Note. I am giving the second lecture; Meg Doucette is giving the first.

1. Functions

Holomorphic functions have a very idiosyncratic geometry. I will explain how some major theorems illustrate this. Henceforth, let $\Omega \subseteq \mathbb{C}$ be an open domain, $D \subseteq \Omega$ an open disk, and $\partial D$ the positively-oriented loop formed by the boundary of $D$.

Let $f : \Omega \rightarrow \mathbb{C}$ be holomorphic. The residue formula shows that if $f'$ has no zeros/poles on $\partial D$, then

$$
\frac{1}{2\pi i} \int_{\partial D} \frac{f'}{f} \, dz = \sum_{\text{poles } z \text{ of } f'} \text{res}_z(f') = \sum_{\text{zeros } z \text{ of } f} \text{ord}_z(f).
$$

(Above, we used the fact that any zero of $f$ of order $n$ corresponds to a simple pole of $f'/f$ of residue $n$.) At the same time, the substitution $w = f(z)$ shows that

$$
\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_{f(\partial D)} \frac{dw}{w} = \text{winding number of } f(\partial D) \text{ around } 0.
$$

In short, the number of zeros of $f$ in $D$, counting multiplicities, is given by a winding number! This result is known as Cauchy’s argument principle.

The following result says that if you perturb a function by a very small amount in a fixed neighborhood, then its zeros can move around, but their total multiplicity is conserved.

**Theorem 1.1 (Rouché).** If $h : \Omega \rightarrow \mathbb{C}$ is holomorphic and $|h| < |f|$ on $\partial D$, then $f + h$ has the same number of zeros as $f$ in $D$.

**Proof.** Let $f_t(z) = f(z) + t h(z)$. The winding number of $f_t$ can’t change discontinuously as $t$ runs through the interval $[0, 1]$.

Let’s use Rouché to prove the fundamental theorem of algebra. We claim that if

$$
f = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,
$$

then $f$ has $n$ zeros in $\mathbb{C}$. Indeed, in the statement of Rouché, let $h(z) = \sum_{0 \leq i \leq n-1} a_i z^i$ and pick a disk $D$ so big that $|z^n| > |h(z)|$ on the boundary $\partial D$.

**Corollary 1.2 (Open Mapping).** Any nonconstant holomorphic function is an open map.

**Proof.** Using Rouché, show that if $f(z) = 0$, then $D_\delta(0) \subseteq f(D_{\delta}(z))$ for some $\delta > 0$ and $\epsilon > 0$.

**Corollary 1.3 (Maximum Modulus).** If $f : \Omega \rightarrow \mathbb{C}$ is nonconstant, then $|f|$ cannot attain a maximum on $\Omega$.

**Proof.** By the open mapping theorem, any $z \in \Omega$ can be perturbed to a nearby value $w \in \Omega$ such that $|f(w)| > |f(z)|$. 


2. Spaces

Let’s use the open mapping theorem to give a second proof of the fundamental theorem of algebra. This proof will involve a new space, namely, the Riemann sphere

\[ \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \]

Let \( f(z) \) be a polynomial. As a map \( \mathbb{C} \to \mathbb{C} \), it extends to a map \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) that fixes \( \infty \). Since \( \hat{\mathbb{C}} \) is compact and Hausdorff, the image of \( \hat{f} \) is a closed set. But by the open mapping theorem, it is also open. Since \( \hat{\mathbb{C}} \) is connected, we deduce that \( \hat{f}(\hat{\mathbb{C}}) = \hat{\mathbb{C}} \), whence \( f(\mathbb{C}) = \mathbb{C} \).

**Theorem 2.1.** Every connected, simply-connected Riemann surface is conformally equivalent to one of the following:

1. The plane \( \mathbb{C} \).
2. The Riemann sphere \( \hat{\mathbb{C}} \).
3. The open unit disk \( \mathbb{D} \).

These options are conformally distinct.

We can learn a lot of mathematics simply by computing the conformal automorphism groups of \( \mathbb{C}, \hat{\mathbb{C}}, \mathbb{D} \). Let’s do them in that order.

**Theorem 2.2.** \( \text{Aut}(\mathbb{C}) \) is the group of affine transformations \( z \mapsto az + b \).

**Proof.** Conformal automorphisms are isometries, so for \( \mathbb{C} \), decompose into the composition of a translation and a homothety (i.e., rotation + dilation).

**Theorem 2.3.** \( \text{Aut}(\hat{\mathbb{C}}) \simeq \text{PGL}_2(\mathbb{C}) \), where the projective equivalence class of \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C}) \) acts by the fractional linear transformation

\[ \gamma \cdot z = \frac{az + b}{cz + d} \]

on \( \hat{\mathbb{C}} \).

**Proof.** A map \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is the same thing as a meromorphic function on \( \hat{\mathbb{C}} \). The latter is determined by its restriction to \( \mathbb{C} \). But we know that meromorphic functions on \( \mathbb{C} \) are rational functions, so take the form \( f(z)/g(z) \) for some polynomials \( f, g \). Invertibility forces \( \deg f = \deg g = 1 \).

**Example 2.4.** The Cayley transform is the fractional linear transformation given by

\[ \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \cdot z = \frac{i - z}{i + z} \]

It restricts to a conformal equivalence from the open upper half-plane \( \mathbb{H} \) to the disk \( \mathbb{D} \) that sends \( i \mapsto 0 \). What’s the inverse?

