and $\mathscr{M}\phi_-$. The statement on the location of poles of $\mathscr{M}\phi_+$ and $\mathscr{M}\phi_-$ follows from the trivial observation that the odd Taylor coefficients of the even Schwartz functions are zero, and so are the even Taylor coefficients of the odd Schwartz functions. □

Let us now pick the Gaussian basic function $b_\infty \in \mathscr{S}_+(\mathbb{R})$ given by

$$b_\infty(t) = e^{-\pi t^2}.$$

By making simple change of variable, we calculate its Mellin transform

$$\int_{\mathbb{R}_+^\times} \phi(t)t^s \frac{\mu(t)}{t} = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$

so that $\mathscr{M}b_\infty$ is the meromorphic function on $\Omega(\mathbb{R}) = \Omega^0(\mathbb{R}) \sqcup \Omega^{\mathrm{sign}}(\mathbb{R})$ given by the formulas

$$\mathscr{M}b_\infty(\omega^s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

It is also clear that $\mathscr{M}b_\infty(\omega^s \mathrm{sign}) = 0$ because of the parity.

Mellin transform over non-archimedean local fields

Let F be a non-archimedean local field of ring of integers $\mathscr{O}_F$. We denote by $\mathscr{S}(F)$ the space of locally constant and compactly supported functions on F. We will consider the Mellin integral

$$\mathscr{M}\phi(\chi) = \int_{F^\times} \phi(x)\chi(x)\mu^\times(x), \tag{4.14}$$

$\mu^\times$ being the invariant measure on $F^\times$ normalized so that $\mathrm{vol}(\mathscr{O}_F^\times, \mu^\times) = 1$, as a function of $\chi \in \Omega(F^\times)$.

Recall that the module character $F^\times \to \mathbb{R}_+$ can be given by the formula $|x| = q^{-\mathrm{val}(x)}$ where q is the cardinal of the residue field. Its image is the discrete subgroup of $\mathbb{R}_+^\times$ generated by q. Let us denote $\Omega^0(F^\times)$ the group of unramified characters $\chi : F^\times \to \mathbb{C}^\times$ i.e those whose restriction to $\mathscr{O}_F^\times$ is trivial. We have an isomorphism

$$\Omega^0(F^\times) \simeq \mathbb{C}^\times$$

by assigning to $t \in \mathbb{C}^\times$, the unique unramified character $\chi_t : F^\times \to \mathbb{C}^\times$ satisfying $\chi_t(\varpi_F) = t$. We have $\chi_t = \omega^s$ if $t = q^{-s}$.

The choice of an uniformizing parameter ϖ_F of F induces an isomorphism

$$F^\times = \mathscr{O}_F^\times \times \varpi_F^{\mathbb{Z}}.$$

It follows that

$$\Omega(F^\times) = \Omega(\mathscr{O}_F^\times) \times \Omega(\mathbb{Z})$$

where $\Omega(\mathbb{Z}) = \Omega^0(F^\times) = \mathbb{C}^\times$ and $\Omega(\mathscr{O}_F^\times) = \Lambda(\mathscr{O}_F^\times)$ is a discrete subgroup, union of finte subgroups

$$\Lambda(\mathscr{O}_F^\times) = \bigcup_{n\in\mathbb{N}} \Lambda((\mathscr{O}_F/\varpi_F^n)^\times).$$

The splitting allows us to identify every component of $\Omega(F^\times)$ with $\mathbb{C}^\times$. A function g on $\Omega(F^\times)$ is said to be polynomial if it vanishes on all but finitely many connected components. Whereas a function g on $\Omega(F^\times)$ is polynomial or not, obviously doesn't depend on the choice of uniformizing parameter ϖ_F.

Proposition 4.6. *For a function $\phi \in \mathscr{C}_c^\infty(F^\times)$, the integral* (4.14) *converges absolutely and $\mathscr{M}\phi$ defines a polynomial function g on $\Omega(F^\times)$. Conversely, if g is a polynomial function on $\Omega(F^\times)$, the integral*

$$\phi(x) = \frac{1}{2i\pi} \int_{|t|=q^{-\sigma}} g(t)\chi_t(x)^{-1}\mu(t), \tag{4.15}$$

which is independent of $\sigma \in \mathbb{R}$, defines a function $\phi = \mathcal{N} g$ in $\mathscr{C}_c^\infty(F^\times)$. The maps $\mathcal{M}$ and $\mathcal{N}$ are inverse of each other.

Proof. Let $\phi \in \mathscr{C}_c^\infty(F^\times)$ be a locally constant function with compact support in $F^\times$. For every $x \in F$, there exists a compact open subgroup K_x of $F^\times$ such that ϕ is constant on xK_x. Since the support of ϕ is compact, it is a finite union of cosets xK_x. In particular, ϕ is invariant under some compact open subgroup K of $F^\times$ and can be expressed as a finite linear combination of characteristic functions of cosets of K. The integral (4.14) can be written as a finite linear combination of integrals over cosets of K. Each of these integrals vanishes unless χ is trivial on K. In other words, $\mathcal{M}\phi$ vanishes outside $\mathbb{C}^\times \times \Lambda(\mathcal{O}_F^\times/K)$.

