MARKOV TRACES AND THE RATIONAL DAHA

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1. INTRODUCTION

Fix coprime integers $m, n > 0$. In [18], Gorsky–Oblomkov–Rasmussen–Shende observed a numerical identity relating

1. The HOMFLY polynomial of the $(m, n)$-torus knot $T_{m,n}$.
2. The rational double-affine Hecke algebra (DAHA) of the symmetric group $\mathfrak{S}_n$ of constant parameter $m/n$, which we denote by $A_{m/n}$.

They proposed that this apparent coincidence is the “Euler characteristic” of a deeper relationship, stated in terms of HOMFLY homology rather than the HOMFLY polynomial. In this note, we pursue a new yet related direction. We give a purely representation-theoretic explanation of the identity, and in doing so, find its broadest possible generalization when $\mathfrak{S}_n$ is replaced by any finite Coxeter group.

1.1. We recall the setup of [18]. Let $t \simeq C^{n-1}$ be the reflection representation of $\mathfrak{S}_n$. For general $\nu \in C$, the algebra $A_{\nu}$ is a deformation of $A_0 = C[\mathfrak{S}_n] \times \mathcal{D}(t)$, the algebra of twisted differential operators on $t$. The $A_{\nu}$-module $\text{Sym}^* (t)$ admits a simple quotient $L_{\nu}$, which is finite-dimensional if and only if $\nu$ takes the form $m/n$ above.

Via the inclusion $C[\mathfrak{S}_n] \subseteq A_{m/n}$, we can view $L_{m/n}$ as an $C[\mathfrak{S}_n]$-module. There is a canonical element $h \in A_{m/n}$ that commutes with $\mathfrak{S}_n$, which lets us decompose $L_{m/n}$ into $\mathfrak{S}_n$-stable eigenspaces for $h$. We write $[L_{m/n}]$ for its graded character:

\begin{equation}
[L_{m/n}] = \sum_{\alpha} q^\alpha \left. L_{m/n} \right|_{h=\alpha}.
\end{equation}

In [1], Berest–Etingof–Ginzburg computed this character explicitly:

\begin{equation}
[L_{m/n}] \simeq q^{-(m-1)(n-1)/2} \sum_{i,j} q^{i-m} \text{Sym}^i (t) \otimes \Lambda^j (t).
\end{equation}

In [20], Jones found a similarly explicit formula for homfly$(T_{m,n}) \in Z[q][a^{\pm1}]$ in terms of $q$-integers. Using these formulæ, the authors of [18] showed by direct computation that

\begin{equation}
\text{homfly} (T_{m,n}) = a^{(m-1)(n-1)} \sum_{0 \leq j \leq n-1} (-a^2)^j \left. \Lambda^j (t), [L_{m/n}] \right|.
\end{equation}

where $\left. \Lambda^j (t), [L_{m/n}] \right|$ means the graded multiplicity of $\Lambda^j (t)$ in $[L_{m/n}]$.

In the rest of the paper, the authors construct three different filtrations on $[L_{m/n}]$ that are compatible with the $q$-grading. They conjecture that all three are the same filtration $F$, and
that their HOMFLY identity can be promoted to an isomorphism

\[
\sum_{i,j,k} q^{i} a^{j} t^{k} \text{HHH}^{i}_{j,k}(T_{m,n}) \\
\simeq a^{(m-1)(n-1)} \sum_{i,j,k} (-a^{2})^{j} \text{Hom}_{S_{n}}(A^{j}(t), \text{gr}^{k}_{F}[L_{m/n}]),
\]

where HHH denotes triply-graded HOMFLY homology.

1.2. In this note, we replace $S_{n}$ with an arbitrary Coxeter group $W$. All of the structures above generalize:

- Let $\text{Br}_{W}$ be the Artin braid group of $W$. It has a canonical central element $\pi$ called the full twist. The analogue of a positive $(m, n)$-torus link is (the conjugacy class of) an element $\beta \in \text{Br}_{W}$ such that $\beta^{n} = \pi^{m}$. When such an element $\beta$ exists, we say that $\beta$ is a regular braid and $m/n$ is a regular slope of $W$. The torus link being a knot is generalized by the slope $m/n$ being elliptic, a property defined in terms of the $W$-action on the reflection representation $t$.

- Let $\mathcal{H}$ be the Hecke algebra of $W$. The analogue of the HOMFLY invariant is a Markov trace, a $\mathbb{Z} \langle q^{1/2} \rangle [\alpha]$-valued class function on $\mathcal{H}$ satisfying certain axioms. Since $\mathcal{H}$ is a quotient of $\mathbb{Z} \langle q^{\pm 1/2} \rangle [\text{Br}_{W}]$, any such function pulls back to a class function on $\text{Br}_{W}$. By the work of Y. Gomi [17], there is a canonical Markov trace defined in terms of Lusztig’s exotic Fourier transform.

- For any $v \in \mathbb{C}$, we can again define the rational DAHA $\mathcal{A}_{v}$ of $W$ of parameter $v$. Its simple modules are indexed by the irreducible characters of $W$; we write $L_{v}(\phi)$ for the simple module corresponding to a character $\phi$. By a theorem of Etingof, the module $L_{v} = L_{v}(1)$ corresponding to the trivial character is finite-dimensional if and only if $v$ is a regular elliptic slope.

We can now explain our key ideas.

For any $\beta \in \text{Br}_{W}$, we introduce a graded $W$-character $\Omega_{\beta} \in \mathbb{Z}[[q^{\pm 1/2}]][\hat{W}]$, depending only on the conjugacy class of $\beta$, that refines Gomi’s construction: Namely,

\[
\sum_{j} (-a^{2})^{j} \langle A^{j}(t), \Omega_{\beta} \rangle
\]

is Gomi’s Markov trace up to a scaling factor. We show that $\Omega_{\beta}$ satisfies integrality and symmetry properties generalizing those of the HOMFLY invariant. When $\beta$ is a regular braid of slope $v$, work of Springer and Lusztig allows us to give an explicit formula for $\Omega_{\beta}$ in terms of polynomials attached to the irreducible characters of $W$ called their generic degrees. From this, we can show that $\Omega_{\beta}$ is the graded $W$-character of a virtual $\mathcal{A}_{v}$-module, which always contains $[L_{v}(1)]$ with multiplicity 1.

Roughly, the extent to which other terms $[L_{v}(\phi)]$ appear is inversely correlated with the size of the denominator of $v$ in lowest terms. For a general regular slope $v$, we do not know how to predict their multiplicities. But we can show that if $[L_{v}(\phi)]$ does appear, then the support of $L_{v}(\phi)$ as a sheaf on $t$ is contained in the discriminant locus. When $v$ is elliptic,
we conjecture that only finite-dimensional simple modules appear in $\Omega_{\beta}$. When $v$ is cuspidal in the sense of Bezrukavnikov–Etingof [2], a strictly stronger condition, we can show that $\Omega_{\beta}$ is precisely the sum of the finite-dimensional simple modules of $A_v$, using work of Geck on the cyclotomic blocks of $H$. For most finite Coxeter groups, including $W = S_n$, the only such module is $L_v(1)$. In this way, we recover the GORS identity.

Finally, in the crystallographic case, we explain how $\Omega_{\beta}$ can be refined to incorporate an additional “$t$-grading,” which recovers the $t$-grading in HOMFLY homology. The construction relies on character sheaves. We propose that the filtration $F$ in [18] can be generalized to the virtual $A_v$-module with $W$-character $\Omega_{\beta}$, and that the GORS conjecture is really a comparison between these two gradings on $\Omega_{\beta}$: the $t$-grading and $\text{gr}_F$.

We point out a curious feature of the story. To generalize the $(q^{1/2} \to -q^{-1/2})$-symmetry of the HOMFLY invariant to $\Omega_{\beta}$, we need an identity describing how Lusztig’s exotic Fourier transform behaves under sign twist. Later, we again use it to match $\Omega_{\beta}$ with the right virtual $A_v$-module. In the crystallographic case, the identity follows from Alvis–Curtis duality; in other types, we need to check it case by case.

1.3. In Section 9 of [18], the authors discuss how their ideas relate to those of Oblomkov–Rasmussen–Shende [29]. For any plane curve germ $\hat{C} \subset \mathbb{C}^2$ with link $L(\hat{C}) = \hat{C} \cap S^3$, the latter have conjectured a precise identity relating the Poincaré polynomials of:

1. The HOMFLY homology of $L(\hat{C})$.
2. The cohomology of the (flagged) compactified Jacobian of any rational curve $C$ whose unique singularity is $\hat{C}$, equipped with a perverse filtration defined in terms of a versal deformation of $C$.

The flagged compactified Jacobians above can be realized as parabolic Hitchin fibers for the groups $SL_n$. Using the work of Oblomkov–Yun [30], one can construct the $A_{m/n}$-module $L_{m/n}(1)$ in terms of the cohomology of such Hitchin fibers. As a result, the ORS conjecture for the germ $\hat{C} = \{y^n = x^m\}$ implies the GORS conjecture for the pair $(m, n)$ at the level of Poincaré polynomials.

In a forthcoming paper, we will use our $W$-character $\Omega_{\beta}$ (equipped with its $t$-grading) to generalize and re-interpret the ORS conjecture, in the same way that we here generalize the GORS identity.

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2. Preliminaries

2.1. The Coxeter System. Let $(W, S)$ be a finite Coxeter system of rank $r$. This means $S$ is a set of size $r$ and $W$ is a finite group freely generated by $S$ modulo two kinds of relations:
\[ s^2 = 1 \text{ for all } s \in S, \text{ and relations of the form} \]
\[
\begin{array}{ccc}
\underbrace{m_{s,t} \text{ terms}}_{s \neq t} & \overbrace{m_{s,t} \text{ terms}}^{s \neq t} \\
sts \cdots & tst \cdots 
\end{array}
\]

(2.1)

for all distinct \( s, t \in S \). Let \( T \) be the set of elements conjugate to some element of \( S \) and set \( N = |T| \).

We write \( \hat{W} \) for the set of irreducible, finite-dimensional complex characters of \( W \) and \( \mathbb{Z}[\hat{W}] \) for the Grothendieck ring of \( W \). Let \( \langle -, - \rangle \) be the multiplicity pairing on \( \mathbb{Z}[\hat{W}] \).

