Introduction

В настоящей работе не содержатся сведения, которые могли бы составить предмет изобретения или открытия.

Анон., "Акт Экспертизы", ок. 1956.[†]

0.0. This book is an exposition of basic (local and global) aspects of chiral algebra theory. Chiral algebras have their origin in mathematical physics; they lie at the heart of conformal field theory[1]; see [**BPZ**]. Mathematicians, since the pioneering work of Borcherds [**B1**], usually look at them through the formalism of vertex algebras incorporated into representation theory of infinite-dimensional algebras. We follow a different approach which tastes more of algebraic geometry than of representation theory.

In the introduction we outline the principal structures involved and their interrelations. More specific information, together with bibliographical comments and references, can be found in brief introductions to sections.

0.1. Chiral algebras are "quantum" objects. Let us describe first the corresponding "classical" objects; we call them *coisson* algebras ("coisson" may be considered as an abbreviation for "chiral Poisson" or "compound Poisson", the word "compound" being related to the notion of compound tensor category which we discuss in 0.4). In fact, "coisson algebra" is a new name for a well-known class of objects. Informally, a coisson algebra is defined by a local Poisson bracket on a space of classical fields. In the simplest case where "classical field" means "a function $u(x)$" a mathematical physicist would define a local Poisson bracket by a formula like

$$\{u(x), u(y)\} = \sum_{i=1}^{n} \varphi_i(x, u(x), u'(x), \ldots)\delta^{(i)}(x-y), \quad \varphi_i \in A. \tag{0.1.1}$$

Here A denotes the algebra of functions of $x, u, u', \ldots, u^{(k)}$ (k is not fixed). Of course in the left-hand side of (0.1.1) $u(x)$ is understood as a functional $u \mapsto u(x)$ on the space of classical fields. The bracket should be skew-symmetric and satisfy the Jacobi identity, so the collection of functions φ_i should satisfy certain differential equations. For $f \in A$ and every x we have the functional $\ell_{f,x}$ on the space of

[†] "This article does not contain any information that could be the subject of an invention or a discovery.", Anon., "Expert Certification", circa 1956. (The signing of this Patriot Act was a precondition for publication of a mathematical paper in yonder Russia.)

[1] More precisely, of its purely holomorphic sector which is "holomorphic quantum mechanics".

classical fields defined by $\ell_{f,x}(u) = f(x, u(x), u'(x), \dots)$. It follows from (0.1.1) that

$$\{\ell_{f,x}, \ell_{g,y}\}(u) = \sum_i \varphi_i^{f,g}(x, u(x), u'(x), \dots)\delta^{(i)}(x-y) \tag{0.1.2}$$

for some $\varphi_i^{f,g} \in A$. Formula (0.1.2) has a more symmetric version:

$$\{\ell_{f,x}, \ell_{g,y}\}(u) = \sum_{i,j} \partial_x^i \partial_y^i \{\psi_{ij}^{f,g}(x, u(x), u'(x), \dots)\delta(x-y)\}, \quad \psi_{ij}^{f,g} \in A. \tag{0.1.3}$$

Of course the $\psi_{ij}^{f,g}$ are not uniquely determined by $\varphi_i^{f,g}$; we must take into account the relations

$$(\partial_x + \partial_y)(a\delta) = (\partial_x a) \cdot \delta, \quad a \in A, \tag{0.1.4}$$

where $\delta = \delta(x-y)$.

Now from the algebraic point of view the primary object is the algebra A rather than the "space of classical fields". For instance, one can take $A = \mathbb{C}[x, u, u', u'', \dots]$. More generally, A can be any commutative differential unital algebra over $\mathbb{C}[x]$, and a "classical field" is a homomorphism of differential unital $\mathbb{C}[x]$-algebras $A \to B$ where B is, e.g., the algebra of C^∞-functions of $x \in \mathbb{R}$. A choice of a set of generators and defining relations for the differential $\mathbb{C}[x]$-algebra A identifies classical fields with collections of functions satisfying certain systems of differential equations.

Let us try to formulate in terms of A what is a "local Poisson bracket on the space of classical fields". A glance at (0.1.4) and the right-hand side of (0.1.3) shows that such an object is defined by a mapping

$$A \underset{\mathbb{C}}{\otimes} A \to V \tag{0.1.5}$$

where V is the module over $\mathbb{C}[\partial_x, \partial_y]$ generated by the symbols $a\delta$, $a \in A$, with the defining relations (0.1.4). We prefer another interpretation of V: V is the module over $\mathbb{C}[x, y, \partial_x, \partial_y]$ generated by the symbols $a\delta$, $a \in A$, with the defining relations (0.1.4) and also the following ones:

$$f(x,y) \cdot (a\delta) = (f(x,x)a) \cdot \delta, \quad f \in \mathbb{C}[x,y],\ a \in A. \tag{0.1.6}$$

