SPENCER BLOCH: HOPF ALGEBRAS

Lecture 8: $\boxed{10/18/2007}$

Eventually we would like to relate iterated integrals on the loop space of a manifold X to the bar complex $B(A^\cdot)$ for a certain dg subalgebra $A^\cdot \subset \Lambda^\cdot X$ such that

$$A^0 = \mathbb{R} \text{ and } A^1 \cap d(\Lambda^0 X) = 0.$$

What we considered last time was the truncated de Rham complex, which is not an algebra at all:

$$\overline{\Lambda}^\cdot X := (\tau_{\geq 1} \Lambda^\cdot X)[1].$$

Last time we were in the process of analyzing the map

$$\sigma : (\overline{\Lambda}^\cdot X)^{\otimes s} \longrightarrow gr_s \mathcal{A}',$$
$$\omega_1 \otimes \ldots \otimes \omega_s \longmapsto \int \omega_1 \circ \ldots \circ \omega_s,$$

where $\mathcal{A}' \subset \Lambda^\cdot(\Omega_{x_0} X)$ was introduced in the previous lecture.

<u>Proposition</u>. The map σ is an isomorphism (not just a quasi-isomorphism!) of complexes.

<u>Proof</u>. First note that σ is surjective by construction.

Now let us check that it is injective. We have a natural pairing

$$C_p(\Omega_{x_0}X) \otimes (\overline{\Lambda}^{p+1}X) \longrightarrow \mathbb{R}$$

p-chains on $\Omega_{x_0}X$

$$(c,\omega) \longmapsto \int_c \underbrace{\int \omega}$$

the iterated integral of ω, which is in $\Lambda^p(\Omega_{x_0}X)$

More generally, we obtain

$$(\overline{\Lambda}X)^{\otimes s} \longrightarrow L_s(C_*(\Omega_{x_0}X), \mathbb{R})$$

all multilinear maps $\underbrace{C_*(\Omega_{x_0}X) \times \cdots \times C_*(\Omega_{x_0}X)}_{s \text{ factors}} \longrightarrow \mathbb{R}$

It is not hard to check directly that this map in injective. If we can complete the commutative triangle

$$\begin{tikzcd}
(\overline{\Lambda}X)^{\otimes s} \arrow[rr, "\sigma"] \arrow[dr, hook] & & gr_s \mathcal{A}' \arrow[dl, dashed] \\
& L_s(C_*(\Omega_{x_0}X), \mathbb{R}), &
\end{tikzcd}$$

the proof of the proposition will be finished.

Let us recall the formula: for $a<b<c$,

$$\int_a^c \omega_1 \circ \dots \circ \omega_r = \int_a^b \omega_1 \circ \dots \circ \omega_r + \left(\int_a^b \omega_1 \circ \dots \circ \omega_{r-1}\right) \wedge \int_b^c \omega_r + \\ + \dots + \int_b^c \omega_1 \circ \dots \circ \omega_r .$$

Given $\alpha_i : U_i \longrightarrow \Omega_{x_0} X$, we use concatenation of paths to define

$$\alpha = \alpha_1 \times \dots \times \alpha_r : U_1 \times \dots \times U_r \longrightarrow \Omega_{x_0} X .$$

Apply the formula above to $r=2$ and $a=0$, $b=\frac{1}{2}$, $c=1$.

We obtain:

$$(\alpha_1 \times \alpha_2)^* \int_0^1 \omega_1 \circ \omega_2 = \int_0^{1/2} \omega_1 \circ \omega_2 + \int_0^{1/2} \omega_1 \wedge \int_{1/2}^1 \omega_2 + \\ + \int_{1/2}^1 \omega_1 \circ \omega_2$$

probably, need to insert $(\alpha_1 \times \alpha_2)^*$ everywhere?

It is not clear to me what the precise meaning of the notation above is.

Probably:

$$(\alpha_1 \times \alpha_2)^* \int \omega_1 \circ \omega_2 = pr_1^* \alpha_1^* \int \omega_1 \circ \omega_2 + \\ + \left(pr_1^* \alpha_1^* \int \omega_1\right) \wedge \left(pr_2^* \alpha_2^* \int \omega_2\right) + pr_2^* \alpha_2^* \int \omega_1 \circ \omega_2 ,$$

where $pr_j : U_1 \times U_2 \longrightarrow U_j$ $(j=1,2)$ are the projections.

Let us consider the case $U_j = \Delta^{p_j}$, the standard simplex of dimension $p_j > 0$!! We also assume that $\deg(\omega_1), \deg(\omega_2) > 0$ and $\deg(\omega_1) + \deg(\omega_2) - 2 = p_1 + p_2$. Then two of the terms in the previous formula die, and we obtain

$$\int_{\alpha_1 \times \alpha_2} \int \omega_1 \circ \omega_2 = \left(\int_{\alpha_1} \int \omega_1\right) \cdot \left(\int_{\alpha_2} \int \omega_2\right)$$

Unfortunately, if one of p_1 or p_2 equals 0, this argument does not work.

The correct general statement is:

<u>Lemma</u>. Consider (smooth) maps

$$c_j : \Delta^{p_j} \longrightarrow \Omega_{x_0} X, \quad j = 1, \ldots, s.$$

For every j, define

$$\bar{c}_j = \begin{cases} c_j, & p_j > 0 \\ c_j - x_0, & p_j = 0 \end{cases}$$

constant path at x_0

These are elements of $C_{p_j}(\Omega_{x_0} X)$. Then

$$\int_{c_1 \times \cdots \times c_s} \int \omega_1 \circ \cdots \circ \omega_r = \begin{cases} \prod_{i=1}^{r} \int_{c_i'} \int \omega_i, & r = s \\ 0, & s > r. \end{cases}$$

This is what we need to complete the commutative triangle at the bottom of page 2.