Let \( t_1 \in \hat{W} \) denote the trivial character, and for any \( \phi \in \hat{W} \), let \( \phi' \) denote the sign twist of \( \phi \). Let \( K \subseteq \mathbb{C} \) be the minimal field over which all characters of \( W \) are defined, i.e., \( K = \mathbb{Q}(\{\phi(w)\}_{w \in W, \phi \in \hat{W}}) \).

The braid monoid of \( (W, S) \), denoted \( Br^+_W \), is the monoid freely generated by elements \( \beta_s \) for each \( s \in S \) modulo the relations

\[
\beta_s \beta_t \beta_s \cdots = \beta_t \beta_s \beta_t \cdots
\]

(2.2)

for distinct \( s, t \). The braid group of \( (W, S) \), denoted \( Br_W \), is the group completion of \( Br^+_W \). Thus there is a surjective homomorphism \( Br_W \to W \) that sends \( \beta_s \mapsto s \).

Example 2.1. For the Coxeter system of type \( A_r \), where \( W \) is the symmetric group \( S_{r+1} \) and \( S \) consists of the transpositions \((i \ i + 1)\) for \( i = 1, \ldots, r \), we can identify \( Br_W \) with the usual \((r + 1)\)-strand braid group \( Br_{r+1} \).

For any \( \beta \in Br_W \), the writhe of \( \beta \) is the length \( \text{wr}(\beta) \) of any word in the elements \( \beta_s \) that represents \( \beta \). Note that \( \text{wr}(\beta) \) only depends on its conjugacy class. Every element \( w \in W \) admits a unique lift \( \beta_w \in Br_W \) of minimal writhe, as we see from comparing the presentations of \( W \) and \( Br_W \).

2.2. The Geometric Realization. Let \( t_R \) be a geometric realization of \( (W, S) \). For us, this means \( t_R \) is a real representation of \( W \), equipped with a Euclidean structure in which \( S \) acts by reflections. We assume that \( t \) is minimal, i.e., if \( H_s \subseteq t \) is the hyperplane fixed by \( s \in S \), then \( t = \bigoplus_{s \in S} H_s \). Let \( \tau = t_R \otimes \mathbb{C} \). Let \( \tau \) be the \( W \)-character of \( t \) and set

\[
S(q) = \sum_{i \geq 0} q^i \text{Sym}^i \tau. 
\]

(2.3)

\[
\Lambda(q) = \sum_{j \geq 0} (-q)^j \Lambda^j \tau. 
\]

(2.4)

We write \( \{d_i\}_{i \in I(W)} \) for the multiset of invariant degrees of the \( W \)-action on \( t \).

Let \( t^o \) be the regular locus of \( t \), i.e., the set of points on which \( W \) acts with trivial stabilizer. Its complement is known as the discriminant locus. An element \( w \in W \) is called regular of slope \([v] \in \mathbb{Q}/\mathbb{Z} \) iff its action on \( t \) admits an eigenvector in \( t^o \) of eigenvalue \( e^{2\pi i v} \). In this case, we say that \( v \in \mathbb{Q} \) is a regular slope of \( W \). We say that \( v \) is elliptic iff, in addition, the only fixed point of \( w \) in \( t \) is zero.
For all \( \phi \in \hat{W} \), the fake degree of \( \phi \) is the ratio
\[
P_\phi(q) = \frac{\langle \phi, S(q) \rangle}{\langle 1, S(q) \rangle},
\]
which belongs to \( \mathbb{Z}[q] \) for all \( \phi \). Above, one can show that
\[
\langle 1, S(q) \rangle = \prod_{i \in I(W)} \frac{1}{1 - q^{a_i}}.
\]

The following is [34, Prop. 4.5]:

**Theorem 2.2** (Springer). If \( w \in W \) is regular of slope \([v]\), then
\[
\phi(w) = P_\phi(e^{-2\pi i v})
\]
for all \( \phi \in \hat{W} \).

2.3. **Hecke Algebras.** The Hecke algebra of \((W, S)\), denoted \( \mathcal{H} = \mathcal{H}(W) \), is the quotient of \( \mathbb{Z}[q^\pm][[\text{Br}_W]] \) by the relations
\[
(\beta_s - q^{1/2})(\beta_s + q^{-1/2}) = 0 \quad \text{for all} \quad s \in S.
\]
The set \( \{\beta_w\}_{w \in W} \) forms a free \( \mathbb{Z}[q^\pm] \)-basis for \( \mathcal{H} \). In the limit \( q^{1/2} \to 1 \), we see that \( \mathcal{H} \to \mathbb{Z}[W] \) and \( \beta_w \to w \).

For any \( \mathbb{Z}[q^\pm] \)-algebra \( R \), we set \( \mathcal{H}_R = \mathcal{H} \otimes R \). We write \( \hat{\mathcal{H}}_R \) for the set of isomorphism classes of simple \( \mathcal{H}_R \)-modules and \( \mathbb{Z}[\hat{\mathcal{H}}_R] \) for the Grothendieck ring of \( \mathcal{H}_R \). Let \( \mathbb{Z}[\hat{\mathcal{H}}_R]_{\geq 0} \subseteq \mathbb{Z}[\hat{\mathcal{H}}_R] \) be the multiplicative submonoid of elements that correspond to actual, not just virtual, \( \mathcal{H}_R \)-modules. There is a multiplicative map
\[
r_R : \mathbb{Z}[\hat{\mathcal{H}}_R]_{\geq 0} \to \prod_{w \in W} R[t]
\]
that sends \( M \mapsto \{\det(1 - t\beta_w \mid M)\}_{w \in W} \).

2.3.1. **The Generic Algebra.** If \( R = \mathbf{K}(q^{1/2}) \), then we abbreviate \( \mathcal{H}_R = \mathcal{H}_r \) and \( r_R = r_r \). The identification \( \mathcal{H}_r \mid q^{1/2} \to 1 \simeq \mathbf{K}[W] \) can be lifted to an isomorphism
\[
\mathcal{H}_r \simeq \mathbf{K}(q^{1/2})[W]
\]
[16, Thm. 9.3.5]. This yields a bijection between \( \hat{\mathcal{H}}_r \) and \( \hat{W} \). For any \( \phi \in \hat{W} \), we write \( \phi_q \) for the character of the corresponding simple \( \mathcal{H}_r \)-module.

The \( \mathbf{K}(q^{1/2}) \)-linear, \( \mathbb{Z} \)-valued function on \( \mathcal{H}_r \) that sends \( \beta_1 \mapsto 1 \) and \( \beta_w \mapsto 0 \) for all \( w \neq 1 \) is conjugation-invariant. Thus it is a \( \mathbf{K}(q^{1/2}) \)-linear combination of the \( \phi_q \), say,
\[
\sum_{\phi} \phi_q(\beta_w) s_\phi = \begin{cases} 1 & w = 1 \\ 0 & w \neq 1 \end{cases}
\]
Above, \( s_\phi \in \mathbf{K}(q^{1/2}) \) is called the Schur element of \( \phi_q \). One can show that \( s_\phi \in q^{-N} \mathbf{K}[q] \) [16, Prop. 7.3.9].
2.3.2. The Cyclotomic Algebra. Fix $[v] \in \mathbb{Q}/\mathbb{Z}$. We will write $\Phi_v$ for the minimal polynomial of $e^{2\pi iv}$ over $K$.

If $R = K(e^{\pi iv})$, viewed as a $\mathbb{Z}[q^{\pm 1/2}]$-algebra under $q^{1/2} \mapsto e^{\pi iv}$, then we abbreviate $\mathcal{H}_R = \mathcal{H}_v$ and $r_R = r_v$. By [16, Thm. 7.4.3], there is a unique additive map

$$d_v : \mathbb{Z}[\mathcal{H}_v]_{\geq 0} \to \mathbb{Z}[\mathcal{H}_v]_{\geq 0},$$

known as a decomposition map, such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{Z}[\mathcal{H}_v]_{\geq 0} & \xrightarrow{r_v} & \prod_w K(q^{1/2})[t] \\
\downarrow d_v & & \downarrow \\
\mathbb{Z}[\mathcal{H}_v]_{\geq 0} & \xrightarrow{r_v} & \prod_w K(e^{\pi iv})[t]
\end{array}$$

In particular, $d_v$ sends the simple $\mathcal{H}_v$-module of trace $\phi_q$ to an $\mathcal{H}_v$-module of trace $\Phi_v = \phi_q|_{q^{1/2} \mapsto e^{\pi iv}}$, no longer necessarily simple.

The Brauer graph of $\mathcal{H}_v$ is the (undirected) graph in which the vertex set is $\mathbb{Z}[\mathcal{H}_v]_{\geq 0}$ and there is an edge between $\phi$ and $\psi$ iff $\phi_v$ and $\psi_v$ share a simple constituent. The connected components of this graph are the blocks of $\mathcal{H}_v$, or $\Phi_v$-blocks. The partition of $\mathbb{Z}[\mathcal{H}_v]_{\geq 0}$ into blocks corresponds to a $\oplus$-decomposition of $\mathcal{H}_v$ into idempotent ideals called block ideals, such that the simple modules of any block ideal are in bijection with the characters in its block. The principal block is the block that contains the trivial character.

**Example 2.3.** Suppose $W$ is of type $A_{n-1}$, i.e., the symmetric group on $n$ letters, and $v = 1/n$. Here, the principal block is a line. It consists precisely of the exterior powers $\Lambda^j \tau$, ordered by $j$ along the line from 0 to $n - 1$. We will refer to its block ideal as the Brauer tree algebra of type $A_{n-1}$. Moreover, all other blocks are singletons.

For all $\phi \in \hat{W}$, the $\Phi_v$-defect of $\phi$ is the multiplicity of $\Phi_v(q)$ in $s_\phi$. It measures the complexity of the $\Phi_v$-block containing $\phi$ in the following sense:

**Theorem 2.4 (Geck).** The characters within a given $\Phi_v$-block have the same $\Phi_v$-defect, so we can speak of the defect of the block itself. Moreover:

1. If the defect is 0, then the block is a singleton.
2. If the defect is 1, then the block is a line ordered by Lusztig’s function $a$ and its block ideal is isomorphic to a Brauer tree algebra of type $A$.