The mapping (0.1.5) should satisfy certain properties. First of all, it should be a morphism of $\mathbb{C}[x, y, \partial_x, \partial_y]$-modules (notice that $A \otimes A$ is a module over $\mathbb{C}\left[x, \frac{\partial}{\partial_x}\right] \otimes \mathbb{C}[x, \partial_y] = \mathbb{C}[x, y, \partial_x, \partial_y]$). Before discussing the other properties, let us rewrite (0.1.5) in geometric terms. An algebra A over $\mathbb{C}[x]$ is the same as a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ where $X := \operatorname{Spec} \mathbb{C}[x]$ is the affine line. Differential $\mathbb{C}[x]$-algebras correspond to $\mathcal{D}_X$*-algebras* (see 0.2). The $\mathcal{D}_{X \times X}$-module corresponding to the $\mathbb{C}[x, y, \partial_x, \partial_y]$-module V from (0.1.5) is canonically isomorphic to $\Delta_* \mathcal{A}$ where $\Delta: X \to X \times X$ is the diagonal embedding. So (0.1.5) is equivalent to a $\mathcal{D}_{X \times X}$-module morphism

$$\mathcal{A} \boxtimes \mathcal{A} \to \Delta_* \mathcal{A}. \tag{0.1.7}$$

We denote by $\boxtimes$ the external tensor product, i.e., $\mathcal{A} \boxtimes \mathcal{A} := p_1^* \mathcal{A} \otimes p_2^* \mathcal{A}$ where p_1 and p_2 are the projections $X \times X \to X$.

Notice that (0.1.7) makes sense for an arbitrary smooth variety X. In 0.6 we will define a coisson algebra on X as a commutative (= commutative associative unital) $\mathcal{D}_X$-algebra $\mathcal{A}$ with a $\mathcal{D}$-module morphism (0.1.7) satisfying certain properties which actually mean that (0.1.3) is a Poisson bracket on the "space(s) of classical fields". To formulate these properties in a natural way, we need some polylinear algebra in the category of $\mathcal{D}_X$-modules (see 0.2 – 0.5).

Remarks. (i) The notions of Poisson $\mathcal{O}_X$-algebra or Poisson $\mathcal{D}_X$-algebra are inadequate for expressing the idea of "Poisson bracket on the space(s) of classical fields": a Poisson bracket on $\mathcal{A}$ is a morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ while (0.1.7) is an object of a different nature. The reason is that $\operatorname{Spec} \mathcal{A}$ is the space of "jets of classical fields", not the space of classical fields.

(ii) The reader can compare our notion of coisson algebra with other ways of formalizing the notion of "Poisson bracket on the space of classical fields" (see [**DG1**], [**DG2**], [**Ku**], [**KuM**], [**Ma**] and references therein). As far as we understand, our approach is essentially equivalent to that of [**KuM**] and [**Ma**].

0.2. Let X be a smooth complex algebraic variety. Consider the category $\mathcal{M}^\ell(X)$ of left $\mathcal{D}$-modules on X (see, e.g., [**Ber**], [**Ba**], or [**Kas2**]). The $\mathcal{O}_X$-tensor product $\otimes$ makes it a tensor category with unit object $\mathcal{O}_X$. A commutative algebra in $\mathcal{M}^\ell(X)$ (*commutative* $\mathcal{D}_X$*-algebra*) is just a quasi-coherent commutative $\mathcal{O}_X$-algebra A^ℓ equipped with a flat connection along X. We call an X-scheme equipped with a flat connection along X a $\mathcal{D}_X$*-scheme*; so the spectra of commutative $\mathcal{D}_X$-algebras are $\mathcal{D}_X$-schemes affine over X.

A standard example: for any X-scheme $Y \to X$ the space $\mathcal{J}(Y/X)$ of ∞-jets of sections of Y/X is a $\mathcal{D}_X$-scheme. Horizontal sections of $\mathcal{J}(Y/X)$ are the same as arbitrary sections of Y/X. One may view the (closed) $\mathcal{D}_X$-subschemes of $\mathcal{J}(Y/X)$ as systems of differential equations on sections of Y/X (see [**G**]).

What makes the tensor category $\mathcal{M}^\ell(X)$ substantially different from, say, the tensor category of $\mathcal{O}_X$-modules is the absence of duals. Precisely, for $F \in \mathcal{M}^\ell(X)$ the dual object in the tensor category sense (see, e.g., [**D1**]) exists iff F is coherent as an $\mathcal{O}_X$-module, i.e., is a vector bundle with an integrable connection. For example, for a group $\mathcal{D}_X$-scheme G one has its Lie coalgebra $\operatorname{CoLie}(G)$ but we cannot dualize it to get the Lie algebra (unless G is finite dimensional as a usual scheme). We will see that the category $\mathcal{M}^\ell(X)$ carries a richer structure (that of *compound tensor category*) which remedies the above flaw.

