Eventually we would like to relate iterated integrals on the loop space of a manifold $X$ to the bar complex $B(A^*)$ for a certain dg subalgebra $A^0 \subset \Lambda^* X$ such that $A^0 = \mathbb{R}$ and $A^1 \cap d(\Lambda^0 X) = 0$.

What we considered last time was the truncated de Rham complex, which is not an algebra at all:

$$\overline{\Lambda}^* X := (\tau_{\geq 1} \Lambda^* X)[1].$$

Last time we were in the process of analyzing the map $\sigma : (\overline{\Lambda}^* X)^{\otimes s} \to \text{gr}_s A'$,

$$\omega_1 \otimes \ldots \otimes \omega_s \mapsto \int \omega_1 \ldots \omega_s,$$

where $A' \subset \Lambda^*(\Omega^*_0 X)$ was introduced in the previous lecture.

**Proposition.** The map $\sigma$ is an isomorphism (not just a quasi-isomorphism!) of complexes.

**Proof.** First note that $\sigma$ is surjective by construction.
Now let us check that it is injective. We have a natural pairing

\[ C_p(\Omega^{x_0}X) \otimes (\Lambda^{p+1}X) \rightarrow IR \]

\[ (c, \omega) \mapsto \int_c \int \omega \]

the iterated integral

of \( \omega \), which is in \( \Lambda^p(\Omega^{x_0}X) \)

More generally, we obtain

\[ (\Lambda X)^{\otimes s} \rightarrow \mathbb{L}_s(C_*(\Omega^{x_0}X), IR) \]

\[ \text{all multilinear maps} \]

\[ \mathbb{L}_s(C_*(\Omega^{x_0}X) \times \ldots \times C_*(\Omega^{x_0}X), IR) \rightarrow IR \]

\[ s \text{ factors} \]

It is not hard to check directly that this map is injective. If we can complete the commutative triangle

\[ (\Lambda X)^{\otimes s} \overset{\sigma}{\rightarrow} \gr_s A' \]

\[ \mathbb{L}_s(C_*(\Omega^{x_0}X), IR), \]

the proof of the proposition will be finished.
Let us recall the formula: for $a < b < c$,

$$
\int_a^c \omega_1 \ldots \omega_r = \int_a^b \omega_1 \ldots \omega_r + \left( \int_a^b \omega_1 \ldots \omega_{r-1} \right) \omega_r + \ldots + \int_b^c \omega_1 \ldots \omega_r.
$$

Given $d_i : U_i \to \Omega X_0 X$, we use concatenation of paths to define

$$
\lambda = d_1 \times \ldots \times d_r : U_1 \times \ldots \times U_r \to \Omega X_0 X.
$$

Apply the formula above to $r = 2$ and $a = 0$, $b = \frac{1}{2}$, $c = 1$.
We obtain:

$$
(d_1 \times d_2)^* \int_0^1 \omega_1 \omega_2 = \int_0^{1/2} \omega_1 \omega_2 + \int_0^{1/2} \omega_1 \wedge \int_{1/2}^1 \omega_2 + \int_{1/2}^1 \omega_1 \wedge \omega_2,
$$

probably, need to insert $(d_1 \times d_2)^*$ everywhere?

It is not clear to me what the precise meaning of the notation above is.

Probably:

$$
(d_1 \times d_2)^* \int_0^1 \omega_1 \omega_2 = \text{pr}_1^* d_1^* \int \omega_1 \omega_2 + 
+ \left( \text{pr}_1^* d_1^* \int \omega_1 \right) \wedge \left( \text{pr}_2^* d_2^* \int \omega_2 \right) + \text{pr}_2^* d_2^* \int \omega_1 \wedge \omega_2,
$$

where $\text{pr}_j : U_1 \times U_2 \to U_j$ $(j = 1, 2)$ are the projections.
Let us consider the case \( \Delta^p_j \), the standard simplex of dimension \( p_j > 0 \). We also assume that \( \deg(w_1), \deg(w_2) > 0 \) and \( \deg(w_1) + \deg(w_2) - 2 = p_1 + p_2 \). Then two of the terms in the previous formula die, and we obtain
\[
\int \int w_1 \circ w_2 = \left( \int \int w_1 \right) \cdot \left( \int \int w_2 \right).
\]
Unfortunately, if one of \( p_1 \) or \( p_2 \) equals 0, this argument does not work.

The correct general statement is:

**Lemma.** Consider (smooth) maps
\[
c_j : \Delta^p_j \to \Omega_{x_0} X, \quad j = 1, \ldots, s.
\]
For every \( j \), define
\[
\bar{c}_j = \begin{cases} 
    c_j, & p_j > 0 \\
    c_j - x_0, & p_j = 0
\end{cases}
\]
constant path at \( x_0 \).

These are elements of \( C_p, (\Omega_{x_0} X) \). Then
\[
\int \int w_1 \circ \cdots \circ w_r = \begin{cases} 
    \int_{\bar{c}} \int \int w_i, & r = s \\
    0, & s > r
\end{cases},
\]
\( c_1 \times \cdots \times c_s \).

This is what we need to complete the commutative triangle at the bottom of page 2.