**Proof.** This combines Prop. 7.4, Prop. 8.2, and Thm. 9.6 of [14].

**Remark 2.5.** By contrast, we do not know of any structure theorem for blocks of defect higher than 1.

2.4. Generic Degrees and the Fourier Transform. For all $\phi \in \hat{W}$, the generic degree of $\phi$ is the ratio

$$D_\phi(q) = \frac{s_1}{s_\phi}.$$


Above, one can show that

\[
    s_1(q) = \prod_{i \in I(W)} \frac{1 - q^{d_i}}{1 - q},
\]

i.e., \(s_1 \in K[q]\) is the Poincaré polynomial of \((W, S)\). More generally, \(D_\phi(q) \in K[q]\) for all \(\phi\), cf. \([16, 307]\). Generic degrees and fake degrees are related by a pairing known as the \textit{exotic Fourier transform} that we now explain.

For every finite Coxeter group \(W\), Lusztig and Malle partitioned \(O_W\) into subsets \(O_W^F\) called \textit{families}. To each family, they assigned a fusion datum (in the sense of modular tensor categories), which gives rise to an inclusion \(\hat{W}_F \to M_F\) of finite sets and a hermitian unitary pairing

\[
    \{-, -\}_F : M_F \times M_F \to C
\]

\([25, 26, 27]\). In this paper, we only use the restriction of \(\bigoplus_F \{-, -\}_F\) to \(\hat{W}\), which we denote by \(\{-, -\}\).

\textbf{Remark 2.6.} In type \(A\), we have \(\{-, -\} = \langle -, - \rangle\). For general \(W\), it turns out that \(\{-, -\}\) always takes values in \(K\), as one can check by reducing to the case of irreducible \(W\).

As we explain in the next section, Lusztig first introduced the exotic Fourier transform in the case where \(W\) is the Weyl group of a reductive group over a finite field. To determine the right generalization to other Coxeter groups, he proposed to characterize it by a list of axioms. Implicit in this list is the property that

\[
    D_\psi(q) = \sum_{\phi} \{\phi, \psi\} P_\phi(q)
\]

for all \(\psi \in \hat{W}\) \([25]\).

### 3. Preliminaries: The Crystallographic Case

In this section, we assume that \((W, S)\) is a crystallographic Coxeter system and explain the new structure that this entails.

Fix a prime power \(q \gg 0\). Let \(\ell\) be a prime not dividing \(q\) and fix an isomorphism \(C \cong \hat{Q}_\ell\). We can choose a split semisimple group \(G_0\) over \(F_q\) with Weyl group \(W\). Let \(G = G_0 \otimes \hat{F}_q\), and let \(B\) be the flag variety of \(G\). We write \(Z[G(\hat{F}_q)]\) for the Grothendieck ring of \(G(\hat{F}_q)\) and \(\langle -, - \rangle\) for its multiplicity pairing.

#### 3.1. Deligne–Lusztig Theory

Recall the Bruhat decomposition

\[
    B \times B \simeq \bigsqcup_{w \in W} O_w,
\]

where \(O_w\) is the left \(G\)-orbit of pairs \((B_1, B_2)\) in relative position \(w\). The \textit{Deligne–Lusztig variety} \(B_w\) is the (transverse) intersection of \(O_w\) with the graph of the \(q\)-Frobenius map of
\( B \), which forms a smooth variety over \( \mathbb{F}_q \). The corresponding unipotent Deligne–Lusztig character is the virtual character of \( G(\mathbb{F}_q) \) given by

\[
R_w = \sum_{i \geq 0} (-1)^i H^i_c(B, \mathbb{Q}_\ell),
\]

where \( H^i_c \) denotes étale cohomology with compact support.

**Example 3.1.** Take \( w = 1 \). We have \( B_1(\bar{\mathbb{F}}_q) = B(\mathbb{F}_q) \), a discrete set of \( q + 1 \) points, and \( R_1 = \mathbb{Q}_\ell[B(\mathbb{F}_q)] \), the vector space of \( \mathbb{Q}_\ell \)-valued functions on \( B(\mathbb{F}_q) \).

An element of \( Z[G(\mathbb{F}_q)] \) is called unipotent iff it occurs in some \( R_w \) with nonzero multiplicity. In particular, for any \( \phi \in \hat{W} \), there is a unipotent character

\[
R_\phi = \frac{1}{|\hat{W}|} \sum_{w \in \hat{W}} \phi(w) R_w,
\]

called the almost-character associated with \( \phi \).

There is another way to construct an unipotent character of \( G(\mathbb{F}_q) \) from \( \phi \). It is a theorem of Iwahori that, if we fix a square root \( q^{1/2} \in \mathbb{Q}_\ell \) and specialize \( q^{1/2} \to q^{1/2} \), then there is an isomorphism of \( \mathbb{Q}_\ell \)-algebras

\[
\text{End}_{\mathbb{Q}_\ell[G(\mathbb{F}_q)]}(\bar{\mathbb{Q}}_\ell[G(\mathbb{F}_q)]) \simeq \mathcal{H}_{\mathbb{Q}_\ell}.
\]

Explicitly, we can identify the left-hand side with the vector space of \( G \)-invariant \( \mathbb{Q}_\ell \)-valued functions on \((B \times B)(\mathbb{F}_q)\). Once we do so, (3.4) takes the indicator function on \( O_w \) to

\[
\bar{\beta}_w = q^{w r(\beta_w)/2} \beta_w.
\]

By viewing \( \bar{\mathbb{Q}}_\ell[B(\mathbb{F}_q)] \) as an \((\mathbb{Q}_\ell[G(\mathbb{F}_q)], \mathcal{H}_{\mathbb{Q}_\ell})\)-bimodule, we obtain a decomposition

\[
\mathbb{Q}_\ell[B(\mathbb{F}_q)] \simeq \bigoplus_{\phi \in \hat{W}} \rho_\phi \otimes \phi_\phi^\vee.
\]

Deligne and Lusztig showed that each \( \rho_\phi \) is an irreducible unipotent character, called the unipotent principal-series character associated with \( \phi \).

It follows from the work of Lusztig [22] that

\[
\rho_{\psi} = \sum_{\phi} \{\phi, \psi\} R_\phi
\]

in \( \mathbb{Q}_\ell[G(\mathbb{F}_q)] \). Moreover,

\[
\rho_\phi(1) = D_\phi(q) \quad \text{and} \quad R_\phi(1) = P_\phi(q)
\]

for all \( \phi \in \hat{W} \). This is the origin of (2.17).
3.2. Character Sheaves. For any $G$-variety $Z$ over $\mathbb{F}_q$, we write $D^b_G(Z)$ for the bounded $G$-equivariant derived category of constructible mixed complexes on $Z$. It is equipped with a cohomological shift $[1]$ and a Tate twist $(-1)$. Fixing $q^{1/2} \in \mathbb{Q}_\ell$ defines a half-twist $(-1/2)$.

We set $h_1 := [1][1/2]$. Just as the space of $G$-invariant functions on $G/B$ forms an algebra (cf. (3.4)), $D^b_G(B/B)$ forms a monoidal category under a natural convolution product. For all $w \in W$, we have an intersection complex

$$IC_w = j_{w,*}(\tilde{Q}_\ell)[\dim \mathcal{O}_w] \in D^b_G(B \times B),$$

where $j_w$ is the open embedding $\mathcal{O}_w \to B \times B$. Let $H = H(G)$ be the full additive subcategory of $D^b_G(B \times B)$ generated by the objects $IC_w(n)$ for $w \in W$, $n \in \mathbb{Z}$. The decomposition theorem implies $H$ is stable under convolution. Thus, the split Grothendieck group of $H$ forms a ring graded under $(-)$. By [35, Thm. 2.8], it is graded-isomorphic to $\mathcal{H}$ under the identification $[1] \leftrightarrow q^{-1/2}$. The image of $\{IC_w\}_{w \in W}$ in $\mathcal{H}$ forms a $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{\gamma_w\}_{w \in W}$ known as the Kazhdan–Lusztig basis.

Letting $G$ act on itself by conjugation, we have an $G$-equivariant diagram

$$B \times B \xleftarrow{act} G \times B \xrightarrow{pr} G,$$

where $pr(g, B) = g$ and $act$ sends $(g, B) \mapsto (\text{Ad}(g)B, B)$. We set $\text{Ind} = pr_* \text{act}!$ and

$$K_w = \text{Ind}(IC_w) \in D_G(G).$$

A unipotent character sheaf is an irreducible $G$-equivariant perverse sheaf on $G$ that occurs as a direct summand of some $K_w(n)$.

Example 3.2. Take $w = 1$. The complex $K_1$ is an analogue of the Springer sheaf; that is, it is the pushforward of $\tilde{Q}_\ell$ along a small map that restricts to a Galois $W$-cover over the regular semisimple locus of $G$. In particular, $W$ acts on $K_1$ and its $W$-isotypic decomposition is

$$K_1 \simeq \bigoplus_{\phi} A_{\phi}^{\oplus \phi(1)}.$$

Above, $A_{\phi}$ is a unipotent character sheaf for all $\phi$.

Theorem 3.3 (Lusztig). Upon identifying $[1] = q^{-1/2}$ in the split Grothendieck group of $D^b_G(G)$, we have

$$[K_w] = \sum_{\phi, \psi} \{\phi, \psi\} a_\phi(\gamma_w)[A_\psi] + [\text{non-Weyl}],$$

where “non-Weyl” denotes the contribution of character sheaves not of the form $A_\phi$.

Proof. This is quoted in [38, 413].
4. Markov Characters

4.1. Markov Traces. The HOMFLY invariant is a map from isotopy classes of links in the 3-sphere to bivariate Laurent series:

\[
\text{homfly} : \{\text{links in } S^3\}/\text{isotopy} \to \mathbb{Z}(q^{\pm 1/2})[a^{\pm 1}].
\]

One way to construct it due to Jones and Ocneanu [13, 20] is to construct, for every \( r \), a certain conjugation-invariant function on \( \mathcal{H}(S_{r+1}) \) called a Markov trace. Later, Y. Gomi generalized these Markov traces to arbitrary finite Coxeter systems \((W, S)\) [17].

