§1.1. Homological algebra is more of a very convenient language, rather than a mathematical theory. Note also that homological algebra is still not fully developed: for example, the notion of a triangulated category that many people use these days is not the "correct one".

§1.2. Today we will give a review of the basic structures of homological algebra. 

$\mathsf{Ab}$ = the category of abelian groups

One has the notion of a complex of abelian groups. There are two standard constructions with complexes.

1) Shifts: if $M = (M^i, d^i)$ is a complex, then for any $a \in \mathbb{Z}$, we define a new complex, $M[a]$, by $M[a]^n = M^{a+n}$ and $d_{M[a]} = (-1)^a d_M$. 
2) Cones: Consider a morphism of complexes $M \xrightarrow{f} N$. We can define a complex $\text{Cone}(f)$ as follows:

\[ \text{Cone}(f)^n = N^n \oplus M^{n+1} \]

\[ d_{\text{Cone}(f)} = \begin{pmatrix} d_N & f \\ 0 & -d_M \end{pmatrix} \]

It is trivial to check that

\[ d_N \circ f = f \circ d_M \implies d^2_{\text{Cone}(f)} = 0. \]

Another way to write this definition:

Consider the total complex of the double complex

\[
\begin{array}{ccccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & N^n & & N^{n+1} & & \cdots \\
& & \uparrow f & & \uparrow f & & \\
& & M^n & & M^{n+1} & & \cdots \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & \\
\end{array}
\]

**Exercise.** Consider an embedding of complexes $N \xrightarrow{g} C$ which has termwise splittings, not necessarily compatible with the differentials.
Note that, in any case, \( C/N \) is also a complex. Set \( M = (C/N)[-1] \).

Choose splittings
\[
C^n \cong N^n \oplus (C/N)^n \quad (\star)
\]
and write the differential on \( C \) in terms of this splitting:
\[
d_C = \begin{pmatrix} dN & \ast \\ 0 & dM \end{pmatrix}
\]

We can consider \( \ast \) as a morphism of graded abelian groups \( M \rightarrow N \).

**Exercise:** Show that \( \ast \) is a morphism of complexes, and the direct sum decomposition \( (\star) \) identifies \( C \) with the cone of the morphism \( \ast \).

This essentially tautological exercise provides a slightly different way of thinking about cones.

### §1.4. Additive categories

An **additive category** is a category where all the Hom-sets are equipped with abelian group structures such that composition of morphisms is bi-additive, and which has finite products.
Remark: In particular, looking at the product of an empty collection of objects, we see that there is a final object.

Exercises. Check that the final object of an additive category is in fact a zero object (i.e., it is also initial). Check that finite products are the same as finite coproducts. Use this to show that being additive is a property of a category (i.e., any given category has at most one additive structure).

§1.5. Image of an idempotent.

Let $A$ be an additive category. Suppose we have $M \in A$ and $e \in \text{End}(M)$ is an idempotent, i.e., $e^2 = e$.

Def: We say that $e$ has an image if there is a decomposition $M = M_1 \oplus M_2$ with respect to which the matrix of $e$ equals $(\text{id}_{M_1}, 0)$. One can check that in this case $M_1$ is unique, and we write $M_1 = \text{Im}(e)$. 
Remark: If \( e \in \text{End}(M) \) is an idempotent, so is \( \text{id}_M - e \). If \( e \) has an image, then, with the notation above, \( \text{Im}(\text{id}_M - e) = M_z \).

Def: We say that an additive category \( \mathcal{A} \) is idempotently complete, or Karoubian, if every idempotent endomorphism of every object of \( \mathcal{A} \) has an image.

§1.6. Exercise. Every additive category \( \mathcal{A} \) admits a unique (in the appropriate sense) fully faithful additive embedding \( \mathcal{A} \hookrightarrow \mathcal{A}_{\text{kar}} \) into a Karoubian category such that every object of \( \mathcal{A}_{\text{kar}} \) is the image of an idempotent in \( \mathcal{A} \).

Remark: It is not always possible to turn an additive category into an abelian one "by brute force."

§1.7. Graded additive categories.

Def: A graded additive category is an additive category \( \mathcal{A} \) equipped with...
gradings on the Hom-groups which are compatible with compositions.