0.3. Consider the category $\mathcal{M}(X)$ of right $\mathcal{D}$-modules on X. Let I be a finite non-empty set; denote by $\Delta^{(I)}\colon X \hookrightarrow X^I$ the diagonal embedding. For an I-family of $\mathcal{D}$-modules $L_i \in \mathcal{M}(X)$ and $M \in \mathcal{M}(X)$ set

$$P_I^*(\{L_i\}, M) := \operatorname{Hom}(\boxtimes L_i, \Delta_*^{(I)} M). \tag{0.3.1}$$

Elements of $P_I^*(\{L_i\}, M)$ are called $*$ I*-operations*; they are X-local (since $\Delta_*^{(I)} M$ sits on the diagonal). The $*$ operations compose in a natural way,[2] just as polylinear maps between vector spaces do (i.e., for a surjective map $\pi\colon J \to I$, a J-family $\{K_j\}$ of $\mathcal{D}$-modules, and $\varphi \in P_I^*(\{L_i\}, M)$, $\psi_i \in P^*_{\pi^{-1}(i)}(\{K_j\}, L_i)$ we have $\varphi(\psi_i) \in P_J^*(\{K_j\}, M)$). The composition is associative; if $|I| = 1$, then $P_I^* = \operatorname{Hom}$. We call such data of operations (or "polylinear maps") between the objects of a category

[2] We switched to right $\mathcal{D}$-modules to make the "sign rule" for the $*$ operations obvious.

a *pseudo-tensor structure*. Thus $\mathcal{M}(X)$ carries a canonical pseudo-tensor structure; $\mathcal{M}(X)$ equipped with this structure is denoted by $\mathcal{M}(X)^*$.

One may view pseudo-tensor categories as a straightforward generalization of operads: an operad is just a pseudo-tensor category with single object.

The notions of algebras, modules over them, etc., make perfect sense in any pseudo-tensor category. For example, a Lie algebra in $\mathcal{M}(X)^*$ (or simply a *Lie* algebra on* X) is a $\mathcal{D}$-module L together with a binary $*$ operation $[\ ] \in P_2^*(\{L, L\}, L)$ which is skew-symmetric and satisfies the Jacobi identity.

For $M \in \mathcal{M}(X)$ set $h(M) := M \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X$ (the sheaf of middle de Rham cohomology). There is a canonical map $P_I^*(\{L_i\}, M) \to \mathrm{Hom}(\otimes h(L_i), h(M))$ compatible with composition of operations. Therefore h sends Lie* algebras to the sheaves of usual Lie algebras. In fact, many important Lie algebras (including the Virasoro and affine Kac-Moody algebras) arise naturally from Lie* algebras.

0.4. Let us identify the categories of left and right $\mathcal{D}$-modules by the usual equivalence $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $M \mapsto M^\ell = M\omega_X^{-1} = M \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$ (here $\omega_X = \Omega_X^{\dim X}$). Therefore $\mathcal{M}(X)$ has the tensor product $M \otimes^! L := (M^\ell \otimes L^\ell)\omega_X$. Now the ! tensor product of two $*$ operations is defined according to the following pattern. Let I, J be finite sets, $i_0 \in I$, $j_0 \in J$; denote by $I \vee J$ the disjoint union of I, J with i_0, j_0 identified. Then there is a natural map

$$\otimes_{i_0,j_0}^{I,J}\colon P_I^*\left(\{M_i\}, L\right) \otimes P_J^*\left(\{N_j\}, K\right) \to P_{I\vee J}^*(\{M_i, N_j, M_{i_0} \otimes^! N_{j_0}\}_{\substack{i\neq i_0\\ j\neq j_0}}, L \otimes^! K).$$

If $I = \{i_0\}$, $J = \{j_0\}$ this is the usual tensor product of morphisms.

A *compound tensor structure* on a category is the data of a tensor and a pseudo-tensor structure related as above by the $\otimes_{i_0,j_0}^{I,J}$ maps which are associative and commutative in the obvious sense. Thus $\mathcal{M}(X)$ (and $\mathcal{M}^\ell(X)$) is a compound tensor category.

0.5. The compound tensor structure makes it possible to do basic differential geometry in $\mathcal{M}(X)$ implementing a complementarity principle: the "functions" multiply according to $\otimes^!$ format, while "operators" (such as infinitesimal symmetries) act in the $*$ sense. In other words, the geometric objects we consider are $\mathcal{D}_X$-schemes, the Lie algebras are actually Lie* algebras, and one defines easily what the action of a Lie* algebra on a $\mathcal{D}_X$-scheme means. Note that an action of a Lie* algebra L on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ yields an action on $\mathcal{Y}$ of the usual Lie algebra $h(L)$.

For example, let G be a group $\mathcal{D}_X$-scheme such that $\mathrm{CoLie}(G)$ is a locally free $\mathcal{D}_X$-module of finite rank. Set $\mathrm{Lie}(G) := \mathrm{CoLie}(G)^\circ = \mathrm{Hom}_{\mathcal{D}_X}(\mathrm{CoLie}(G), \mathcal{D}_X)$ (the usual duality of $\mathcal{D}$-modules theory). Then $\mathrm{Lie}(G)$ is a Lie* algebra. A G-action on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ yields a $\mathrm{CoLie}(G)$-coaction on $\mathcal{O}_\mathcal{Y}$ (in the $\otimes^!$ sense). Dualizing, we get a $\mathrm{Lie}(G)$-action on $\mathcal{Y}$.