If \( S' \subseteq S \) and \( W' \) is the subgroup of \( W \) generated by \( S' \), then \((W', S')\) is again a Coxeter system and \( \mathcal{H}(W') \subseteq \mathcal{H}(W) \). A Markov trace on \( \mathcal{H}(W) \) is a \( \mathbb{Z}[q^{\pm 1/2}] \)-linear function \( \text{tr} : \mathcal{H}(W) \to \mathbb{Z}[q^{\pm 1/2}][z] \) such that:

1. \( \text{tr}(1) = 1 \).
2. \( \text{tr}(\beta \gamma) = \text{tr}(\gamma \beta) \) for all \( \beta, \gamma \in \mathcal{H}(W) \).
3. \( \text{tr}(\beta \gamma) = z \text{tr}(\gamma) \) for all \( s \in S \) and \( \gamma \in \mathcal{H}(W') \), where \( W' \) is the subgroup of \( W \) generated by \( S \setminus s \).

By property (2), we must have

\[
\text{tr} = \sum_{\phi \in \hat{W}} (\text{tr}, \phi_q) \phi_q
\]

for some weights \((\text{tr}, \phi_q) \in \mathbb{K}(q^{1/2})[z]\).

**Theorem 4.1** (Gomi). Make the change of variables

\[
z = -a^{-1} \left( \frac{q^{1/2} - q^{-1/2}}{a - a^{-1}} \right).
\]

Then there is a Markov trace on \( \mathcal{H} \) with the weights

\[
(\text{tr}, \phi_q) = \left( -q^{1/2} a \left( \frac{q^{1/2} - q^{-1/2}}{a - a^{-1}} \right) \right)^r \sum_{\psi \in \hat{W}} \{\phi, \psi\} \{\psi, S(q) \otimes \Lambda(a^{-2})\},
\]

where \( S \) and \( \Lambda \) are defined by (2.3) and (2.4), respectively, and the pairing \( \{-, -\} \) is Lusztig’s exotic Fourier transform.

Henceforth, we write \( \text{tr} : \text{Br}_W \to \mathbb{Z}[q^{\pm 1/2}][a^{\pm 1}] \) for the class function on \( \text{Br}_W \) obtained by pulling back the Markov trace in Gomi’s theorem.

**Theorem 4.2** (Jones–Ocneanu). The HOMFLY invariant of a link \( L \) satisfies

\[
\text{homfly}(L) = (-a)^{\text{wr}(&) \left( \frac{a - a^{-1}}{q^{1/2} - q^{-1/2}} \right)^r \text{tr}(\beta)
\]

for any \( r \geq 1 \) and braid \( \beta \in \text{Br}_{r+1} \) whose closure is \( L \).

**Remark 4.3.** By a theorem of Markov, every link in the 3-sphere is the closure of some braid in some braid group \( \text{Br}_{r+1} \). However, it is difficult to determine the minimal such \( r \).
4.2. From Traces to Characters. We want a refinement of $\text{tr}$ whose output is a graded character of $W$, rather than a bivariate series. First, we introduce a normalization:

\[
\text{tr}(\beta) = (-1)^{wr(\beta)} a^r \left( \frac{a - a^{-1}}{q^{1/2} - q^{-1/2}} \right)^r \text{tr}(\beta).
\]

Equivalently, in type A,

\[
\text{homfly}(L) = a^{wr(\beta) - r} \text{tr}(\beta)
\]

where $L$ is the link closure of $\beta$.

**Lemma 4.4.** For any $\beta \in \text{Br}_W$, we have

\[
\text{tr}(\beta) = (-1)^{wr(\beta)} q^{r/2} \sum_{\phi, \psi \in \hat{W}} \{\phi, \psi\} \phi_q(\beta) \{\psi', S(q) \otimes \Lambda(a^2)\}.
\]

**Proof.** By definition,

\[
\text{tr}(\beta) = (-1)^{wr(\beta)} q^{r/2} a^2 \sum_{\phi, \psi \in \hat{W}} \{\phi, \psi\} \phi_q(\beta) \{\psi, S(q) \otimes \Lambda(a^{-2})\}.
\]

To prove the formula, it remains to show that $a^{2r} \Lambda(a^{-2}) = 1' \otimes \Lambda(a^2)$. This follows from the fact that $\Lambda^{r-j} \tau = 1' \otimes \Lambda^j \tau$ for all $j$. \hfill \Box

Within formula (4.8), observe that

\[
\langle \psi', S(q) \otimes \Lambda(a^2) \rangle = \langle \Lambda(a^2), S(q) \otimes \psi' \rangle
\]

because Sym$i$ $\tau$ is self-dual for all $i$. This inspires the following definition.

**Definition 4.5.** For any $\beta \in \mathcal{H}$, let

\[
R_\beta = \sum_{\phi, \psi \in \hat{W}} \{\phi, \psi\} \phi_q(\beta) \psi,
\]

an element of $K[q^{\pm 1/2}][\hat{W}]$. If $\beta \in \text{Br}_W$, then we define the Markov character of $\beta$ to be

\[
\Omega_\beta = (-1)^{wr(\beta)} q^{r/2} S(q) \otimes (R_\beta)',
\]

an element of $K((q^{1/2}))[[\hat{W}]]$.

**Corollary 4.6.** For all $\beta \in \text{Br}_W$, we have $\text{tr}(\beta) = \langle \Lambda(a^2), \Omega_\beta \rangle$.

Two well-known properties of the HOMFLY invariant are integrality and symmetry. In terms of $\text{tr}$, they can be stated as:

1. $\text{tr}(\beta) \in (q^{1/2})^{r-wr(\beta)} Z[[q]][a^2]$ for all $\beta \in \text{Br}_W$.
2. $\text{tr}(\beta)|_{q^{1/2} \rightarrow -q^{-1/2}} = \text{tr}(\beta)$. 

In the rest of this subsection, we generalize these properties to \( \Omega_\beta \). This amounts to studying corresponding statements about the graded multiplicities \( \langle \psi, R_\beta \rangle \). Throughout, we reduce the arguments to the case where \( W \) is irreducible, by means of the following fact: If \( W \cong W_1 \times W_2 \), then the exotic Fourier transform of \( W \) is the product of the exotic Fourier transforms of \( W_1 \) and \( W_2 \); as a result,

\[
R_{\beta_1 \times \beta_2} = R_{\beta_1} \cdot R_{\beta_2}
\]

for all \( \beta_i \in Br_W \).

In the crystallographic case, we let \( F_q, \bar{Q}_\ell, G_0, G \), etc. be defined as in Section 3, and give uniform proofs using the material in that section. The remaining irreducible types are \( H_3, H_4, \) and \( I_2(h) \) for \( h = 5 \), which we check case by case.

**Proposition 4.7** (Integrality). For all \( \beta \in Br_W \), we have \( R_\beta \in q^{-wr(\beta)/2}Z[q][\hat{W}] \).

**Proof for irreducible crystallographic types.** We show that \( q^{wr(\beta)/2} \langle \psi, R_\beta \rangle \) is an element of both \( Z[q^{\pm 1/2}] \) and \( Q[q] \).

In the notation of §3.2, we can write \( \beta \) as a \( Z \)-linear combination of the Kazhdan–Lusztig basis elements \( \gamma_w \). By Thm. 3.3,

\[
\langle \psi, R_{\gamma_w} \rangle = \langle [A_\psi], [K_w] \rangle
\]

in the Grothendieck group of \( D^b_G(G) \), and the right-hand side belongs to \( Z[q^{\pm 1/2}] \) for all \( w \). Thus the same is true of \( q^{wr(\beta)/2} \langle \psi, R_\beta \rangle \).

Next, in the notation of §3.1, we can write \( q^{wr(\beta)/2} \beta \) as a \( Z \)-linear combination of the elements \( \bar{\beta}_w \). It remains to show that

\[
\sum_\phi \{\phi, \psi\} \phi_q(\bar{\beta}_w) \in Q[q]
\]

for all \( w \).

Since \( \{ -, - \} \) takes values in \( Q \) when \( W \) is crystallographic, it would be enough to show that \( \phi_q(\bar{\beta}_w) \in Q[q] \) for all \( w \) and \( \phi \). If \( W \) is not of type \( E_7 \) or \( E_8 \), then this holds by a theorem of Benson–Curtis [3]. Otherwise, it fails: In types \( E_7 \) and \( E_8 \), there is a unique two-element family \( \hat{W}_F \subseteq \hat{W} \) such that, for \( \phi \in \hat{W}_F \), we can only ensure \( \phi_q(\bar{\beta}_w) \in Q[q^{1/2}] \). However, it turns out (4.15) holds for these cases as well.

**Corollary 4.8.** For all \( \beta \in Br_W \), we have \( \Omega_\beta \in (q^{1/2})^{r-wr(\beta)}Z[[q]][\hat{W}] \).

**Remark 4.9.** As the sign character always forms a one-element family, we can check that \( \{1', R_\beta \} = (-q^{1/2})^{-wr(\beta)} \). It follows that \( \{1, \Omega_\beta \} = (q^{1/2})^{r-wr(\beta)} \).

**Proposition 4.10** (Symmetry). For all \( \beta \in \mathcal{H} \), we have \( R_\beta |_{q^{1/2} \rightarrow -q^{-1/2}} = R'_\beta \).

**Lemma 4.11.** We have \( \{\phi', \psi'\} = \{\phi, \psi\} \) for all \( \phi, \psi \in \hat{W} \).
Proof for crystallographic types. Recall that Alvis–Curtis duality is an involution on the ring $\mathbb{Z}[G(F_q)]$, which is an isometry with respect to $(\cdot, \cdot)$ and exchanges

\begin{equation}
\rho_\phi \leftrightarrow \rho_{\phi'} \quad \text{and} \quad R_\phi \leftrightarrow R_{\phi'}
\end{equation}

for all $\phi$. Since $\{\phi, \psi\} = \langle \rho_\phi, R_\psi \rangle$, these properties imply the lemma. □

Proof of Prop. 4.10 for crystallographic types. For all $\psi \in \hat{W}$, we compute

\begin{align*}
\sum_{\phi, \psi} \{\phi, \psi\}_q(\beta) &= \sum_{\phi, \psi} \{\phi', \psi\}_q(\phi')_q(\beta) \\
&= \sum_{\phi, \psi} \{\phi, \psi\}(\phi')_q(\beta) \\
&= \sum_{\phi, \psi} \{\phi, \psi\}_q(\beta)|_{q^{1/2} \rightarrow -q^{-1/2}}.
\end{align*}

We conclude that $(\psi', R_\beta) = (\psi, R_\beta)|_{q^{1/2} \rightarrow -q^{-1/2}}$, as needed. □

Corollary 4.12. For all $\beta \in \text{Br}_W$, we have $\Omega_\beta|_{q^{1/2} \rightarrow -q^{-1/2}} = \Omega_\beta$.