Def: A differential graded (DG) structure on an additive category $\mathcal{A}$ is a grading in the sense above together with differentials of degree 1 on the Hom-groups satisfying:

$$d(f \circ g) = (df) \circ g + (-1)^{\deg(f)} (f \circ (dg)),$$

whenever $f$ is homogeneous and $f \circ g$ is defined.

Remark: If $\mathcal{A}$ is a DG additive category and $M \in \mathcal{A}$, then $\text{End}_{\mathcal{A}}(M)$ is a d.q.a. differential graded algebra.

§1.8. If $\mathcal{A}$ is a DG category, the homotopy category $\text{Ho}(\mathcal{A})$ is the additive category defined by

- Objects $\text{Ho}(\mathcal{A}) = \text{Objects}(\mathcal{A})$
- $\text{Hom}_{\text{Ho}(\mathcal{A})}(M, N) = \text{Ho}(\text{Hom}_{\mathcal{A}}(M, N))$.

§1.9. Example of a DG category. Fix an additive category $\mathcal{B}$, and let $C(\mathcal{B})$ be the category of complexes in $\mathcal{B}$. 
If we only consider morphisms of complexes of degree 0, then $C(B)$ becomes a plain additive category. However, it is better to define $C(B)$ as a DG category.

**Namely:** If $M, N \in C(B)$, we define a complex of abelian groups $\text{Hom}^\cdot(M, N)$ by

$$\text{Hom}^n(M, N) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_B(M^a, N^{a+n}).$$

The differential

$$d : \text{Hom}^n(M, N) \to \text{Hom}^{n+1}(M, N)$$

is given by

$$d(f) = dN \circ f + (-1)^n f \circ dM.$$

It is easy to check that this makes $C(B)$ into a DG category.

**Remark:** $Z^0(\text{Hom}^\cdot(M, N)) = \{ \text{usual (degree 0) morphisms of complexes } M \to N \}$

and $H^0(\text{Hom}^\cdot(M, N)) = \{ \text{homotopy classes of degree 0 morphisms of complexes } M \to N \}$.

**3.1.10** As usual, we have full DG subcategories $C^+(B), C^-(B), C^b(B) \subset C(B)$. 
§1.11. Remark. Translations and cones can be defined abstractly in any DG category (but they do not always exist):

\[ C = \text{any DG category} \]

1) \[ M \in C \Rightarrow M[a] \quad \text{(if it exists)} \]
   determined by \[ \text{Hom} (L, M[a]) = \text{Hom} (L, M)[a] \]

2) \[ f : M \to N \text{ a DG morphism in } C \]
   by definition, this means an element of \[ Z^0 \text{Hom} (M, N) \]
   \[ \Rightarrow \text{Cone}(f) \quad \text{(if it exists)} \]
   by \[ \text{Hom} (L, \text{Cone}(f)) = \text{Cone} \left( \text{Hom} (L, M) \to \text{Hom} (L, N) \right) \].

§1.12. Definition. A DG category \( C \) is said to be strongly pre-triangulated if all objects of \( C \) have shifts, and all DG morphisms in \( C \) have cones.

(Roughly speaking, just "pre-triangulated" means that the functors defining shifts and cones are representable at the level of the homotopy category.)
§1.13  Distinguished triangles.
Fix a strongly pre-triangulated DG category $C$. By taking all the cohomology of the Hom's in $C$, we obtain a graded additive category, which we will also denote by $\text{Ho}(C)$.

Alternatively, we can define $\text{Ho}(C)$ in the old way, and consider

\[
\text{Hom}^n_{\text{Ho}(C)}(M, N) = \text{Hom}_{\text{Ho}(C)}(M, N)
\]

**Def:** A triangle in $\text{Ho}(C)$ is a diagram of morphisms of degree 0:

\[
M \to N \to C \to M[1]
\]

Given such a triangle, we expand it to a long sequence (infinite on both ends) as follows:

\[
\cdots \to C[-1] \xrightarrow{-w[-1]} M \to N \xrightarrow{w} C \xrightarrow{w} M[1]
\]

\[
\xrightarrow{w} M[1] \xrightarrow{-w[1]} N[1] \xrightarrow{v[1]} C[1] \xrightarrow{-w[1]} M[2]
\]

\[
\]