The support of ϕ is contained in a compact set of the form $m \leq \mathrm{val}(x) \leq n$. On each component of $\mathbb{C}^\times \times \Lambda(\mathcal{O}_F^\times/K)$, $\mathcal{M}\phi$ is a linear combination of $\{t^i | m \leq i \leq n\}$. It follows that $\mathcal{M}\phi$ is a polynomial function on $\Omega(F^\times)$. □

Proposition 4.7. *For every $\phi \in \mathscr{S}(F)$, the integral* (4.14) *converges absolutely in the domain $|t| < 1$. Let $\mathscr{L}(\Omega(F^\times))$ denote the space of rational functions on $\Omega(F^\times)$ with at most a simple pole at $\chi = 1$. Then for every $\phi \in \mathscr{S}(F)$, $\mathcal{M}\phi$ has a meromorphic continuation $g \in \mathscr{L}(\Omega(F^\times))$. Conversely, if $g \in \mathscr{L}(\Omega(F^\times))$, for $\sigma > 0$, the integral*

$$\phi(x) = \frac{1}{2i\pi} \int_{|t|=q^{-\sigma}} g(t)\chi_t(x)^{-1}\mu(t), \tag{4.16}$$

being the sum of integrals over circles of radius $q^{-\sigma}$ in every component of $\Omega(F^\times)$, defines a function $\phi = \mathcal{N} g$ in $\mathscr{C}_c^\infty(F^\times)$. The maps $\mathcal{M}$ and $\mathcal{N}$ are inverse of each other.

Proof. The basic function $b_F = 1_{\mathcal{O}_F}$ is invariant under the maximal compact subgroup $\mathcal{O}_F^\times$. Its Mellin transform $\mathcal{M}\phi$ vanishes on all component except the unramified component $\Omega^0(F^\times)$. In the open disc $|t| < 1$, the integral (4.14) converges to

$$\sum_{i=0}^{\infty} t^n = (1-t)^{-1}$$

This function can be extended as a rational function on $\mathcal{O}_F^\times$ with a simple pole at $t = 1$.

For every $\phi \in \mathscr{S}(F)$ can be expressed in the form $\phi = c b_F + \phi'$ where $c = \phi(0)$ is a constant and $\phi' \in \mathscr{C}_c^\infty(F^\times)$. It follows that the Mellin transform $\mathcal{M}\phi$ is defined as a convergent series on all components other than the unramified component, and on the unramified component it is absolutely convergent on the disc $|t| < 1$. It also implies that the Mellin transform $\mathcal{M}\phi$ can be meromorphically continued as a rational function of the form $c(1-t)^{-1} + \mathcal{M}\phi'$ where $\mathcal{M}\phi'$ is a polynomial function on $\Omega(F^\times)$. □

Eigendistributions and γ-function

The multiplicative group $F^\times$ acts on F by dilation $(t,x) \mapsto tx$. It acts on functions on F by $\phi \mapsto \phi_t$ where $\phi_t(x) = \phi(t^{-1}x)$. We have equivalent formulas

$$\phi_t(x) = \phi(t^{-1}x) \Leftrightarrow \phi_t(tx) = \phi(x).$$

If f is a Schwartz (or Schwartz-Bruhat) function then so is f_t, or in other words $F^\times$ acts on $\mathscr{S}(F)$. By duality, $F^\times$ acts on the space $\mathscr{S}'(F)$ of tempered distributions. The action $a \mapsto a_t$ is given by equivalent formulas

$$\langle a_t, \phi\rangle = \langle a, \phi_{t^{-1}}\rangle \Leftrightarrow \langle a_t, \phi_t\rangle = \langle a, \phi\rangle.$$

In particular, for Dirac distributions, we have $(\delta_x)_t = \delta_{tx}$.

For every character $\chi : F^\times \to \mathbb{C}^\times$, we consider the space of tempered distributions which are χ-eigenvectors with respect to the dilation action of $F^\times$ on $\mathscr{S}'(F)$:

$$\mathscr{S}'(F)^\chi = \{a \in \mathscr{S}'(F) | \forall t \in F^\times, a_t = \chi(t)a\}. \tag{4.17}$$

Examples of such distributions can be constructed out of characters themselves under an assumption on the exponent.

Proposition 4.8. *A character $\chi : F^\times \to \mathbb{C}^\times$ extended by 0 as a function $\chi : F \to \mathbb{C}$ is locally integrable if $\Re(\chi) > -1$. The distribution χ given by the formula*

$$\phi \mapsto \int_F \chi(x)\phi(x)\mu(x)$$

for compactly supported smooth function is tempered i.e. also makes sense for $\phi \in \mathscr{S}(F)$. It is an eigenvector for the action of $F^\times$ on $\mathscr{S}'(F)$ of eigenvalue $\chi^{-1}\omega^{-1}$ i.e.

$$\chi \in \mathscr{S}'(F)^{\chi^{-1}\omega^{-1}}$$

where $\omega^{-1}(x) = |x|^{-1}$ is the inverse of the module character. As tempered distribution, the Fourier transform $\hat{\chi}$ is also an eigendistribution

$$\hat{\chi} \in \mathscr{S}'(F)^\chi.$$

Proof. One can check case by case that χ extended by zero as a function $F \to \mathbb{C}$ is locally integrable under the assumption $\Re(\chi) > -1$. For instance, in the real case, this boils down to the finiteness of the definite integral

$$\int_0^1 t^\sigma \mu(t) = (\sigma+1)^{-1}.$$

We split the integral into two domains

$$\int_{F^\times} \phi(x)\chi(x)\mu(x) = \int_{|x|\le 1} \phi(x)\chi(x)\mu(x) + \int_{|x|>1} \phi(x)\chi(x)\mu(x).$$

On the compact domain $|x| \le 1$, the integral $\int_{|x\le 1|} \phi(x)\chi(x)\mu(x)$ is well defined since χ is locally integrable. On the non-compact domain $|x| > 1$, the integral is absolutely convergent as $f(x)$ is of rapid decay as $|x| \to \infty$, and certainly decay faster than $|x|^{-\Re(s)-1-\epsilon}$.