Proof. Consider what the involution $q^{1/2} \rightarrow -q^{1/2}$ does to each piece of $\Omega_\beta$. It sends:

- $q^{r/2} \mapsto (-q)^{-r} q^{r/2}$.
- $S(q) \mapsto (-q)^r S(q')$, as one can see from the identity $S(q) \otimes \Lambda(q) = 1$ in the Grothendieck ring $\mathbb{Z}[[q]][\hat{W}]$.
- $R_\beta \mapsto R_\beta = (R_\beta')'$, by the proposition.

Multiplying these together gives the result. □

Example 4.13. Let $W = S_4$. We label the irreducible characters $1, \tau, \phi, \tau', \phi'$, where $\phi$ is of degree 2. Let $\beta_i \in \text{Br}_4$ be the braid generator corresponding to the transposition $(i \ i + 1) \in S_4$, and let $\beta = (\beta_1 \beta_2 \beta_3)\beta_1$. It turns out that

\begin{equation}
R_\beta = q^{19/2} - q^{13/2} \tau + (q^{1/2} - q^{-1/2})\phi + q^{-13/2} \tau' - q^{-19/2} \phi'.
\end{equation}

The $q$-coefficients of $\Omega_\beta$ are given by the following table.

<table>
<thead>
<tr>
<th></th>
<th>$-8$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>$\phi$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$\tau'$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\phi'$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Above, $\tau' = \Lambda^2 \tau$ and $\phi' = \Lambda^3 \tau$. The $(\Lambda^j \tau)$-rows list out the coefficients of the HOMFLY invariant of the link closure of $\beta$. Explicitly, this link is the $(2, 13)$-cable of the blackboard framing of the trefoil.

Remark 4.14. The example above exhibits the striking feature that all of the coefficients of $\Omega_\beta$ are non-negative. It would be interesting to determine necessary or sufficient conditions on $\beta$ for this to happen in general.
4.3. **Regular Braids.** In [7, §3], Broué–Michel gave a new characterization of the regular elements of $W$. In doing so, they also defined a similar property for the elements in $\text{Br}_W$.

Let $w_0 \in W$ be the longest element of $(W, S)$ with respect to Bruhat length, and let

$$\pi = \beta_{w_0}^2.$$  

Then $\pi$ is a canonical central element of $\text{Br}_W$ called the *full twist*, which maps to 1 under $\text{Br}_W \rightarrow W$. For any $v \in \mathbb{Q}_{>0}$, we say that $\beta$ is a *positive $v$th power* of $\pi$ iff, writing $v = m/n$ in lowest terms, we have

$$\beta^n = \pi^m.$$  

In this situation, we also say that $\beta$ is *regular of slope* $v$. For example, $\beta_{w_0}$ is regular of slope $1/2$, whereas the braid in Example 4.13 is not regular.

**Theorem 4.15** (Broué–Michel). Fix $v \in \mathbb{Q}_{>0}$ and let $v_0 = v - [v]$. For any $w \in W$, the following are equivalent:

1. $w$ is regular of slope $[v]$.
2. $w$ admits some lift to $\text{Br}_W$ that is regular of slope $v$.
3. $\beta_w$ is regular of slope $v_0$.

**Proof.** This combines Prop. 3.11, Th. 3.12, and Cor. 3.13 of [7]. □

Let $\beta$ be a regular braid of slope $v$, and let $w$ be the image of $\beta$ in $W$. In the rest of this section, we derive a formula for $R_\beta$ in terms of the values of generic degrees at $e^{2\pi i v}$. First, a definition:

**Definition 4.16.** We define the *content of* $\phi$ to be

$$c(\phi) = \frac{1}{\phi(1)} \sum_{t \in T} \phi(t).$$  

**Remark 4.17.** When $W = S_n$ and $\phi$ is the character of the Specht module that corresponds to the partition $\lambda \vdash n$, one can check that $c(\phi)$ is the content of the standard Young tableau of $\lambda$.

**Lemma 4.18.** The function $c$ is constant in Lusztig’s families.

**Proof.** It essentially follows from [7, Cor. 4.2], cf. [15, §6.2], that

$$c(\phi) = N - a(\phi) - A(\phi),$$  

where $a(\phi) = \text{val}_q D_\phi(q)$ and $A(\phi) = \text{deg}_q D_\phi(q)$. Lusztig showed that the functions $a$ and $A$ are constant in families [24, Ch. 4]. □

**Lemma 4.19.** If $\beta$ is a regular braid of slope $v$ and $w$ is its image in $W$, then $\phi_q(\beta) = q^{ve(\phi)} \phi(w)$ for all $\phi \in \check{W}$.
Proof. The following proof is inspired by Jones’s argument in Section 9 of [20].

By a theorem of Springer [16, Thm. 9.4.3], \( \pi \) acts on the underlying \((H_\phi \otimes \mathbb{C})\)-module of \( \phi_q \otimes \mathbb{C} \) by the scalar \( q^{\epsilon(\phi)} \). Specialize \( q \) to a generic complex number \( q \) and write \( v = m/n \) in lowest terms. Then \( q^{-me(\phi)\pi^m} \) acts on the same module by the identity matrix \( I \), and \( q^{-ve(\phi)} \beta \) acts by an \( n \)th root of \( I \).

Since the (diagonalizable) roots of \( I \) do not deform, the deformation \( q \to 1 \) shows that the eigenvalue spectrum of \( q^{-ve(\phi)} \beta \) must match that of \( \phi \) acting on the underlying \( \mathbb{C}[\hat{W}] \)-module of \( \phi \). \( \square \)

By combining Lemma 4.19, Lemma 4.18, Theorem 2.2, and (2.17), we compute:

\[
\sum_{\phi} \{ \phi, \psi \} \phi_q(\beta) = \sum_{\phi} q^{ve(\phi)} \{ \phi, \psi \} \phi(w)
\]

(4.23)

\[
= q^{ve(\psi)} \sum_{\phi} \{ \phi, \psi \} \phi(w)
\]

\[
= q^{ve(\psi)} D_\psi(e^{-2\pi iv}).
\]

We can actually get rid of the negative sign. Proposition 4.7 implies that \( D_\psi(e^{-2\pi iv}) \in \mathbb{Z} \), so by Galois theory, \( D_\psi(e^{-2\pi iv}) = D_\psi(e^{2\pi iv}) \). We arrive at:

**Theorem 4.20.** If \( \beta \) is a regular braid of slope \( v \), then

(4.24)

\[
R_\beta = \sum_{\phi} q^{ve(\phi)} D_\phi(e^{2\pi iv}) \phi
\]

in \( \mathbb{Z}[q^{\pm \frac{1}{2}}][\hat{\mathbb{W}}] \).

**Example 4.21.** Suppose that \( \beta = \pi^m \) for some \( m \in \mathbb{Z}_{>0} \). Then \( D_\phi(e^{2\pi im}) = \phi(1) \) for all \( \phi \), so the above result says

(4.25)

\[
R_{\pi^m} = \sum_{\phi} q^{me(\phi)} \phi(1) \phi.
\]

If instead \( \beta = 1 \), then \( \beta \) is no longer regular. But by replacing Lemma 4.19 with the fact that \( \phi_q(1) = \phi(1) \), we see that the formula remains true at \( m = 0 \):

(4.26)

\[
R_1 = \sum_{\phi} \phi(1) \phi.
\]

In other words, \( R_1 \) is the character of the regular representation of \( \mathbb{W} \). We claim that this implies

(4.27)

\[
\Omega_1 = \left( \frac{q^{1/2}}{1-q} \right)^r \sum_{\phi} \phi(1) \phi.
\]
Indeed, $R_1' = R_1$ and

$$S(q) \otimes R_1 = \sum_{\phi} \langle \phi, S(q) \rangle (\phi \otimes R_1)$$

$$= \sum_{\phi} \langle \phi, S(q) \rangle \phi(1) \cdot R_1$$

$$= \sum_{i} q^i (\text{Sym}^i \tau)(1) \cdot R_1$$

$$= (1 - q)^{-r} R_1,$$

(4.28)

as needed.

We conclude this section with some further expectations about regular braids. Compare part (1) with Remark 4.14.

**Conjecture 4.22.** Let $\beta$ be a regular braid of slope $v$.

1. The coefficients of $\Omega_\beta$ are non-negative.
2. If $v$ is elliptic, then $\Omega_\beta$ is finite-dimensional, i.e., $\langle \psi, \Omega_\beta \rangle$ is a Laurent polynomial for all $\psi \in \hat{W}$.

5. **Representations of the Rational DAHA**

5.1. **The Rational DAHA.** Let $(-, -) : t \times t^\vee \to \mathbb{C}$ be the evaluation pairing. For every reflection $t \in T$, fix a root $\alpha_t \in t^\vee$ and coroot $\alpha_t^\vee \in t$ such that $\langle \alpha_t^\vee, \alpha_t \rangle = 2$.

For any $v \in \mathbb{C}$, let $A_v = A_v(W)$ be the rational double affine Hecke algebra (DAHA), or rational Cherednik algebra, of $W$ of constant parameter $v$. By definition, it is the quotient of $Q[W \ltimes (t \oplus t^\vee)]$ by the relations

$$xy - yx = \langle x, y \rangle + v \sum_{t} \langle x, \alpha_t \rangle \langle \alpha_t^\vee, y \rangle t$$

(5.1)

for all $(x, y) \in t \times t^\vee$.

Like the universal enveloping algebra of a semisimple Lie algebra, $A_v$ admits a triangular decomposition and a “category $O$,” denoted $O_v$. The standard/Verma modules in $O_v$ are indexed by $\hat{W}$. For all $\phi \in \hat{W}$, we write $\Delta_v(\phi)$ for corresponding standard module and $L_v(\phi)$ for its simple quotient. In particular, $\Delta_v(1)$ is known as the polynomial module of $A_v$ and $L_v(1)$ is known as the simple spherical module of $A_v$. For general $\phi$, the module $\Delta_v(\phi)$ can be constructed explicitly by inflating $\phi$ from $Q[W]$ to $Q[W \times t^\vee]$, then tensoring up to $A_v$.