In fact, for locally free $\mathcal{D}_X$-modules of finite rank the duality yields an identification $P_I^*(\{M_i\}, L) \xrightarrow{\sim} \mathrm{Hom}(L^\circ, \otimes M_i^\circ)$. So, on an appropriate derived category level, the $*$ operations can be recovered from the usual tensor structure and the duality functor $^\circ$. This is a very awkward thing to do however, when dealing with both ! and $*$ operations simultaneously (as happens in coisson algebras), and it precludes chiral quantization (see 0.8 and 0.9).

0.6. Now a *coisson algebra* is simply a Poisson algebra in the compound setting. Namely, this is a commutative $\mathcal{D}_X$-algebra A^ℓ together with a Lie* bracket on A (*coisson bracket*) such that the adjoint action is compatible with the multiplicative structure on A^ℓ. From this we get an action of $h(A)$ on A^ℓ in the usual sense. Derivations of A^ℓ that come from $h(A)$ are called hamiltonian vector fields. Coisson brackets localize nicely so one knows what is a coisson structure on any $\mathcal{D}_X$-scheme.

0.7. Let $\mathcal{Y} = \operatorname{Spec} A^\ell$ be an affine $\mathcal{D}_X$-scheme. The space of horizontal sections $X \to \mathcal{Y}$ is an ind-affine ind-scheme $\langle \mathcal{Y} \rangle = \langle \mathcal{Y} \rangle(X) = \operatorname{Spf} \langle A \rangle$; if X is compact, this is actually a scheme. This construction also makes sense locally. For example, for $x \in X$ the space of horizontal sections of $\mathcal{Y}$ over the formal punctured disc at x is an ind-affine scheme $\operatorname{Spf} A_x^{as}$. Suppose that X is compact; set $U_x := X \smallsetminus \{x\}$. The evaluation map which assigns to a global section on U_x its restriction to the punctured disc is a closed embedding of the ind-schemes of sections $\langle \mathcal{Y} \rangle(U_x) \hookrightarrow \operatorname{Spf} A_x^{as}$; its ideal is generated by the image of a certain canonical map $r_x \colon \Gamma(U_x, h(A)) \to A_x^{as}$. We also have an embedding $\mathcal{Y}_x \hookrightarrow \operatorname{Spf} A_x^{as}$ whose image consists of horizontal sections that extend to x, so $\langle \mathcal{Y} \rangle(X) = \langle \mathcal{Y} \rangle(U_x) \cap \mathcal{Y}_x$.

If A is a coisson algebra finitely generated as a $\mathcal{D}_X$-algebra, then Aas_x is a topological Poisson algebra, $\Gamma(U_x, h(A))$ is a Lie algebra, and r_x commutes with brackets. Therefore r_x is a hamiltonian action of $\Gamma(U_x, h(A))$ on $\operatorname{Spf} A_x^{as}$, and $\langle \mathcal{Y} \rangle(U_x)$ is the zero fiber of the momentum map. When we pass to quantization and chiral homology (see 0.9 and 0.10), the zero fiber changes into the Hamiltonian reduction.

The algebra $\langle A \rangle$ is also denoted by $H_0^{ch}(X, A)$. The derived version of this construction yields a graded commutative superalgebra $H_\cdot^{ch}(X, A)$.

0.8. Now let us pass to chiral algebras. There are two complementary (equivalent) approaches to this notion: Lie algebra style and commutative algebra style.[3] We begin with the "Lie algebra" approach; for the "commutative algebra" picture see 0.12.

From now on we assume that $\dim X = 1$. Let $j^{(I)} \colon U^{(I)} \hookrightarrow X^I$ be the complement to the diagonal divisor. Set

$$P_I^{ch}(\{L_i\}, M) := \operatorname{Hom}(j_*^{(I)} j^{(I)*} \boxtimes L_i, \Delta_*^{(I)} M). \tag{0.8.1}$$

These are *chiral I-operations*. One defines their composition in the obvious manner. We get the *chiral pseudo-tensor structure* on $\mathcal{M}(X)$; the corresponding pseudo-tensor category is denoted by $\mathcal{M}(X)^{ch}$.

Now a *chiral algebra* on X is simply a Lie algebra in $\mathcal{M}(X)^{ch}$ with an additional property (existence of unit). We refer to the corresponding Lie^{ch} bracket as the *chiral product*.

0.9. Let us explain why the notion of chiral algebra is a quantization of that of coisson algebra. For every $\mathcal{D}_X$-module A we have a canonical exact sequence

$$0 \to \operatorname{Hom}(A^{\ell \otimes 2}, A^\ell) \to P_2^{ch}(\{A, A\}, A) \to P_2^*(\{A, A\}, A). \tag{0.9.1}$$

The right arrow assigns to every chiral product μ on A a Lie* bracket $[\ ]_\mu$. If the latter vanishes (we say then that our chiral algebra is *commutative*), then μ can be considered as a binary operation on A^ℓ with respect to $\otimes$. This way one

[3] A poetically-minded reader may call them "dynamic" and "static" points of view.

identifies commutative chiral algebra structures on A with commutative $\mathcal{D}_X$-algebra structures on A^ℓ.

