Iterated integrals

\( X = \) a fixed \( C^\infty \) manifold

We will be interested in the path space \( \mathcal{P}(X) \) of \( X \).

There are some technical issues that we are ignoring for the moment (our paths should have some degree of differentiability, because eventually we will need to integrate differential \( 1 \)-forms along these paths).

Also, we would like to talk about differential forms on \( \mathcal{P}(X) \). However, \( \mathcal{P}(X) \) is infinite dimensional, so it is not clear how to define differential forms on it. On the other hand, if \( U \) is a finite dimensional manifold, it is clear what we mean by a smooth map \( U \to \mathcal{P}(X) \), namely, it is the same as a map \( U \times I \to X \) satisfying some appropriate (piecewise) smoothness conditions.

Given a smooth map \( d: U \to \mathcal{P}(X) \) and a differential form \( \omega \in \Lambda^*(\mathcal{P}(X)) \), we should be able to define \( d^*\omega \in \Lambda^*(U) \). This observation, in fact, suggests a definition of \( \Lambda^*(\mathcal{P}(X)) \).
Namely, an element \( \omega \in \Lambda^* (\mathcal{P}(X)) \) is a collection
\[
\omega = \{ \omega_{U, x} \mid \text{U = smooth finite dim. manifold, } \notag \\
x: U \rightarrow \mathcal{P}(X) \notag \\
\text{is a smooth map} \}
\]
satisfying an obvious functoriality property.

Next, let \( X \) and \( X' \) be finite dimensional manifolds, and \( F: I \times X \rightarrow X' \) a smooth map (where \( I \) always stands for \([0, 1]\)). We obtain an operator
\[
\int_{\mathcal{F}}: \Lambda^* (X') \rightarrow \Lambda^{*-1} (X)
\]
given by
\[
\int_{\mathcal{F}} \omega := \int_{\mathcal{F}} F^* \omega,
\]
where
\[
\begin{array}{ccc}
I \times X & \xrightarrow{F} & X' \\
\text{pr}_2 \downarrow & & \downarrow \text{pr}_2 \\
X & & \\
\end{array}
\]

Explicitly, if \( F^* \omega = \nu + \nu' \wedge dt \),
where \( \nu, \nu' \) are forms on \( I \times X \) which do not involve \( dt \), and \( t \) is the coordinate on \( I \),
then
\[
\int_{\mathcal{F}} \omega := \int_0^1 \nu' dt \in \Lambda^{*-1} (X).
\]
Lemma. Let $F_0, F_1 : X \to X'$ be the smooth maps obtained by restricting $F$ to $\{0\} \times X$ and $\{1\} \times X$, respectively. Then
\[
d \int_{F_0} + \int_{F_1} = F_1^* - F_0^* : \Lambda^*(X) \to \Lambda^*(X)
\]
The proof is a simple exercise.

Let us relate the previous discussion to path spaces. If $X$ is a finite dimensional manifold as before, consider
\[
F : I \times \mathcal{P}(X) \to \mathcal{P}(X)
\]
\[
(t, \gamma) \mapsto (s \mapsto \gamma(ts))
\]
Then $F_1 = \text{id}_{\mathcal{P}(X)} : \mathcal{P}(X) \to \mathcal{P}(X)$,

whereas $F_0 = \eta \circ p_0 : \mathcal{P}(X) \to X \to \mathcal{P}(X)$,

where $p_0 : \mathcal{P}(X) \to X$ is $\gamma \mapsto \gamma(0)$

and $\eta : X \to \mathcal{P}(X)$ is $x \mapsto$ the constant path at $x$.

Goal: We would like to consider
\[
\int_{F} = \int' : \Lambda^*(P) \to \Lambda^{* - 1}(P)
\]
However, it is not obvious how to make sense out of this, because given a smooth map \( \lambda : U \rightarrow \mathcal{P}(X) \), there is no natural way to complete the diagram

\[
\begin{array}{ccc}
I \times \mathcal{P}(X) & \xrightarrow{F} & \mathcal{P}(X) \\
\uparrow \text{id} \times \lambda & & \uparrow \lambda \\
I \times U & \rightarrow & U
\end{array}
\]

However, what we do have is the commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}(X) & \xleftarrow{pr_2} & I \times \mathcal{P}(X) \\
\uparrow \lambda' & & \uparrow \text{id} \times \lambda' \\
I \times U & \xleftarrow{pr_{23}} & I \times I \times U \\
\downarrow \mu \times \text{id} & & \downarrow \lambda' \\
I \times U & \rightarrow & I \times U
\end{array}
\]

where \( \mu : I \times I \rightarrow I \) is the multiplication map, \((t, s) \mapsto t \cdot s\), and \( \lambda' \) is the composition

\[
I \times U \xrightarrow{id \times \lambda} I \times \mathcal{P}(X) \xrightarrow{F} \mathcal{P}(X)
\]

\[\lambda' = F \circ (id \times \lambda)\]

Now, given \( w \in \Lambda^\circ(\mathcal{P}(X)) \), we will produce a differential form on \( I \times U \) which plays the role of \( \lambda'*(F^*w) \). Thus, \( \int' = \int_F \) cannot
quite be defined as an operator
\[ \Lambda^* (\mathcal{P}(\mathcal{X})) \rightarrow \Lambda^{* -1} (\mathcal{P}(\mathcal{X})). \]

However, \( \Lambda^* (\mathcal{F}'(\omega)) \) enjoys the "remarkable" property that it does not involve \( dt \) (although its coefficients may depend on it).