There is a canonical “Euler” element $h \in A_v$, such that $h$ commutes with $W$ and its action on any module $M \in O_v$ is locally finite. This endows the module $M$ with a decomposition into finite-dimensional, $W$-stable $h$-eigenspaces $M_\alpha$. We write $[M]$ for the resulting $q$-graded $W$-character, i.e.,

$$[M] = \sum_{\alpha} q^{\alpha} M_\alpha |_{\mathbb{C}[W]}.$$
By construction, we have
\[ [\Delta_v(\phi)] = q^{r/2 - w(\phi)} S(q) \otimes \phi \]
for all \( \phi \) [11, Ch. 11].

### 5.2. Supports and the KZ Functor.

Any module \( M \in \mathcal{O}_v \) can be viewed as a \( W \)-equivariant quasicoherent sheaf on \( t = \text{Spec } \mathbb{C}[t^\vee] \). From this viewpoint, the support of \( M \) is the \( W \)-stable subvariety of \( t \) that forms its support as a sheaf. One can show that the support of \( M \) is \( \emptyset \) if and only if \( M \) is finite-dimensional.

Let \( \text{Mod}(\mathcal{H}_v) \) be the category of finite-dimensional \( \mathcal{H}_v \)-modules. In [15], the authors introduced the Knizhnik–Zamolodchikov (KZ) functor, an exact, monoidal functor
\[ \text{KZ} : \mathcal{O}_v \to \text{Mod}(\mathcal{H}_v). \]

We summarize its relevant properties:

**Theorem 5.1** (Ginzburg–Guay–Opdam–Rouquier).

1. \( \text{KZ} \) induces an equivalence \( \mathcal{O}_v/\mathcal{O}_v^{\text{tor}} \simeq \text{Mod}(\mathcal{H}_v) \), where \( \mathcal{O}_v^{\text{tor}} \subseteq \mathcal{O}_v \) is the full subcategory of modules supported on the discriminant locus \( t \setminus t^\circ \).
2. \( \text{KZ} \) induces an isomorphism between the center of \( \mathcal{O}_v \) and the center of \( \mathcal{H}_v \). Thus it induces a bijection between the blocks of \( \mathcal{O}_v \) and the blocks of \( \text{Mod}(\mathcal{H}_v) \) (i.e., \( \Phi_v \)-blocks).
3. \( \text{KZ}(\Delta_v(\phi)) = \phi_v \).

**Proof.** (1) is [15, Thm. 5.14], (2) is [15, Cor. 5.18], and (3) is [15, Thm. 6.8]. \( \square \)

While the result below follows from part (2), we believe that [32, Thm. 5.15] is the first time it is stated formally.

**Theorem 5.2.** Suppose that
\[ \{ \phi_0 > \phi_1 > \cdots > \phi_{n-1} \} \]
is a block of \( \mathcal{H}_v \) of defect 1, ordered by Lusztig’s function \( a \). Then, in \( \mathcal{O}_v \), there is a BGG resolution of \( \Delta_v(\phi_0) \) of the form
\[ \Delta_v(\phi_{n-1}) \to \cdots \to \Delta_v(\phi_1) \to \Delta_v(\phi_0) \to L_v(\phi_0) \to 0. \]

In particular, \( |L_v(\phi_0)| = \sum_{0 \leq k \leq n-1} (-1)^i [\Delta_v(\phi_k)] \).

### 5.3. Singular Slopes.

When \( v \) is a generic complex number, \( L_v(\phi) = \Delta_v(\phi) \), but when \( v \) takes special values, \( L_v(\phi) \) can be much smaller. To wit, suppose that \( v \in \mathbb{Q}_{>0} \). Let
\[ I_v(W) = \{ i \in I(W) : v \in \frac{1}{d_i} \mathbb{Z} \}. \]

Concretely, if \( n \) is the denominator of \( v \) in lowest terms, then \( I_v(W) \) is the set of indices \( i \) such that \( n \) divides \( d_i \). We say that \( v \) is a singular slope of \( W \) iff \( |I_v(W)| \geq 1 \). It is a theorem of Springer [34, Thm 4.2(iii)] that any regular slope is a singular slope, though the converse is false.
(1) Work of Dunkl–de Jeu–Opdam [10] shows that \( v \) is a singular slope of \( W \) if and only if \( L_v(1) \in \mathcal{O}_v^{tor} \).

(2) Work of Etingof [12], extending work of Varagnolo–Vasserot [36] in the crystallographic case, shows that \( v \) is a regular elliptic slope of \( W \) if and only if \( L_v(1) \) is finite-dimensional.

Our first result in this subsection is a slight strengthening of (1). Consider the formal linear combination of standard modules

\begin{equation}
M_v = \sum_{\phi} D_\phi(e^{2\pi i v}) \Delta_\phi(\phi).
\end{equation}

By Proposition 4.7 and Theorem 4.20, the coefficients in the sum are integers, so

\begin{equation}
M_v = \sum_{\phi} m_v(\phi) L_v(\phi)
\end{equation}

for some \( m_v(\phi) \in \mathbb{Z} \). Note that we always have \( m_v(1) = 1 \). Indeed, \( D_1(q) = 1 \) and the only standard module to contain \( L_v(1) \) as a simple constituent is \( \mathcal{O}_v^{tor} \).

**Theorem 5.3.** If \( v \) is a singular slope, then \( L_v(\phi) \in \mathcal{O}_v^{tor} \) for all \( \phi \) such that \( m_v(\phi) \neq 0 \).

**Lemma 5.4.** Let \( \mu_q = \sum_{\phi} D_\phi(q) \phi_q \), an element of \( K[q][\hat{\mathcal{H}}_\omega] \). Then

\begin{equation}
\mu_q(\beta_w) = \begin{cases} 
s_1(q) & w = 1 \\
0 & w \neq 1
\end{cases}
\end{equation}

To prove the lemma, we reduce to the case where \( W \) is irreducible via the following: If \( W \cong W_1 \times W_2 \), then

\begin{equation}
D_{\phi_1 \times \phi_2}(q) = D_{\phi_1}(q) \cdot D_{\phi_2}(q)
\end{equation}

for all \( \phi_i \in \hat{W}_i \). Once again, we give a uniform proof for crystallographic types using the ideas of Section 3, and check the other types case by case.

**Proof for crystallographic types.** In the notation of §3.1, we see that \( \mu_q|_{q^{1/2} \to q^{1/2}} \) is the character of the \( \mathcal{H}_{\hat{Q}_\ell} \)-module \( \hat{Q}_\ell[B(F_q)] \). Recall that the \( \mathcal{H}_{\hat{Q}_\ell} \)-action on this module is given by the identifications

\begin{equation}
\mathcal{H}_{\hat{Q}_\ell} \cong \hat{Q}_\ell[G(F_q)]/(B \times B)(F_q)] \cong \text{End}_{G(F_q)}(\hat{Q}_\ell[B(F_q)]).
\end{equation}

where the first isomorphism takes \( \tilde{\beta}_w \) to the indicator function on the \( G(F_q) \)-orbit \( O_w \). For \( w \neq 1 \), this means \( \tilde{\beta}_w \) acts on \( \hat{Q}_\ell[B(F_q)] \) by an operator with zeros along its diagonal. We deduce that \( \mu_q(\tilde{\beta}_w) = 0 \), whence \( \mu_q(\beta_w) = 0 \).

For \( w = 1 \), we instead observe that \( \mu_q(\beta_1) = |B(F_q)| \). It is well-known that \( |B(F_q)| \) is the Poincaré polynomial of \( W \) in the variable \( q \).

**Proof of Thm. 5.3.** We claim it suffices to show that the virtual \( \mathcal{H}_v \)-module \( \text{KZ}(M_v) \) is the zero module. For, by Theorem 5.1(1), the KZ functor restricts to a bijection between the
simple objects of $O_v/O_v^{tor}$ and the simple modules of $\mathcal{H}_v$. The simple objects of $O_v/O_v^{tor}$ are precisely the images of the simple objects of $O_v$ that do not come from $O_v^{tor}$.

By Theorem 5.1(3), we have

$$KZ(M_v) = \sum_{\phi} D_\phi(e^{2\pi i v}) \phi_v. \quad (5.13)$$

This is the image, under the decomposition map $d_\psi$ of §2.3.2, of the $\mathcal{H}_\sigma$-module whose character is $\mu_{q_1}$. So we want to show that $\mu_{q_1}(\beta_w)|q_{1/2} \to e^{\pi iv} = 0$ for all $w$. By the lemma, the only nontrivial case is $w = 1$, where we must show that if $v$ is a singular slope, then $q = e^{2\pi iv}$ is a root of $s_1(q)$. This follows from (2.15).

Our second result shows that when $v$ is regular, not just singular, $[M_v]$ is a Markov character.

**Theorem 5.5.** $[M_v] = \Omega_\beta$ for any regular braid $\beta$ of slope $v$.

**Lemma 5.6.** If $v$ is a regular slope of $W$, then $D_\psi(e^{2\pi iv}) = (-1)^{2vN} D_\phi(e^{2\pi iv})$ for all $\psi \in \hat{W}$.

**Proof.** Let $w \in W$ be regular of slope $[v]$. By Lemma 4.11,

$$D_\psi(e^{2\pi iv}) = \sum_{\phi} \{\phi, \psi\} \phi(w)$$

$$= \sum_{\phi} \{\phi, \psi\} \phi'(w)$$

$$= 1'(w) \cdot D_\psi(e^{2\pi iv}). \quad (5.14)$$

Finally, $1'(w) = (-1)^{2vN}$ because the Bruhat length of $w$ is $2N(v - [v])$. \hfill \square

**Proof.** Using the lemma above, we compute

$$\sum_{\phi} D_\phi(e^{2\pi iv})[\Delta_v(\phi)] = q^{r/2} S(q) \otimes \sum_{\phi} q^{-\nu(\phi)} D_\phi(e^{2\pi iv}) \phi$$

$$= q^{r/2} S(q) \otimes \sum_{\phi} q^{-\nu(\phi')} D_\phi'(e^{2\pi iv}) \phi'$$

$$= (-1)^{2vN} q^{r/2} S(q) \otimes \sum_{\phi} q^{\nu(\phi)} D_\phi(e^{2\pi iv}) \phi'. \quad (5.15)$$

We know $wr(\beta) = 2vN$, so Theorem 4.20 completes the proof. \hfill \square

The conjecture below is to Etingof’s result as Theorem 5.3 is to Dunkl–de Jeu–Opdam’s result. In conjunction with Theorem 5.5, it would also imply Conjecture 4.22(2).