Before we compute \( \text{Aut}(\mathbb{D}) \), we prepare some notation. For all \( \alpha \in \mathbb{D} \), the Blaschke factor of \( \alpha \) is the fractional linear transformation

\[ \psi_{\alpha}(z) = \begin{pmatrix} -1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix} \cdot z = \frac{\alpha - z}{1 - \bar{\alpha} z}. \]

One can check directly that \( \psi_{\alpha} \) restricts to an involution of \( \mathbb{D} \) that swaps 0 and \( \alpha \).
Theorem 2.5. Every automorphism \( f \in \text{Aut}(D) \) takes the form

\[
f(z) = u \psi_u(z) = \begin{pmatrix} u & \alpha \\ 1 & -\bar{\alpha} \end{pmatrix} \cdot z
\]

for some \( u \in S^1 = \partial D \) and \( \alpha \in D \).

Lemma 2.6 (Schwarz). If holomorphic \( f : D \to D \) satisfies \( f(0) = 0 \), then \( |f(z)| \leq |z| \) for all \( z \). If moreover equality holds for some nonzero value of \( z \), then \( f \) is a rotation.

Proof of the theorem. Let \( \alpha = f^{-1}(0) \) and \( g = f \circ \psi_{\alpha} \). Then Schwarz’s lemma applies to both \( g \) and \( g^{-1} \), giving

\[
|g(z)| \leq |z| \quad \text{and} \quad |g^{-1}(z)| \leq |z|
\]

for all \( z \in D \). We deduce that \( |g(z)| \leq |z| \leq |g(z)| \), whence \( |g(z)| = |z| \), for all \( z \). Now the second part of the lemma says that \( g \) is a rotation.

Corollary 2.7. \( \text{Aut}(H) \simeq \text{PSL}_2(\mathbb{R}) \), where the action is by fractional linear transformations.

Proof. Since the Cayley transform is an equivalence \( H \to D \), it induces an isomorphism \( \text{Aut}(H) \simeq \text{Aut}(D) \). Now check what it does at the level of fractional linear transformations.

Corollary 2.8. There is a diffeomorphism \( H \simeq \text{PSL}_2(\mathbb{R})/\text{PSO}(2) \).

Proof. The stabilizer of \( i \in \text{SL}_2(\mathbb{R}) \) consists of the matrices of the form \( \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \). This is precisely \( \text{SO}(2) \subseteq \text{SL}_2(\mathbb{R}) \).

Another Proof. Under the Cayley transform, the stabilizer of \( i \in H \) corresponds to the stabilizer of \( 0 \in D \). The latter is the rotation group \( S^1 \subseteq \text{Aut}(D) \), which corresponds to \( \text{SO}(2) \subseteq \text{Aut}(H) \).

Corollary 2.9. \( \pi_1(\text{SL}_2(\mathbb{R})) \simeq \mathbb{Z} \).

3. Metrics

There is a stronger version of Schwarz’s lemma due to Georg Pick. To state it, define the hyperbolic metric on \( D \) by

\[
d(z, w) = \tanh^{-1} \left( \frac{|z - w|}{1 - \bar{z}w} \right).
\]

(Recall that \( \tanh^{-1}(x) = \frac{1}{2} \log \frac{1+x}{1-x} \), a monotonic function.) To give you some intuition for \( d \), I’ll draw the geodesics and horocycles in \( D \). It is similarly easy to draw the pictures for \( H \).

Lemma 3.1 (Schwarz–Pick). Any holomorphic \( f : D \to D \) is a contraction in the hyperbolic metric: \( d(f(z), f(w)) \leq d(z, w) \). Equivalently,

\[
\left| \frac{f(z) - f(w)}{1 - f(z)f(w)} \right| \leq \left| \frac{z - w}{1 - \bar{z}w} \right|.
\]

Moreover, \( f \) is an isometry for \( d \) if and only if \( f \) is an automorphism.

The elements of \( \text{PSL}_2(\mathbb{R}) \) can be classified according to what they do to the hyperbolic geometry of \( H \). For convenience, I’ll work with \( \text{SL}_2(\mathbb{R}) \) instead in what follows. If \( \gamma \in \text{SL}_2(\mathbb{R}) \), then we say that:
(1) Elliptic when \(|\text{tr } \gamma| < 2\). Equivalently, \(\gamma\) is of finite order and its eigenvalues are distinct complex conjugates.

(2) Parabolic when \(|\text{tr } \gamma| = 2\). Equivalently, \(\gamma\) is unipotent.

(3) Hyperbolic when \(|\text{tr } \gamma| > 2\). Equivalently, \(\pm \gamma\) can be conjugated into the diagonal torus \(T\) of \(\text{SL}_2(\mathbb{R})\).

To be more precise about (3): If \(\gamma\) is hyperbolic with positive eigenvalues, then we can write

\[
(3.3) \quad \beta \gamma \beta^{-1} = \begin{pmatrix} a \\ 1/a \end{pmatrix} \in T
\]

for some \(\beta \in \text{SL}_2(\mathbb{R})\) and \(a > 0\).

Let \(\tilde{T}\) be the image of \(T\) in \(\text{PSL}_2(\mathbb{R})\). The torus \(\tilde{T}\) acts on \(\text{PSL}_2(\mathbb{R})/\text{PSO}(2)\) by left multiplication. Since \(\tilde{T} \simeq \mathbb{R}\) and \(\text{PSL}_2(\mathbb{R})/\text{PSO}(2) \simeq \mathbb{H}\), this defines a flow on \(\mathbb{H}\). It turns out to be precisely the geodesic flow.