**Def:** \( \Lambda^* (\mathcal{F}'(\omega)) = \int (\mu \times \text{id}_U)^* \Lambda^* (\omega) \)

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**Exercise:** Show that this does not involve \( dt \), and that \( \mathcal{F}' \circ \mathcal{F}' = 0 \).

Let \( \pi_1 : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{X} \) be the map given by \( y \mapsto y(1) \). If \( \varphi \in \Lambda^*(\mathcal{X}) \), it is clear how to define \( \pi_1^* \varphi \in \Lambda^* (\mathcal{P}(\mathcal{X})). \)

Let us compute \( \int \pi_1^* \varphi \).

Let \( \mathcal{L} : \mathcal{U} \rightarrow \mathcal{P}(\mathcal{X}) \) be a smooth map as before. Then \( \mathcal{L}^* \int \pi_1^* \varphi \in \Lambda^*(\text{I} \times \mathcal{U}) \).

Explicitly, \( \mathcal{L}^* \int \pi_1^* \varphi = \frac{\text{the function } I \rightarrow \Lambda^{* -1} (\mathcal{U}) \text{ given by}}{s \mapsto \int_0^s \text{eval}_y^*(\varphi)} \)

where \( \text{eval}_y : \text{I} \times \mathcal{U} \rightarrow \mathcal{X} \) is the "evaluation" map which determines, and is determined by, \( \mathcal{L}. \)
Let us verify this. We have

\[ P \xrightarrow{p_1} X \]
\[ \lambda' = \text{eval}_x \]
\[ I \times I \times U \xrightarrow{\mu \times \text{id}_U} I \times U \]

Now \( \int (\mu \times \text{id}_U) \text{eval}_x^* (\tau) \) is the differential form on \( I \times U \), not involving \( ds \) (\( s \) is the coordinate on \( I \)), corresponding to the function \( s \mapsto \int_0^s \text{eval}_x^* (\tau) \).

Remark. Write, as usual,

\[ \text{eval}_x^* (\tau) = \varphi (t) + \varphi' (t) \lambda dt \]

where \( \varphi, \varphi' : I \to \Lambda^0 (U) \) are functions.

Then

\[ \int_0^1 \varphi' (st) d(st) = \int_0^s \varphi' (t) dt \]
Definition of iterated integrals

Given \( \omega_1, \ldots, \omega_r \in \Lambda^*(X) \), we want to define \( \int \omega_{r \circ \ldots \circ \omega_1} \in \Lambda^*(\mathcal{P}(X)) \),

not quite, actually.

The definition is inductive:

\[
\int \omega_1 := \int_{p_1^*} \omega_1
\]
\[
\int \omega_{r \circ \ldots \circ \omega_1} := \int \left( \int \omega_{r \circ \ldots \circ \omega_{r-1}} \right) \wedge p_1^* \omega_1
\]

Here, \( J \) is the "number operator," which makes sense on any graded vector space \( V = \bigoplus_{n \in \mathbb{Z}} V^n \), and is given by \( J(\nu) = (-1)^n \nu \) for \( \nu \in V^n \).

Example. Let \( U = pt \) be the manifold consisting of one point; a smooth map \( \lambda : pt \to \mathcal{P}(X) \) is just a path \( A : I \to X \).

Given \( \omega_1, \omega_2 \in \Lambda^1(X) \), we see that \( \lambda^* \int_{p_1^*} \omega_1 \) is the following function on \( I \):

\[
\lambda^* \int_{p_1^*} \omega_1 \quad \text{is the following function on } I: \quad s \mapsto \int_0^s A^*(\omega_1).
\]
Now let us try to understand the iterated integral $\int_A w_1 \wedge w_2$. It is the function $I \to \mathbb{R}$,

$\quad t \mapsto \int_0^t \left( \int_0^s A^* w_1 \right) A^* w_2.$

More generally, consider $w_1, \ldots, w_r \in \Lambda^1(\mathcal{X})$.

Write $A^* w_j = f_j(s) \, ds$. Then

$\int_A w_1 \wedge \cdots \wedge w_r : t \mapsto \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_r} f_1(s_1) \, ds_1 \, f_2(s_2) \, ds_2 \cdots f_r(s_r) \, ds_r.$

Even more generally, we can allow $U$ to be more than a point. It is easy to write down the correct formula in this more general case. In fact, if $w \in \Lambda^{n_i}(\mathcal{X})$, then $\int w_1 \wedge \cdots \wedge w_r$ is a differential form on $I \times U$ which does not involve $dt$ and has degree $(\sum n_i) - r$. 
Next, we need to relate iterated integrals to the exterior derivative,
\[ d : \Lambda^\circ \mathcal{P}(X) \rightarrow \Lambda^{\circ+1} \mathcal{P}(X) \].

**Proposition.** We have
\[
d \int \omega_1 \wedge \cdots \wedge \omega_r = \sum_{i=1}^r (-1)^i \int \left( \int \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge d\omega_i \right) \omega_{i+1} \wedge \cdots \wedge \omega_r \\
- \sum_{i=1}^r (-1)^i \int \left( \int \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge (\omega_i \wedge \omega_{i+1}) \right) \omega_{i+2} \wedge \cdots \wedge \omega_r \\
- \rho^* \omega_1 \wedge \int \omega_2 \wedge \cdots \wedge \omega_r + \int \left( \int \omega_1 \wedge \cdots \wedge \omega_r \right) \wedge \rho^* \omega_r.
\]

**Remark.** This formula is very similar to what we would obtain if we tried to define the bar complex for differential graded algebras.

This will be explained next time.