**Conjecture 5.7.** If $v$ is a regular elliptic slope, then $L_v(\phi)$ is finite-dimensional for all $\phi$ such that $m_v(\phi) \neq 0$. 

5.4. Cuspidal Slopes. From the Bott–Solomon formula (2.15), we have

\[ s_1(q) = \prod_{n \geq 1} \Phi_n(q)^{i \in I(W) : n \mid d_i}. \] (5.16)

In the terminology of §2.3.2, we deduce that:

**Lemma 5.8.** The \( \Phi_\nu \)-defect of the principal block of \( \mathcal{H}_\nu \) is precisely \( |I_\nu(W)| \).

Following Bezrukavnikov–Etingof in [2], we say that \( \nu \in \mathbb{Q}_{>0} \) is a cuspidal slope of \( W \) iff \( |I_\nu(W)| = 1 \). Any cuspidal slope is a regular elliptic slope, giving a hierarchy:

cuspidal \( \subseteq \) regular elliptic \( \subseteq \) regular \( \subseteq \) singular.

The table below lists the cuspidal numbers, i.e., denominators in lowest terms of cuspidal slopes, of the irreducible Coxeter groups.

<table>
<thead>
<tr>
<th>( W )</th>
<th>( {d_i}_i )</th>
<th>cuspidal numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_r )</td>
<td>2, 3, 4, \ldots, ( r + 1 )</td>
<td>( r + 1 )</td>
</tr>
<tr>
<td>( BC_r )</td>
<td>2, 4, 6, \ldots, 2( r )</td>
<td>2( r )</td>
</tr>
<tr>
<td>( D_r )</td>
<td>2, 4, 6, \ldots, 2(( r - 1 )), ( r )</td>
<td>2(( r - 1 ))</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>2, 5, 6, 8, 9, 12</td>
<td>9, 12</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>2, 6, 8, 10, 12, 14, 18</td>
<td>14, 18</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>2, 8, 12, 14, 18, 20, 24, 30</td>
<td>15, 20, 24, 30</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>2, 6, 8, 12</td>
<td>8, 12</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>2, 6, 10</td>
<td>6, 10</td>
</tr>
<tr>
<td>( H_4 )</td>
<td>2, 12, 20, 30</td>
<td>12, 15, 20, 30</td>
</tr>
<tr>
<td>( I_2(h) )</td>
<td>2, ( h )</td>
<td>( k \mid h ) such that ( k &gt; 2 )</td>
</tr>
</tbody>
</table>

As the next result shows, the cuspidal slopes are the singular slopes where we attain the greatest control over the blocks of \( \mathcal{H}_\nu \).

**Lemma 5.9.** The \( \Phi_\nu \)-defect of any \( \Phi_\nu \)-block is bounded above by \( |I_\nu(W)| \). Equality holds if and only if \( D_\phi(e^{2\pi i \nu}) \neq 0 \).

Moreover, if \( |I_\nu(W)| \leq 1 \), then there are two kinds of blocks:

1. Any block of defect 0 is a singleton. If \( \phi \) belongs to such a block, then \( D_\phi(e^{-2\pi i \nu}) = 0 \).
2. Any block of defect 1 is a line ordered by \( a \). If

\[ \{\phi_0 > \phi_1 > \cdots > \phi_{n-1}\} \]

is such a block, then \( D_{\phi_j}(e^{2\pi i \nu}) = (-1)^j \) for all \( j \).

**Proof.** The only claim that does not follow from Theorem 2.4 and Lemma 5.8 is the claim about the generic degrees in case (2).

By definition, \( D_{\phi_j}(e^{2\pi i \nu}) = (s_1/s_{\phi_j})|_{q \rightarrow e^{2\pi i \nu}} \). The work of Broué [6, §3.7] shows that such Schur-element ratios are Morita invariants of the block ideal. (Strictly speaking, Broué considered group algebras, not cyclotomic Hecke algebras, but see the footnote on page 286.)
of \([8]\). Therefore, by part (2) of Theorem 2.4, they match the corresponding ratios for the Brauer tree algebra of type \(A_{n-1}\), letting us reduce to the case where \(W\) is the symmetric group on \(n\) letters and \(v = 1/n\). Here, \(\phi_j = \Lambda^j \tau\) and the identity \(D_{\Lambda^j \tau}(e^{2\pi i/n}) = (-1)^j\) is well-known. \(\square\)

**Theorem 5.10.** If \(v\) is a cuspidal slope of \(W\), then

\[
[M_v] = \sum_{\phi_0} [L_v(\phi_0)],
\]

where \(\phi_0\) runs over the initial elements of the \(\Phi_v\)-blocks of defect 1.

**Proof.** Combine the lemma with Theorem 5.2. \(\square\)

As Bezrukavnikov–Etingof observed, there are only two cases where \(W\) is irreducible, \(v\) is a cuspidal slope, and there is a \(\Phi_v\)-block of defect 1 besides the principal block: Namely, \(W\) is either of type \(E_8\) or of type \(H_4\), and the denominator of \(v\) in lowest terms is 15. In these cases,

\[
[M_v] = [L_v(1)] + [L_v(\tau)].
\]

In all other cases where \(v\) is cuspidal,

\[
[M_v] = [L_v(1)].
\]

Combined with Theorem 5.5, this recovers the Gorsky–Oblomkov–Rasmussen–Shende identity when \(W\) is of type \(A\) and we take \((\Lambda^j \tau)\)-isotypic components.

**Remark 5.11.** If \(W \simeq W_1 \times W_2\), then:

- \(|I_v(W)| = |I_v(W_1)| + |I_v(W_2)|\).
- The category \(\mathcal{O}\) of \(A_v(W)\) is the \(\otimes\)-product of the categories \(\mathcal{O}\) of \(A_v(W_1)\) and \(A_v(W_2)\).

Using these facts, one can generalize Theorem 5.10 to any situation where \(|I_v(W')| \leq 1\) for every irreducible factor \(W' \subseteq W\).

6. The \(t\)-Grading

In this section, we assume \(W\) is crystallographic and fix notation as in Section 3. We will explain how to refine \(\Omega_\beta\) to accommodate a second grading that in type \(A\) recovers the \(t\)-grading of HOMFLY homology.

6.1. Soergel Bimodules. Fix a split pinning \(T_0 = B_0 \subseteq G_0\), i.e., \(B_0\) is a Borel of \(G_0\) and \(T_0\) its split maximal torus. Let \(B = B_0 \otimes \hat{F}_q\) and \(T = T_0 \otimes \hat{F}_q\). Let \(R\) be the ring

\[
\hat{Q}_e[[t^\nu]] \simeq H_T(\cdot) \simeq H_B(\cdot).
\]

In what follows, we write \(H(\cdot)\) to denote the hypercohomology of a complex.
We have identifications $B \simeq G/B$ and $G \setminus (B \times B) \simeq B \setminus G/B$. Thus, for any $w \in W$, the $G$-equivariant hypercohomology

$B_w = H^*_G(\text{IC}_w)$

(6.2)

can be identified with the $(B \times B)$-equivariant hypercohomology of a complex on $G$. The latter admits the structure of a bimodule over $R$, or equivalently, the structure of a $R \otimes R^{\text{op}}$-module. One can show that $B_w$ is always free as a left $R$-module.

We view $R$ as a graded $\mathbb{Q}_t$-algebra, with $t$ placed in degree 2. Thus $B_w$ inherits a grading from $R$. We further define an internal grading shift $[m]$ on $B_w$ by setting

$B_w(m)^i = B_w^{i+m}$.

(6.3)

Soergel showed that the map $\text{IC}_n \mapsto B_w(m)$ extends to a fully-faithful monoidal functor from $H$ to the category of $R$-bimodules graded as left $R$-modules. Its essential image is known as the category of Soergel bimodules attached to $(W, S)$ and $t$.

Given any $R$-bimodule $M$, its $m$th Hochschild homology is defined by

$\text{HH}_m(M) = \text{Tor}_{R}(R, M)$.

(6.4)

If $M$ is a graded left $B_1$-module, then $\text{HH}_m(M)$ receives an internal grading from $M$. We denote its $n$th graded component by $\text{HH}_m(M)^n$. With this notation, the following is the main result of [37], once combined with equation (5) in [38].

**Theorem 6.1** (Webster–Williamson). For all $w \in W$, let $H^{i,j}_G(K_w)$ be the $j$th weight space of $H^*_G(K_w)$, i.e., its $q^j$-eigenspace under Frobenius. Then

$H^{i,j}_G(K_w) \simeq \text{HH}_{j-i}(B_w)^j$

(6.5)

or equivalently,

$\text{HH}_m(B_w)^n \simeq H^n_{G}^{n-m,n}(K_w)$.

(6.6)

These isomorphisms are compatible with the shifts $(-)$ and $(-)$ via $(1) \mapsto (1)$.

**Example 6.2.** Since $K_1$ can be viewed as the pushforward of the constant sheaf along $B \setminus B \to G \setminus G$, we have

$H^*_G^{n+m}(K_1) \simeq H^*_T^{n-m}(\cdot) \otimes H^m(T)$

$\simeq \text{Sym}^{n-m}(t^\vee) \otimes \Lambda^m(t^\vee)$.

(6.7)

In fact, this is an isomorphism of $\mathbb{Q}_t[W]$-algebras. Cf. equation (7) in [38].

6.2. **Rouquier Complexes.** Recall that $H$ is the full additive subcategory of $D^b_G(B \times B)$ generated by the objects $\text{IC}_w(n)$. Let $K^b(H)$ be its homotopy category of complexes.

For any $s \in S$, consider the objects of $K^b(H)$ defined by

$F_{\beta_s} = 0 \to [\text{IC}_s] \to \text{IC}_1(1) \to 0$.