Provided that the integral

$$\langle \chi, \phi \rangle = \int_{F^\times} f(x)\chi(x)\mu(x)$$

converges, the effect of the dilation can be easily calculated:

$$\langle \chi, \phi_{t^{-1}} \rangle = \int_{F^\times} \phi(tx)\chi(x)\mu(x) = \chi(t)^{-1}|t|^{-1}\langle \chi, \phi \rangle.$$

It follows that $\chi\mu^\times$ is an eigenvector for the action of $F^\times$ on $\mathscr{S}'(F)$ with respect to the character $t \mapsto \chi(t)^{-1}|t|^{-1}$.

If $\Re(\chi) > -1$, χ is a tempered distribution and so is its Fourier transform. The fact that the Fourier transform of χ is an eigenvector for the action of $F^\times$ with respect to the character χ is to be derived from the following proposition. □

Proposition 4.9. *For every $t \in F^\times$ and $\phi \in \mathscr{S}(F)$, we have*

$$\hat{\phi}_t = |t|\widehat{\phi_{t^{-1}}}. \tag{4.18}$$

For every tempered distribution $a \in \mathscr{S}^\times(F)$, we have

$$\hat{a}_t = |t|^{-1}\widehat{a_{t^{-1}}}. \tag{4.19}$$

Proof. The calculation needed to prove these equalities are completely straightforward. We will write it down only for the sake of record.

We calculate the effect of dilation by t on the Fourier transform $\hat{\phi}$ of a Schwartz function $\phi \in \mathscr{S}(F)$ by writing as the Fourier integral on a variable y and then using the change of variable $z = t^{-1}y$:

$$\begin{aligned}
\hat{\phi}_t(x) &= \hat{\phi}_1(t^{-1}x) \\
&= \int_F \phi(y)\mathrm{e}_1(t^{-1}xy)\mu(y) \\
&= |t| \int_F \phi_1(tz)\mathrm{e}_1(xz)\mu(z) \\
&= |t|\widehat{\phi_{t^{-1}}}(x).
\end{aligned}$$

The equality (4.18) follows.

We calculate the effect of dilation by t on the Fourier transform $\hat{a}$ of a tempered distribution $a \in \mathscr{S}'(F)$ by coupling it with a test function:

$$\begin{aligned}
\langle \hat{a}_t, \phi \rangle &= \langle \hat{a}, \phi_{t^{-1}} \rangle \\
&= \langle a, \widehat{\phi_{t^{-1}}} \rangle \\
&= \langle a, |t|^{-1}\hat{\phi}_t \rangle \\
&= |t|^{-1}\langle \widehat{a_{t^{-1}}}, \phi \rangle.
\end{aligned}$$

The equality (4.19) follows. □

Proposition 4.10. *For every character $\chi : F^\times \to \mathbb{C}^\times$, the space of eigendistributions $\mathscr{S}'(F)^\chi$ is one-dimensional.*

Proof. For the sake of notation clarity, we fix a character $\chi_1 \in \Omega(F^\times)$ and use the letter $\chi \in \Omega(F^\times)$ to denote a variable character. Let $a \in \mathscr{S}'(F)^{\chi_1}$. For every $t \in F^\times$ and $\phi \in \mathscr{S}(F)$, we have $a_t(\phi) = \chi_1(t)a(\phi)$. In other words the linear form a on $\mathscr{S}(F)$ annihilate functions of the form $\phi_{t^{-1}} - \chi_1(t)\phi$. Using the isomorphism $\mathscr{M} : \mathscr{S}(F) \to \mathscr{L}(\Omega(F^\times))$, we can see a as a linear form annihilating functions of the form

$$\mathscr{M}(\phi_{t^{-1}} - \chi_1(t)\phi)(\chi) = (\chi(t) - \chi_1(t))\mathscr{M}\phi(\chi).$$

Conversely, every linear form on $\mathscr{L}(\Omega(F^\times))$ annihilating the submodule generated by function $\chi \mapsto (\chi - \chi_1)g(\chi)$ for $\chi \in \mathscr{L}(\Omega(F^\times))$ gives rise to an element of $\mathscr{S}'(F)^{\chi_1}$. The proposition follows from the equality

$$\dim_{\mathbb{C}}\left(\mathscr{L}(\Omega(F^\times))/(\chi(t) - \chi_1(t))\mathscr{L}(\Omega(F^\times))\right)$$

which can be checked directly upon the definition of $\mathscr{L}(\Omega(F^\times))$.