Now assume we have a family A_t of chiral algebras that depends on a parameter $t \in \mathbb{C}$ such that A_0 is commutative. Then $\{\ \} := \left(t^{-1}[\ ]_{\mu_t}\right)\big|_{t=0}$ is a coisson bracket on A_0^ℓ. Thus chiral algebras are quantizations of coisson algebras as promised.

In fact, the whole chiral pseudo-tensor structure can be considered as a quantization of the compound tensor structure (see 3.2).

The problem of quantization of coisson algebras is fairly interesting. We do not know how to solve it in general. In the main body of this work we treat the simplest cases of linear brackets. In particular, we construct chiral enveloping algebras of Lie* algebras and Lie* algebroids (chiral algebras of twisted differential operators). We also discuss (in 3.9.10) quantizations modt^2. The general theory of deformations of chiral algebras was recently developed in [**Tam1**].

In a sense, in the "chiral world" chiral algebras are parallel to associative algebras while Lie* algebras and commutative $\mathcal{D}_X$-algebras play, respectively, the roles of Lie and commutative algebras.

0.10. The constructions from 0.7 generalize to arbitrary chiral algebras. For a chiral algebra A one can define its *chiral homology* $H_i^{ch}(X, A)$ (see 0.12). Here $H_0^{ch}(X, A)$ is what mathematical physicists usually call "the space of conformal blocks". For any $x \in X$ we have an associative algebra A_x^{as}; the fiber A_x is naturally a quotient of A_x^{as} modulo a left ideal. $H_0^{ch}(X, A)$ identifies canonically with a quotient of A_x modulo a right ideal generated by the image of a Lie algebra morphism $r_x : \Gamma(U_x, h(A)) \to A_x^{as}$. In the classical limit we return to the objects from 0.7.

0.11. Let us sketch now the "commutative algebra" style description of chiral algebras. This approach is essential for certain subjects, in particular, for the definition of chiral homology.

Consider the space $\mathcal{R}(X)$ of finite non-empty subsets of X; for such a subset $S \subset X$ we denote the corresponding point of $\mathcal{R}(X)$ by $[S]$. So $\mathcal{R}(X)$ carries a natural filtration $\mathcal{R}(X)_n$; the open stratum $\mathcal{R}(X)_n^\circ$ is the space of configurations of n (distinct) points on X. Remarkably enough, $\mathcal{R}(X)$ is contractible.

Informally, a *factorization algebra* on X is an $\mathcal{O}$-module $A^\ell_{\mathcal{R}(X)}$ on $\mathcal{R}(X)$ such that for every two *non-intersecting* subsets $S, T \subset X$ there is a canonical identification of fibers $A^\ell_{[S\cup T]} = A^\ell_{[S]} \otimes A^\ell_{[T]}$ which is associative and commutative in the obvious sense. We also demand the existence of a unit section (the definition is left to the reader). Such $A^\ell_{\mathcal{R}(X)}$ provides an $\mathcal{O}$-module A^ℓ on $X = \mathcal{R}(X)_1$. In fact, A^ℓ is automatically a left $\mathcal{D}_X$-module, and $A^\ell_{\mathcal{R}(X)}$ amounts to a certain structure on A^ℓ of local origin (referred to as factorization algebra structure).

Now a factorization algebra structure on $\mathcal{A}^\ell$ amounts to a chiral algebra structure on $\mathcal{A}$. Namely, the chiral product corresponding to a factorization algebra structure is the composition $j_*j^*A \boxtimes A = j_*j^*A_{X\times X} \to \Delta_*\Delta^! A_{X\times X} = \Delta_* A$. Here $A_{X\times X}$ is the pull-back of $A_{\mathcal{R}(X)}$ by the obvious map $X \times X \to \mathcal{R}(X)_2$, $\Delta : X \hookrightarrow X \times X$ the diagonal, $j : U \hookrightarrow X \times X$ its complement, the equalities are structure identifications, and the arrow is a canonical morphism.

The $\mathcal{O}$-tensor product of factorization algebras is evidently a factorization algebra, so chiral algebras form a tensor category.

0.12. Therefore chiral algebras can be considered as geometric objects on $\mathcal{R}(X)$. In this vein, the chiral homology of A is defined as the de Rham homology of $\mathcal{R}(X)$ with coefficients in $A^{\ell}_{\mathcal{R}(X)}$. The chiral homology functor has many remarkable properties; e.g., it commutes with tensor products. In particular, higher chiral homology of the unit chiral algebra vanishes (which also follows from contractibility of $\mathcal{R}(X)$).

Chiral homology is naturally defined for DG chiral algebras and it is preserved by quasi-isomorphisms. In the exposition we do not pave a road across the morass of homotopy theory of chiral algebras, but we resort to an unworthy (yet solid) path of functorial resolutions. Namely, chiral homology can be realized as homology of certain functorial *chiral chain* complexes $C^{ch}(X,A)_{\mathcal{P}\mathcal{Q}}$ which depend on appropriate auxiliary resolutions $\mathcal{P}$, $\mathcal{Q}$ of $\mathcal{O}_X$ (see 4.2.12).[4] These complexes resemble Chevalley homology complexes of Lie algebras: for example, they carry a canonical BV (Batalin-Vilkovisky) algebra structure. In the classical limit it becomes an odd Poisson bracket.