(6.8)

$F_{\beta_s^{-1}} = 0 \to \text{IC}_1(-1) \to [\text{IC}_s] \to 0$.

(6.9)
where the boxes indicate the terms in degree 0 and the morphisms are induced by pullback and Gysin pushforward along the inclusion map \( O_1 \to \hat{O}_s = O_1 \cup O_{r} \). (Recall that \( \text{IC}_s \) is already shifted to be perverse.) More generally, for any braid \( \beta = \beta_{s_1}^{e_1} \beta_{s_2}^{e_2} \cdots \in \text{Br}_W \), we set

\[
F_\beta = F_{\beta_{s_1}^{e_1}} \otimes F_{\beta_{s_2}^{e_2}} \otimes \ldots
\]

Rouquier essentially proved \([31]\) that the map \( \beta \mapsto F_\beta \) defines a monoidal functor from \( \text{Br}_W \) to \( K^b(\mathcal{H}) \) under \( \otimes \). That is: \( F_\beta \) is independent of the word used to represent \( \beta \) up to homotopy, and the complexes \( F_{\beta_{s}^{\pm 1}} \) satisfy the braid relations up to homotopy. One can check that the map \( \text{Br}_W \to \mathcal{H} \) factors as the composition

\[
\text{Br}_W \to K^b(\mathcal{H}) \to \mathcal{H} \to \mathcal{H},
\]

where the first map sends \( \beta \mapsto F_\beta \), the second takes the Euler characteristic of a complex, and the third is decategorification to the split Grothendieck group.

**Remark 6.3.** If \( \beta \in \text{Br}_W^+ \), then \( F_\beta \) and \( F_{\beta^{-1}} \) are the images, under the chromatographic functor of Webster–Williamson, of explicit sheaf complexes on \( B \times B \) of geometric origin, closely related to the generalized Deligne–Lusztig varieties of Broué–Michel in \([7]\). For details, see Section 6 of \([33]\).

To be precise, Rouquier worked with the images of the complexes \( F_\beta \) under \( H_G(\mathcal{H}) \), i.e., with complexes of Soergel bimodules. Abusing notation, we will also denote the latter by \( F_\beta \). In \([19]\), Khovanov used these Soergel-bimodule complexes to give a new construction of triply-graded HOMFLY homology, which he and Rozansky had originally defined using matrix factorizations.

**Theorem 6.4 (Khovanov).** Let \( \beta \) be a braid in type \( A \) and let \( L \) be the link closure of the braid \( \beta \). Then

\[
\sum_{i,j,k} q^i a^j t^k HH_H^{i,j,k}(L) \simeq q^{r/2}(at)^{\text{wr}(\beta)} + r \sum_{i,m,k} q^{i/2}(q a^2 t)^{-m} t^{-k} H_k(H_H^{n}(F_\beta))^i
\]

as triply-graded vector spaces.

Let \( \text{Ch} \) denote the full additive subcategory of \( D^b(G/G) \) generated by the objects \( A_\phi(n) \) for \( \phi \in \hat{W} \) and \( n \in \mathbb{Z} \). The functor \( \text{Ind} : \mathcal{H} \to \text{Ch} \) induces a functor at the level of homotopy categories:

\[
\text{Ind} : K^b(\mathcal{H}) \to K^b(\text{Ch}).
\]

We set \( K_\beta = \text{Ind}(F_\beta) \). Theorem 6.1 implies that we can replace the right-hand side of (6.12) with

\[
q^{r/2}(at)^{\text{wr}(\beta)} + r \sum_{n,m,k} q^{n/2}(q^{1/2} a^2 t)^{-m} t^{-k} H_k(H_H^{n+m}(K_\beta)).
\]

As in Section 4, we will refine the \( a \)-grading of this series to a character decomposition.
6.3. **Categorifying the Markov Character.** Observe that \( H^*_G(-) = \text{Ext}^*_G(A_1, -) \), since \( A_1 \) is precisely the constant local system on \( G \). Extending earlier notation, let \( \text{Ext}^{i,j}_G(-, -) \) be the weight-\( j \) subspace of \( \text{Ext}^i_G(-, -) \) in general. We define a graded additive functor \( E^* \) from \( \text{Ch} \) under \( \langle - \rangle \) to (singly)-graded vector spaces by setting

\[
E^i(-) = \text{Ext}^{i,i}_G(K_1, -)
\]

for all \( n \). We claim that \( E^*(K) \) admits the structure of a graded \( (\hat{Q}_\ell[W] \rtimes R) \)-module. Indeed, the Yoneda product gives an associative, \( \hat{Q}_\ell \)-linear action map

\[
\text{Ext}^{i,j}_G(K_1, K_1) \times \text{Ext}^{i',j'}_G(K_1, -) \to \text{Ext}^{i+i', j+j'}_G(K_1, -),
\]

and above,

\[
\bigoplus_{i,j} \text{Ext}^{i,j}_G(K_1, K_1) \simeq \hat{Q}_\ell[W] \rtimes \bigoplus_{i,j} H^{i,j}_G(K_1)
\]

as bigraded \( \hat{Q}_\ell[W] \)-algebras, cf. Remark 2 of [38].

The functor \( E^* \) is our proposed refinement of the hypercohomology functor \( H^*_G \), as justified by the following results.

**Lemma 6.5.** We have

\[
\text{Ext}^{i,j}_G(A_\phi, A_\psi(n)) \simeq \text{Hom}_W(\phi, \psi \otimes H^{i+n,j+n}_G(K_1))
\]

for all \( \phi, \psi \in \hat{W} \).

**Proof.** This is proven by an isotypic enrichment of the proof of (6.17). For the case \( \phi = 1 \), see Proposition 11 of [38] (note that \( H^*_G(K_1) \) is self-dual as a \( W \)-representation).

\[\square\]

**Remark 6.6.** The isomorphisms of the lemma are functorial in the sense that, for any element of

\[
\text{Hom}^m_{\text{Ch}}(A_\psi, A_\xi(n)) \simeq \text{Hom}_W(\psi, \xi \otimes H^{n+m}_G(K_1)),
\]

the following coincide:

- The morphism \( \text{Ext}^{i,j}_G(A_\phi, A_\psi) \to \text{Ext}^{i,j+m}_G(A_\phi, A_\xi(n)) \) induced by the morphism \( A_\psi \to A_\xi(n) \).
- The morphism

\[
\text{Hom}_W(\phi, \psi \otimes H^{i,j}_G(K_1)) \to \text{Hom}_W(\phi, \xi \otimes H^{i+n,j+n+m}_G(K_1))
\]

induced by the morphism \( \psi \to \xi \otimes H^{n+m}_G(K_1) \).

**Corollary 6.7.** There is an isomorphism

\[
\text{Hom}_W(A^j \tau, E^i(-)) \simeq H^{i+j}_G(-)
\]

of additive functors from \( \text{Ch} \) to \( \hat{Q}_\ell \)-vector spaces.
Proof. We want to show
\[
\text{Hom}_W(\Lambda^j \tau, \text{Ext}_\text{Ch}^{i;i}(K_1, -)) \simeq \text{Ext}_\text{Ch}^{i;i+j}(A_1, -). 
\]
For all $\psi$ and $n$, we compute
\[
\text{Hom}_W(\Lambda^j \tau, \text{Ext}_\text{Ch}^{i;i}(K_1, A_\psi(n))) 
\simeq \bigoplus_\phi \text{Hom}_W(\Lambda^j \tau, \phi^\vee \otimes \text{Ext}_\text{Ch}^{i+n;i+n}(A_\phi, A_\psi))
\]
\[
\simeq \bigoplus_\phi \text{Hom}_W(\Lambda^j \tau, \psi \otimes \text{Sym}^{i+n-j}(t^\vee))
\]
\[
\simeq \text{Hom}_W(\Lambda^j \tau, \text{Sym}^{i+n-j}(t^\vee))
\]
\[
\simeq \text{Hom}_W(1, \psi \otimes \text{Sym}^{i+n-j}(t^\vee) \otimes \Lambda^j(t^\vee)),
\]
as needed. To check that the isomorphism is functorial, use the remark.

Applying $E^*$ to each term of $K_\beta$ gives us a complex $E_\beta$ of graded representations of $W$:
\[
E_\beta = \cdots \to E^*(K_\beta^1) \to \left[ E^*(K_\beta^0) \right] \to E^*(K_\beta^{-1}) \to \cdots
\]
As in §3.2, we use the variable $q$ to record the grading on each term above, identifying $h_1$ with $q^{1/2}$. This grading descends to a $q$-grading on the homology groups of $E_\beta$.

Definition 6.8. For any $\beta \in \text{Br}_W$, we define the $(q, t)$-Markov character of $\beta$ to be
\[
\Omega_\beta = q^{r/2} t^{wr(\beta)} \sum_{n,k} q^{n/2} t^{-k} H_k(E_\beta)^n,
\]
viewed as an element of $\mathbb{Z}[[q^{\pm 1/2}]][[t^{\pm 1}]][[\hat{W}]]$.

Theorem 6.9. $\Omega_\beta$ satisfies the following properties:
(1) $\Omega_\beta|_{t \to 1} = \Omega_\beta$ in the notation of Section 4.
(2) In type $A$, if $L$ is the link closure of the braid $\beta$, then
\[
a^{wr(\beta) + r} \sum_j \left( q^{1/2} a^2 t^{-j} \right)^{-j} \langle \Lambda^j \tau, \Omega_\beta \rangle
\]
is the Poincaré polynomial of the $(q, a, t)$-graded HOMFLY homology of $L$.

Proof. Part (1) follows from the identity $E^*(A_\phi) \simeq \phi \otimes \text{Sym}^*(t^\vee)$ and the Hopf trace formula, viz., the Euler characteristic of a complex agrees with the Euler characteristic of its total homology. Part (2) follows from Theorem 6.4 and Corollary 6.7.

Our last conjecture generalizes both Proposition 4.10 and a symmetry conjecture of Dunfield–Gukov–Rasmussen for HOMFLY homology [9].

Conjecture 6.10. If $\Omega_\beta$ is finite-dimensional, i.e., $\Omega_\beta \in \mathbb{Z}[[q^{\pm 1/2}]][[t^{\pm 1}]][[\hat{W}]]$, then it is invariant under $q^{1/2} \mapsto q^{-1/2} t^{-1}$.
REFERENCES