For instance, if χ_1 does not lie in the set of prescribed poles, a linear form on $\mathscr{L}(\Omega(F^\times))$ annihilating functions of the form $\chi \mapsto (\chi(t)-\chi_1(t))g(\chi)$ must be a multiple of the linear form $g \mapsto g(\chi_1)$. If χ_1 lies in the set of prescribed poles, instead of evaluating g at χ_1, we must take the residue of g at χ_1. $\square$

According to Proposition 4.8, for $\Re(\chi) > -1$, we have a non-zero elements

$$\chi \in \mathscr{S}'(F)^{\chi^{-1}\omega^{-1}}, \hat{\chi} \in \mathscr{S}'(F)^{\chi}.$$

Thus for $-1 < \Re(\chi) < 0$, we thus have two non-zero vectors χ' and $\hat{\chi}$ of $\mathscr{S}'(F)^\chi$ where $\chi' = \chi^{-1}\omega^{-1}$. There exists a unique non-zero scalar $\gamma(\chi\omega) \in \mathbb{C}^\times$ such that

$$\hat{\chi} = \gamma(\chi\omega)\chi'. \tag{4.20}$$

Let us unravel this definition to give an integral representation to $\gamma(\chi)$. Formally, we can rewrite (4.20) as

$$\gamma(\chi\omega)\chi'(y) = \int_F \chi(x)\mathrm{e}(xy)\mu(x).$$

Setting $y = 1$, and writing χ for $\chi\omega$ we get

$$\begin{aligned} \gamma(\chi) &= \int_F \mathrm{e}(x)\chi(x)|x|^{-1}\mu(x) && (4.21)\\ &= \int_{|x|\le 1} \mathrm{e}(x)\chi(x)|x|^{-1}\mu(x) + \int_{|x|>1} \mathrm{e}(x)\chi(x)|x|^{-1}\mu(x). && (4.22)\end{aligned}$$

The integral over the region $|x| \le 1$ converges under the assumption $\Re(\chi) > 0$ and the integral over $|x| > 1$ converges under the assumption that $\Re(\chi) < 1$. Thus (4.21) gives rise to a convergent integral representation of the function γ in the region $-1 < \Re(\chi) < 0$, which also shows that it is holomorphic on this region. Each integral $\int_{|x|\le 1}$ and $\int_{|x|>1}$ can be meromorphically continued, and therefore γ also admits a meromorphic continuation to $\Omega(F^\times)$.

By unraveling (4.20) using an arbitrary test function ϕ we get the formula

$$\mathscr{M}\hat{\phi}(\chi) = \gamma(\chi)\mathscr{M}\phi(\chi^{-1}\omega^1) \tag{4.23}$$

where again the Mellin integrals $\mathscr{M}\hat{\phi}(\chi)$ converges for $\Re(\chi) > 0$ and $\mathscr{M}\phi(\chi^{-1}\omega^1)$ converges for $\Re(\chi) < 1$. This gives rise to a convergent integral representation of γ in the region $0 < \Re(\chi) < 1$ after choosing an arbitrary test function ϕ. This also gives rise to meromorphic continuation of $\gamma(\chi)$ as both Mellin integrals have meromorphic continuation.

Let us calculate the γ-function for different local fields:

- For $F=\mathbb{R}$, we calculate γ on the neutral component $\Omega^0(\mathbb{R}^\times)$ of $\Omega(\mathbb{R}^\times)$:

$$\gamma(\omega^s)=\int_{\mathbb{R}}\exp(2i\pi x)x^{s-1}\mu(x)$$

which is similar to but different from the classical Γ-function

$$\Gamma(s)=\int_{\mathbb{R}^+}\exp(-x)x^{s-1}\mu(x).$$

Let us pick for test function the Gaussian function $b_\infty(x)=e^{-\pi x^2}$ which is its own Fourier transform. We have also calculated its Mellin transform

$$\mathscr{M}b_\infty(\omega^s)=\pi^{-s/2}\Gamma(s/2).$$

The formula (4.23) implies then

$$\gamma(\omega^s)=\frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(1-s)/2}\Gamma((1-s)/2)}=2^{1-s}\pi^{\frac{s}{2}}\cos\left(\frac{\pi s}{2}\right)\Gamma(s).$$

- For $F=\mathbb{Q}_p$, we calculate γ on the neutral component $\Omega^0(F^\times)=\mathbb{C}^\times$ by picking for test function $b_p=1_{\mathbb{Z}_p}$ which is its own Fourier transform. We have also calculate the Mellin transform of b_p

$$\mathscr{M}b_p(\chi_t)=(1-t)^{-1}.$$

The formula (4.23) implies then

$$\gamma(\chi_t)=\frac{1-p^{-1}t^{-1}}{1-t};$$

or in other words for $t=p^{-s}$ we have

$$\gamma(\omega^s)=\frac{1-p^{s-1}}{1-p^{-s}}.$$

5 Zeta integrals

Adelic multiplicative measures

Recall that for every local field F, we have defined a preferred additive character $\mathrm{e}_1^F:F\to\mathbb{C}^1$ which gives rise to a canonical self-dual Haar measure μ. For every continuous function with compact support in $F^\times$ can be extended by zero and defines a continuous function with compact support in F, μ defines a Radon measure on $F^\times$. We claim that for $\omega^{-1} : F^\times \to \mathbb{C}^\times$ being the inverse of the module character, $\omega^{-1}\mu$ defines an invariant measure on $F^\times$. Indeed, for every $t \in F^\times$, we denote $\phi_t(x) = \phi(t^{-1}x)$, then the equality

$$\int_{F^\times} \phi(t^{-1}x)|x|^{-1}\mu(x) = \int_{F^\times} \phi(y)|y|^{-1}\mu(y)$$

holds by the virtue of the change of variable $y = t^{-1}x$.