The comeuppance is the lack of understanding of the structure of the homotopy category of chiral algebras (see Remarks in 3.3.13). By the way, the latter lies outside of Quillen's model category framework due to the absence of cofibrant objects (e.g., the chiral algebra freely generated by $\mathcal{D}_X \in \mathcal{M}(X)$ does not exist).

0.13. Physicists usually describe a chiral algebra structure in terms of *operator product expansions*, ope for short (see, e.g., [**BPZ**]). The same approach is common in the literature on vertex algebras; see 0.15.

To explain what ope is, one needs to consider some non-quasi-coherent $\mathcal{D}$-modules. Thus, for a left $\mathcal{D}_X$-module A^{ℓ}, let $\hat{\Delta}_* A^{\ell}$ be a sheaf of left $\mathcal{D}_{X\times X}$-modules supported on the diagonal $X \hookrightarrow X \times X$ which is I-adically complete and satisfies $\hat{\Delta}_* A^{\ell}/I\hat{\Delta}_* A^{\ell} = A^{\ell}$ (here $I \subset \mathcal{O}_{X\times X}$ is the ideal of the diagonal). Such $\hat{\Delta}_* A^{\ell}$ exists and is unique. Note that for any local section $a \in A^{\ell}$ there is a unique section $a^{(1)} \in \hat{\Delta}_* A^{\ell}$ such that $a^{(1)}$ is horizontal along the second variable and $a^{(1)} \bmod I\hat{\Delta}_* A^{\ell} = a$. Therefore, if t is a local coordinate on X, one may write any section φ of $\hat{\Delta}_* A^{\ell}$ as a formal power series $\sum\limits_{i\geq 0} a_i^{(1)}(t_2-t_1)^i$, $a_i = \frac{1}{i!}\partial^i_{t_2}\varphi \bmod I\hat{\Delta}_* A^{\ell}$. Now let $\tilde{\Delta}_* A^{\ell}$ be the localization of $\hat{\Delta}_* A^{\ell}$ with respect to the equation of the diagonal. Its section is a Laurent power series $\sum\limits_{i\gg -\infty} a_i^{(1)}(t_2-t_1)^i$; i.e., we can write $\tilde{\Delta}_* A^{\ell}$ as $A^{\ell}(\!(t_2-t_1)\!)$. Notice that $\tilde{\Delta}_* A^{\ell}/\hat{\Delta}_* A^{\ell} = \Delta_* A^{\ell}$.

Now an ope is a morphism of $\mathcal{D}_{X\times X}$-modules $\circ : A^{\ell} \boxtimes A^{\ell} \to \tilde{\Delta}_* A^{\ell}$. Composing it with the above projection to $\Delta_* A^{\ell}$, we get a binary chiral operation $\mu = \mu_{\circ}$ on A. One can explain what it means for $\circ$ to be commutative and associative. In fact, $\circ \mapsto \mu_{\circ}$ is a bijective correspondence between the set of commutative and associative ope and that of chiral algebra structures on A.

From the point of view of factorization algebras, our ope is the gluing isomorphism that reconstructs $A^{\ell}_{X\times X}$ from its restriction to U (which is $A^{\ell} \boxtimes A^{\ell}|_U$) and to the formal neighbourhood of the diagonal (which is $\hat{\Delta}_* A^{\ell}$).

Notice also that the Lie* bracket of a chiral algebra is just the the polar part of the ope.

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[4] A poetically-minded reader may say that $\mathcal{P}$ resolves ultraviolet problems of A, while $\mathcal{Q}$ takes care of its infrared behavior.

0.14. To see an example of a non-commutative chiral algebra, let us describe in geometric terms the chiral enveloping algebra of a Kac-Moody Lie algebra.

Let G be an algebraic group. For $x \in X$ consider the set of pairs $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a G-bundle on X and α is a trivialization of $\mathcal{F}$ on $X \smallsetminus \{x\}$. This is the set of points of a formally smooth ind-scheme $\mathcal{GR}_x$; a choice of a parameter t_x at x identifies $\mathcal{GR}_x$ with $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. Our $\mathcal{GR}_x$ are fibers of an ind-scheme $\mathcal{GR}_X$ over X called the *affine Grassmannian*. This is a $\mathcal{D}_X$-ind-scheme: when x varies infinitesimally, $X \smallsetminus \{x\}$, hence $\mathcal{GR}_x$, do not change.

The trivialized G-bundle $(\mathcal{F}, \alpha)$ on X defines the section $e\colon X \to \mathcal{GR}$ (which is the only horizontal section of $\mathcal{GR}$). Denote by A^ℓ_x the vector space of distributions on $\mathcal{GR}_x$ supported at $e(x)$; i.e., A^ℓ_x is the (topological) dual to the formal completion of the local ring $\mathcal{O}_{\mathcal{GR}_x, e(x)}$. When x varies, the A^ℓ_x form a left $\mathcal{D}_X$-module A^ℓ_X.