If $F = \mathbb{Q}_p$ is the field of p-adic number, the volume of $\mathbb{Z}_p$ with respect to the self-dual Haar measure μ in one. It follows that the measure of the compact open subset $\mathbb{Z}_p^\times$ has volume

$$\mathrm{vol}(\mathbb{Z}_p^\times, \mu) = 1 - p^{-1}.$$

Since $\mathbb{Z}_p^\times$ is precisely the set of $x \in \mathbb{Q}_p$ of module one, it also has the volume

$$\mathrm{vol}(\mathbb{Z}_p^\times, \omega^{-1}\mu) = \frac{p-1}{p}$$

$(p-1)/p$ with respect to the invariant measure $\omega^{-1}\mu$ of $\mathbb{Q}_p$. More generally, we record the fact:

Proposition 5.1. *If F is is a non-archimedean local field, we have*

$$\mathrm{vol}(\mathcal{O}_F^\times, \omega^{-1}\mu) = (1 - q^{-1})\mathrm{discr}_{F/\mathbb{Q}_p}^{-1/2}$$

where q is the cardinality of the residue field of F, and $\mathrm{discr}_{F/\mathbb{Q}_p}$ *is the discriminant.*

Proof. This is the combination of two identities

$$\mathrm{vol}(\mathcal{O}_F^\times, \omega^{-1}\mu) = \frac{q-1}{q}\mathrm{vol}(\mathcal{O}_F, \mu)$$

and

$$\mathrm{vol}(\mathcal{O}_F, \mu) = \mathrm{discr}_{F/\mathbb{Q}_p}^{-1/2}$$

the latter one being established in ... □

This fact creates a difficulty to normalize Haar measure on the group of idèles $\mathbb{A}^\times$ using self-dual Haar measure μ on $\mathbb{A}$. For instance, for $k = \mathbb{Q}$, we note that the infinite product

$$\prod_{p\in\mathscr{P}}(1-p^{-1})$$

converges to 0. As a result, the product of measure $\omega^{-1}\mu$ on the multiplicative group of non-archimedean local field $F^\times$ gives rise to the measure zero on $\mathbb{A}_{\text{fin}}^\times$. A related and equivalent fact is that $\mathbb{A}_{\text{fin}}^\times$ is a subset of measure 0 in $\mathbb{A}_{\text{fin}}$, where $\mathbb{A}_{\text{fin}}$ is the ring of finite adèles.

To overcome this inconvenient truth, we will normalize multiplicative Haar measures as follows:

- If F is a non-archimedean local field, we denote $\mu^\times$ the unique Haar measure on $F^\times$ such that the maximal compact open subgroup $\mathscr{O}_F^\times$ of invertible integers has volume one. In other words, for $F = \mathbb{Q}_p$ we have

  $$\mu^\times = (1-p^{-1})^{-1}\omega^{-1}\mu.$$

  and more generally for every non-archimedean local field F we have

  $$\mu^\times = (1-q^{-1})\text{discr}_{F/\mathbb{Q}_p}^{-1/2}\omega^{-1}\mu \tag{5.1}$$

  where q is the cardinality of the residue field of F, and $\text{discr}_{F/\mathbb{Q}_p}$ is the discriminant.

- If F is an archimedean local field, we set

  $$\mu^\times = \omega^{-1}\mu. \tag{5.2}$$

Although this normalization of Haar measure on multiplicative groups lacks the elegance of self-dual Haar measure on additive groups, it will give rise of arithmetic interest.

We claim that local multiplicative measures $\mu^\times$ normalized as above gives rise to a Haar measure $\mu_{\mathbb{A}}$ on the group of idèles $\mathbb{A}^\times$. As $\mathbb{A}^\times = \mathbb{A}_{\text{fin}}^\times \times k_{\mathbb{R}}^\times$. A Haar measure $\mu^\times$ on $\mathbb{A}^\times$ can be decomposed as a product $\mu^\times = \mu_{\text{fin}}^\times \times \mu_{k_R}^\times$ where $\mu_{\text{fin}}^\times$ is a Haar measure on $\mathbb{A}_{\text{fin}}^\times$ and $\mu_{k_R}^\times$ is a Haar measure on $k_R^\times$.

Lemma 5.2. *Let G be a totally disconnected locally compact abelian group, and H a compact open subgroup of G. Then the measure of H with respect to any Haar measure of G is positive.*

Proof. We will prove that if H has measure zero, then every compact set K in G has measure zero, which would imply that the measure is degenerate. By replacing K with KH which is also compact, we may assume that K is invariant under H-translation, in other words, K is a disjoint union of H-cosets. Since H is open, all H-cosets are open subset of G. Since K is compact, and is a union of H-cosets, it is a finite union of H-cosets. If H is of measure zero, K is also of measure zero. □

If we apply this lemma to $G = \mathbb{A}_{\text{fin}}^\times$, for every Haar measure $\mu_{\text{fin}}^\times$ on this group, the measure of the maximal compact open subgroup $\prod_{v\in\mathscr{P}_k} \mathscr{O}_v^\times$ is positive. We may normalize it by imposing

$$\mathrm{vol}\left(\prod_{v\in\mathscr{P}_k} \mathscr{O}_v^\times, \mu_{\text{fin}}^\times\right) = 1$$

in which case $\mu_{\text{fin}}^\times$ is the product of normalized local measures on $F_v^\times$ as in formula (5.1).

On $k_{\mathbb{R}}^\times$ we can normalize Haar measure as the finite product of normalized local measures $\omega^{-1}\mu$, as many as archimedean places of k, as in formula (5.2). We thus obtain a Haar measure on $\mathbb{A}^\times$ as the product of normalized local Haar measures

$$\mu_{\mathbb{A}}^\times = \prod_{v\in V} \mu_v^\times \tag{5.3}$$

where $\mu_v^\times$ is the invariant measure on $k_v^\times$ defined by formulas (5.1) and (5.2).

Normalized Haar measure on $\mathbb{A}^\times$ yields some constant of arithmetic interest. Recall that for every number field k, we have an exact the module character gives rise to an exact sequence

$$0 \to \mathbb{A}^1 \to \mathbb{A}^\times \to \mathbb{R}_+^\times \to 0.$$

With a splitting provided by a choice of an archimedean place of k, we have

$$\mathbb{A}^\times = \mathbb{A}^1 \times \mathbb{R}_+^\times.$$

There exists a unique Haar measure $\mu_{\mathbb{A}}^1$ on $\mathbb{A}^1$ such that

$$\mu_{\mathbb{A}}^\times = \mu_{\mathbb{A}}^1 \times \mu_{\mathbb{R}}^\times$$

where $\mu_{\mathbb{R}}^\times$ is defined as in (5.2). According to theorem 1.28, $k^\times$ is a discrete cocompact subgroup of $\mathbb{A}^1$. The invariant measure μ_A^1 on $\mathbb{A}^1$ induces an invariant measure $\bar{\mu}_{\mathbb{A}}^1$ on its compact quotient $\mathbb{A}^1/k^\times$ and one may ask what is the volume of $\mathbb{A}^1/k^\times$ with respect to that measure. The answer is that the volume is equal to the constant appearing in Dirichlet's analytic class number formula.

Proposition 5.3. *The volume of* $\mathbb{A}^1/k^\times$ *is given by the formula:*

$$\mathrm{vol}(\mathbb{A}^1/k^\times, \bar{\mu}^1_{\mathbb{A}}) = \frac{2^{r_1}(2\pi)^{r_2} h R}{\sqrt{|d|} w}$$

where r_1 *is the number of real places of* k*,* r_2 *the number of complex places,* h *is its class number,* R *its regulator, and* w *the number of roots of unit in* k*.*

Proof. This formula is to be derived from the exact sequence (1.36). □

Characters of idèles classes

Let us first consider the case of rational numbers $\mathbb{Q}$ and its ring of adèles $\mathbb{A}$. Let $\Omega_{\mathbb{Q}}$ denote the group of characters of $\mathbb{A}^\times/\mathbb{Q}^\times$. According to explicit description of the idèle class group $\mathbb{A}^\times/\mathbb{Q}^\times = \hat{\mathbb{Z}}^\times \times \mathbb{R}_+$, $\Omega_{\mathbb{Q}}$ is the direct product of the Pontryagin dual of $\hat{\mathbb{Z}}^\times$ and $\Omega(\mathbb{R}_+) = \mathbb{C}$, in other words,

$$\Omega_{\mathbb{Q}} = \Lambda(\hat{\mathbb{Z}}^\times) \times \mathbb{C}.$$

Every element $\hat{\mathbb{Z}}^\times \to \mathbb{C}^1$ factorizes through a character of a finite quotient $(\mathbb{Z}/m\mathbb{Z})^\times$, in other words the group $\Lambda(\hat{\mathbb{Z}}^\times)$ is the injective limit of $\Lambda((\mathbb{Z}/m\mathbb{Z})^\times)$ for m running over the set of nonzero integers ordered by divisibility. A character $\chi : \hat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ factorizing through $\chi_m : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^1$ for monimal m is said to be a primitive Dirichlet character modulo m. Thus elements of $\Omega_{\mathbb{Q}}$ are pairs $\chi = (\chi_m, s)$ where χ_m is a primitive Dirichlet character modulo m and s is a complex number. The real part function of χ is given by the formula $\Re(\chi) = \Re(s)$.