Let us show that A is naturally a chiral algebra. As in 0.11 we have to define a factorization algebra $A^\ell_{\mathcal{R}(X)}$. We have a $\mathcal{D}$-ind-scheme $\mathcal{GR}_{\mathcal{R}(X)}$ over $\mathcal{R}(X)$ with fibers $\mathcal{GR}_{[S]}$ equal to the space of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a G-bundle on X and α is a trivialization of $\mathcal{F}$ on $X \smallsetminus S$. There is a canonical horizontal section e of $\mathcal{GR}_{X^n}$, and we set $A^\ell_{[S]}$ to be the vector space of distributions on $\mathcal{GR}_{[S]}$ supported at e. The factorization property for $A^\ell_{\mathcal{R}(X)}$ follows from the similar property of $\mathcal{GR}_{\mathcal{R}(X)}$ itself: there is a canonical identification $\mathcal{GR}_{[S \cup T]} = \mathcal{GR}_{[S]} \times \mathcal{GR}_{[T]}$ if $S \cap T = \emptyset$.

To get the chiral envelope of a Kac-Moody algebra of a non-zero level, one must twist A^ℓ by an appropriate canonical line bundle on $\mathcal{GR}$.

0.15. Chiral algebras are related to (various versions of) vertex algebras as follows.

The category of vertex algebras in the sense of [**B1**] and [**K**] is equivalent to the category of chiral algebras on $X = \mathbb{A}^1$ equivariant with respect to the group T of translations.[5] Namely, the vertex algebra V_A corresponding to a T-equivariant chiral algebra A is the vector space of T-invariant sections of A^ℓ with the vertex algebra structure given by the operator product expansion (see 0.13); of course, V_A identifies naturally with the fiber A^ℓ_x at any point $x \in \mathbb{A}^1$. In the same manner the category of V_A-modules is identified with that of (weakly) T-equivariant A-modules or, more conveniently, with that of A-modules supported at x.[6]

Replacing T on the chiral side by the group Aff of affine transformations, we get on the vertex side essentially an object called the "vertex algebra" in [**FBZ**], "$\mathbb{Z}$-graded vertex algebra" in [**GMS2**], and "graded vertex algebra" in [**K**].[7] Adding to the structure a Virasoro vector, a.k.a. stress-energy tensor, which is a morphism from the Virasoro Lie* algebra (of some central charge) to A compatible with the T- or Aff-action, we get essentially a "conformal vertex algebra" from [**K**] and [**FBZ**] or the "vertex operator algebra" of [**FLM**], [**FHL**], [**DL**], [**Hu**].

Similarly, a chiral algebra on the "coordinate disc" $\operatorname{Spec} \mathbb{C}[[t]]$ equivariant with respect to the action of the group ind-scheme $\operatorname{Aut} \mathbb{C}[[t]]$ is the same as the "quasi-conformal vertex algebra" from [**FBZ**]. The equivariance means that such an object yields a chiral algebra on any non-coordinate formal disc, and, in fact, by patching,

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[5] Here "equivariant" means that the group T of translations acts on A as on the $\mathcal{O}_X$-module, and the action of each $t \in T$ is compatible with the chiral algebra structure. In particular, A is a *weakly* T-equivariant $\mathcal{D}$-module.

[6] One identifies a weakly T-equivariant A-module M with the A-module $(j_{x*}j^*_x M)/M$ supported at x; here j_x is the embedding $\mathbb{A}^1 \smallsetminus \{x\} \hookrightarrow \mathbb{A}^1$.

[7] Here "essentially" means that we discard the varying finiteness conditions of the references.

on any curve (see [**FBZ**] 18.3.3 and [**HL2**]). In this way it amounts to a "universal" chiral algebra, i.e., a rule that assigns to any curve a chiral algebra on it, in a way compatible with the étale localization

When one is interested in local questions, such as the usual representation theory, chiral algebras are essentially equivalent to vertex algebras. In practice, however, chiral algebras are considerably more flexible: for example, twists of vertex algebras (see 3.4.17) and constructions like that of A_x^{as} (see 3.6.4) are painful in the pure vertex algebra setting.

Let us mention that all kinds of vertex/chiral algebras can be seen as chiral algebras on appropriate *c-stacks* (see 2.9, 3.1.16, 3.3.14), the above-mentioned functors and equivalences being mere base change. In the exposition we stick to the usual $\mathcal{D}$-module setting.

0.16. We started this work with the modest intent of understanding the ingenious formal power series manipulations that haunt the books on vertex operator algebras. For a VOA insider, untrammelled by algebro-geometric affections, the mode of the output might resemble though *задопасомый вол*'s[8] features.

A challenging problem is to define chiral algebras on higher-dimensional X (see [**B2**] and [**Tam2**] in this respect). The definitions from 0.8 and 0.11 formally work also for $\dim X > 1$, but to make them sensible, one must plunge at once into the homotopy setting which we had no courage to do (see 0.12). Notice that coisson algebras live in any dimension, so it may be reasonable to look first for the corresponding quasi-classical objects.

Let us also mention that while the general format of the classical setting is covered nicely by the compound tensor category axiomatics (see 1.4), chiral algebras proper defy such treatment.

We consider chiral algebras on usual curves, while some applications demand the setting of super curves. Presumably, the rendition should not be difficult.

0.17. The book consists of four chapters. The first one discusses some relevant abstract nonsense. The classical (coisson) story occupies the second chapter. The third chapter deals with the local theory of chiral algebras proper (some readers may find the exposition of the textbooks [**K**], [**FBZ**] livelier,[9] and we do recommend parallel reading). It divides into two parts: the first considers basic structures and their interrelations while the second deals with elementary methods of constructing chiral algebras. The final chapter treats global theory, i.e., the formalism of chiral homology. It contains an exposition of the general machinery and some results

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[8] *"В барбари же суть волы глаголются епистоны, зане не пасутся обычайно как вол места сего, идуще вперед, но оные пасутся назад, потому что роги у них суть крюковаты и склоняются наперед ... утыкаются роги их в землю", из "Дамаскина архиерея Студита собрания от древних философов о неких собствах естества животных".* "In Barbary there are oxen called epistomi for they do not graze walking forward in the usual manner of the ox of our land, but move backwards, since horns of theirs are crooked and bent forward ... sticking into the earth," from "A collection (of excerpts) from ancient philosophers on certain properties of the nature of animals" by Damaskin, the high priest of the Studiy monastery.

[9] *"У нас дважды два тоже четыре, да выходит как-то бойчее." М. Гаспаров, "Записки и Выписки"; комментарий к понятию "Народность".* "At ours two by two is also four, yet it comes livelier", from M. Gasparov, "Notes and Excerpts", a comment on the notion *narodnost'* (which has no adequate English analog).

showing that the chiral homology functor transforms the constructions of chiral algebras from Chapter 3 to parallel constructions for BV algebras.

It is known that distilled axioms are pretty indigestible. We strongly suggest the reader follow the complicated alembics of Chapter 1 simultaneously with respective sections of more wholesome Chapters 2 and 3.

0.18. We were particularly motivated by the observation that chiral algebras provide a natural tool for tackling geometric automorphic forms in the $\mathcal{D}$-module setting. This can be seen already in the oper construction at the critical level (see [**BD**]). One hopes that general automorphic $\mathcal{D}$-modules come as (higher) chiral homology of twisted chiral Hecke algebras (where the moduli of G-bundles and de Rham $G^\vee$-local systems are parameters of the twists). A related local problem is to describe a "spectral decomposition" of the category of representations of an affine Kac-Moody algebra (at negative integral level) with spectral parameters being moduli of $G^\vee$-local systems on the punctured formal disc (see [**Be**]). The chiral Hecke algebras arise from the global geometry of the affine Grassmannian; when G is a torus, they amount to lattice Heisenberg algebras. We will return to these subjects elsewhere.

A general matter of special interest is the factorization property of chiral homology for degenerating families of curves. At the moment, it is to some extent understood only for H_0^{ch} of rational field theories (the Verlinde rules). Presumably, there should be an interesting theory beyond the rational situation with Verlinde's summation over the finite set of irreducibles replaced by "integration" over an appropriate "compact space". The (twisted) chiral Hecke algebra may provide an example of such a situation.

0.19. Our sincere gratitude is due to Sasha Belavin and Pierre Deligne. Apart from an obvious influence of their ideas, it was in their apartments in 1992–1995 where the prime part of the work was done. During the (all too long) course of writing, we were greatly helped by many mathematicians; we are very grateful to S. Arkhipov, J. Bernstein, R. Bezrukavnikov, P. Bressler, B. Feigin, E. Frenkel, V. Ginzburg, V. Hinich, Y.-Z. Huang, M. Kapranov, D. Kazhdan, J. Lepowski, Yu. Manin, B. Mazur, V. Schechtman, I. Shapiro, G. Segal, J. Wiennfield, G. Zuckerman, and, especially, D. Gaitsgory, for their interest, inspiration, and correction of mistakes. The first author would also like to thank his wife and children[10] for their ability to endure the rewritings of the draft. We are grateful to Dottie Phares of IAS and Richard Lloyd of MIT for the careful typing of the first version of the first part of the manuscript back in 1994–1995, and to Sergei Gelfand and Arlene O'Sean of the AMS for the editing of the manuscript. The authors were partially supported by NSF grant DMS-0100108.

0.20. A few words about notation commonly used in the book. "$x \in Y$" means that either x is an element of a set Y or x is an object of a category Y or x is a local section of a sheaf Y. For a category $\mathcal{M}$ we denote by $\mathcal{M}^\circ$ the dual category; *Sets* is the category of all sets. For a smooth variety X we denote by Θ_X and $\mathcal{D}_X$ the tangent sheaf and the algebra of differential operators.

[10] *"И слоновую же оустрашает ярость малаа квичащаа свинаа поросята", из "Похвалы Богу о сотворении всей твари Георгия Писиды."* "And the elefant rage is terrified by tiny squealing swine piggies," from "A praise unto the Lord for the creation of all living creatures" by George Pisida.