If $\mathbb{A}$ is the ring of adèles of a number field k, the kernel of $\omega^1_{\mathbb{A}} : \mathbb{A}^\times/k^\times \to \mathbb{R}^\times_+$ is the compact group $\mathbb{A}^\times/k^\times$ by Theorem 1.28. We have a splitting $\mathbb{A}^\times/k^\times = \mathbb{A}^1/k^\times \times \mathbb{R}^\times_+$ where a section of $\omega^1_{\mathbb{A}} : \mathbb{A}^\times/k^\times \to \mathbb{R}^\times_+$ can be produced upon the choice of an archimedean place of k. Let us denote $\Omega_k = \Omega(\mathbb{A}^\times/k^\times)$. We have then

$$\begin{aligned} \Omega_k &= \Omega(\mathbb{A}^1/k^\times) \times \Omega(\mathbb{R}^\times_+) \\ &= \Lambda(\mathbb{A}^1/k^\times) \times \mathbb{C} \end{aligned}$$

for all characters of the compact group $\mathbb{A}^1/k^\times$ are necessarily unitary, and the group $\Lambda(\mathbb{A}^1/k^\times)$ being the Pontryagin dual of a compact group, is a discrete group.

Zeta integrals

Let k be a number field and $\mathbb{A}$ its ring of adèles. The space $\mathscr{S}(\mathbb{A})$ of Schwartz functions on $\mathbb{A}$ consists in finite linear combinations of functions of the form

$$\phi = \bigotimes_{v\in|k|} \phi_v$$

where $\phi_v \in \mathscr{S}(k_v)$ is a Schwartz function which is equal to $1_{\mathscr{O}_v}$ for almost all v.

Proposition 5.4. *For $\phi \in \mathscr{S}(\mathbb{A})$ and character $\chi \in \Omega_k$ with exponent $\Re(\chi) > 1$, the integral*

$$\mathscr{M}\phi(\chi) = \int_{\mathbb{A}^\times} \phi(x)\chi(x)\mu^\times(x) \tag{5.4}$$

with respect to the normalized Haar measure $\mu^\times$ of $\mathbb{A}^\times$, converges absolutely. It defines a holomorphic function of variable χ in the domain $\Re(\chi) > 1$ satisfying the identity

$$\mathscr{M}\phi(\chi) = \prod_v \mathscr{M}\phi_v(\chi_v) \tag{5.5}$$

if $\phi = \bigotimes_v \phi_v$ and $\chi_v : k_v^\times \to \mathbb{C}^\times$ being the restriction of χ to $k_v^\times$.

Proof. Let us denote $\sigma = \Re(\chi) > 1$. We need to prove that the integral

$$\int_{\mathbb{A}^\times} |\phi(x)||x|^\sigma \mu^\times(x)$$

is absolutely convergent. Assume that $\phi = \bigotimes_{v\in|k|} \phi_v$ where $\phi_v = 1_{\mathscr{O}_v}$ for all $v \in |k|-S$ where S is a finite subset of $|k|$. For all $v \in S$, we have

$$\int_{k_v^\times} |\phi_v(x)||x|^\sigma \mu^\times(x) = (1-q_v^{-\sigma})^{-1}.$$

The assumption $\sigma > 1$ guarantee that the infinite product

$$\prod_{v\in|k|-S} (1-q_v^{-\sigma})^{-1}$$

is convergent. It follows that the function $\phi(x)\chi(x)$ is integrable over $\mathbb{A}^\times$ as long as $\sigma = \Re(\chi) > 1$. We derive from the absolute convergence of the integral (5.4) that the function $\mathscr{M}\phi(\chi)$ of variable χ in the domain $\sigma > 1$ is holomorphic and satisfied the product formula (5.5) every time ϕ is a pure tensor. $\square$

Proposition 5.5. *For every $x \in \mathbb{A}^\times$, let us denote $\bar{x}$ its image by the quotient map $\mathbb{A}^\times \to \mathbb{A}^\times/k^\times$. For every $\phi \in \mathscr{S}(\mathbb{A})$, the series*

$$\phi_+(\bar{x}) = \sum_{\alpha \in k^\times} f(\alpha x)$$

is absolutely convergent. It defines a function $\phi_+ : \mathbb{A}^\times/k^\times \to \mathbb{C}$ of rapid decay as $|x| \to \infty$ i.e for every $\sigma > 0$, the function $\phi_+(x)|x|^\sigma$ is bounded.

Proof. Let us prove that this summation is absolutely convergent for $k = \mathbb{Q}$. Recall that for all p, ϕ_p is a locally constant functions with compact support in $\mathbb{Q}_p$, and for almost all p, $\phi_p = 1_{\mathbb{Z}_p}$ for all p. It follows there exists $m_x \in \mathbb{N}$ such that $f(\alpha x) \neq 0$ implies $m_x\alpha \in \mathbb{Z}$. Moreover m_x can be made locally constant with respect to x. What we needs to prove comes to the same as $\sum_{m_x\alpha \in \mathbb{Z}-\{0\}} \phi_\infty(\alpha x_\infty)$ is absolutely convergent that we knows because ϕ_∞ is a Schwartz function. The same argument shows that

$$\phi_+(\bar{x}) = O(|\bar{x}|^{-N})$$

as $|\bar{x}| \to \infty$ for every integer $N \in \mathbb{Z}$. □

For every character $\chi \in \Omega_{\mathbb{Q}}$, we can break the integral $\int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi_+(\bar{x})\chi(\bar{x})\mathrm{d}\bar{x}$ into two parts:

$$\int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi_+(\bar{x})\chi(\bar{x})\mathrm{d}\bar{x} = \int_{|\bar{x}|\geq 1} \phi_+(\bar{x})\chi(\bar{x})\mathrm{d}\bar{x} + \int_{|\bar{x}|<1} \phi_+(\bar{x})\chi(\bar{x})\mathrm{d}\bar{x}. \tag{5.6}$$

Because of ϕ_+ decays rapidly as $|x| \to \infty$, the integral over the region $|\bar{x}| \geq 1$ is absolutely convergent for all $\chi \in \Omega_{\mathbb{Q}}$. The integral over the other region $\int_{|\bar{x}|<1}$ is absolutely convergent as long as $\Re(\chi) > 1$.

Let $\hat{\phi}$ denote the Fourier transform of ϕ. For every $x \in \mathbb{A} - \{0\}$, the Poisson summation formula

$$\sum_{\alpha \in \mathbb{Q}} \phi(\alpha x) = |x|^{-1} \sum_{\alpha \in \mathbb{Q}} \hat{\phi}(\alpha x^{-1})$$

holds. It follows that for every $x \in \mathbb{A}^\times$, we have

$$\phi_+(x) = -\phi(0)|x|^{-1}(\hat{\phi}_+(x^{-1}) + \hat{\phi}(0)).$$

Plugging this identity into the integral over $\int_{|\bar{x}|<1} \phi_+(\bar{x})\chi(\bar{x})$, we get

$$-\phi(0)\int_{|\bar{x}|<1} \chi(\bar{x})\mathrm{d}\bar{x} + \hat{\phi}(0)\int_{|\bar{x}|<1} |x|^{-1}\chi(\bar{x})\mathrm{d}\bar{x} + \int_{|\bar{x}|>1} \hat{\phi}_+(\bar{x})\chi^{-1}(\bar{x})\mathrm{d}\bar{x}.$$

The third term in this sum converges absolutely for all $\chi \in \Omega$ since $\hat{f}$ also decays rapidly as $|x| \to \infty$. The first two terms are directly computable. If we write $\mathbb{A}^\times/\mathbb{Q}^\times$ as $\hat{\mathbb{Z}} \times \mathbb{R}_+$ and $\chi = (\bar{\chi}, s)$, the integral $\int_{|\bar{x}|<1} \chi(\bar{x}) \mathrm{d}\bar{x}$ splits as product

$$\int_{\hat{\mathbb{Z}}} \bar{\chi}(z) \mathrm{d}z \int_0^1 t^{s-1} \mu(t)$$

which is convergent for $\Re(s) > 0$, in which case it is equal to

$$\frac{1}{s} \int_{\hat{\mathbb{Z}}} \bar{\chi}(z) \mathrm{d}z.$$

This expression can obviously be meromorphically continued over Ω. It vanishes on all component with $\bar{\chi} \neq 1$, and over the neutral component with $\bar{\chi} = 1$, it has a simple pole at $s = 0$. The integral $\int_{|\bar{x}|<1} |x|^{-1} \chi(\bar{x}) \mathrm{d}\bar{x}$ is

$$\int_{\hat{\mathbb{Z}}} \bar{\chi}(z) \mathrm{d}z \int_0^1 t^{s-1} \mu(t)$$

is convergent for $\Re(s) > 1$, in which case it is equal to

$$\frac{1}{s-1} \int_{\hat{\mathbb{Z}}} \bar{\chi}(z) \mathrm{d}z.$$

This expression can obviously be meromorphically continued over X; it vanishes on all component $\bar{\chi} \neq 1$, and over the neutral component $\bar{\chi} = 1$, it has a simple pole at $s = 1$.

Putting together the four terms, we see that (5.6) can be meromorphically continued with a simple pole at $s = 0$ of residue $f(0)$, a simple pole at $s = 1$ of residue $\hat{f}(0)$. It can also be put in the symmetrical form

$$\int_{|\bar{x}|>1} (\phi(0) + \phi_+(\bar{x}))\, \chi(\bar{x}) + \left(|\bar{x}|\hat{\phi}(0) + |\bar{x}|\hat{\phi}_+(\bar{x})\right) \chi^{-1}(\bar{x}) \mathrm{d}\bar{x}$$

the integrals $\int_{|\bar{x}|>1} \chi(\bar{x}) \mathrm{d}\bar{x}$ being given the meaning as $-\int_{|\bar{x}|<1} \chi(\bar{x}) \mathrm{d}\bar{x}$ calculated above. We thus derived the functional equation relating the Mellin transforms of ϕ and $\hat{\phi}$

$$\mathscr{M}\phi(\chi) = \mathscr{M}\hat{\phi}(\omega^1 \chi^{-1}) \tag{5.7}$$

where ω^1 is the module character of $\mathbb{A}^\times/\mathbb{Q}^\times$. This formula can be recorded formally as a product formula for γ factors

$$\prod_v \gamma_v(\chi) = 1$$

that holds all over Ω_K. This formula doesn't make a proper sense as the product of local γ-factor does not converge.

Glossary of notations

- $\mathbb{Z}$ is the ring of integers
- $\mathbb{Z}_+$ is the set of non-negative integers
- $\mathbb{N}$ is the set of positive integers
- $\mathbb{Q}$ is the field of rational numbers
- $\mathbb{R}$ is the field of real numbers
- $\mathbb{R}_+$ is the set of non-negative real numbers
- $\mathbb{R}^\times$ is the multiplicative group of non-zero real numbers
- $\mathbb{R}_+^\times$ is the multiplicative group of positive real numbers