**Remark:** When we shift a DG morphism of objects: $f \mapsto f[a]$, there is no sign change!
Def: A triangle in $\text{Ho}(\mathcal{C})$ is said to be distinguished if it is isomorphic to a triangle of the form

$$M \xrightarrow{u} N \xrightarrow{} \text{Cone}(u) \xrightarrow{} M[1]$$

the canonical embedding and projection

\[\square\ 0.14\] Definition. A \textbf{triangulated category} $\mathcal{D}$ is such that all is a graded additive category $\mathcal{D}$ such that all shifts in $\mathcal{D}$ exist. (So we have an actual shift functor $\mathcal{T}: \mathcal{D} \rightarrow \mathcal{D}$) together with a chosen collection of triangles in $\mathcal{D}$ (in the sense of the definition given above), called the \textbf{distinguished triangles}, satisfying the following list of axioms.

1. Every triangle isomorphic to a distinguished one is distinguished.
2. Every triangle of the form
   $$M \xrightarrow{\text{id}_M} M \xrightarrow{} 0 \xrightarrow{} M[1]$$
   is distinguished.
3. "\textbf{Rotation invariance}" of distinguished triangles: a triangle
   $$M \xrightarrow{u} N \xrightarrow{v} C \xrightarrow{w} M[1]$$
   is distinguished $\iff$ the triangle
   $$N \xrightarrow{v} C \xrightarrow{w} M[1] \xrightarrow{-u[1]} N[1]$$
   is distinguished.
(3) Every morphism \( M \xrightarrow{f} N \) in \( D \) can be included in a distinguished triangle
\[
M \xrightarrow{f} N \rightarrow C \rightarrow M[1]
\]
and every morphism between morphisms, i.e., a commutative diagram
\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M' & \xrightarrow{f'} & N'
\end{array}
\]
\[
\begin{array}{ccc}
& C & \rightarrow M[1] \\
& \swarrow & \\
C' & \rightarrow M'[1]
\end{array}
\]
these have already been chosen
\( C \rightarrow M[1] \)
\( C' \rightarrow M'[1] \)
can be included in a commutative diagram of distinguished triangles
\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M' & \xrightarrow{f'} & N'
\end{array}
\]
\[
\begin{array}{ccc}
& C & \rightarrow M[1] \\
& \swarrow & \\
C' & \rightarrow M'[1]
\end{array}
\]
(4) \underline{Octahedron Axiom}. (The last one; the most complicated and in some sense the most important one.)

Look up the definition in any textbook on derived categories.
Remark. In fact, one should view the octahedron axiom as the analogue for triangulated categories of the "Second Isomorphism Theorem": if $M \subseteq N \subseteq L$ are abelian groups (or objects in any abelian category), then

$$L/N \cong (L/M)/(N/M)$$

canonically.

§1.15. Theorem. Let $E$ be any strongly pre-triangulated DG category. Then the homotopy category $Ho(E)$, with the graded structure and the collection of distinguished triangles defined in §1.14, is a triangulated category.

Remark: The same theorem is true if $E$ is assumed to be merely pre-triangulated.

§1.16. Key Lemma. Let $E$ be a strongly pre-triangulated DG category.

(1) Consider a DG morphism $M \longrightarrow N$ which is a split embedding (but the splitting is not necessarily a DG morphism). Then we have $N/M \in E$, and the natural projection $\text{Cone}(\phi) \longrightarrow N/M$ is a homotopy equivalence.
(2) Any DG morphism in $\mathcal{C}$ is homotopic to a split embedding as above. That is, if $M \xrightarrow{f} L$ is any DG morphism in $\mathcal{C}$, there is a (naturally constructed) commutative diagram of DG morphisms

$$
\begin{array}{ccc}
M & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{g} \\
N & \xrightarrow{?} & L
\end{array}
$$

where $u$ is a split embedding and $g$ is a homotopy equivalence (i.e., becomes an isomorphism in $\text{Ho}(\mathcal{C})$).

(Hint for (2): take $N = L \oplus \text{Cone}(\text{id}_M)$.)

§1.17. Now we sketch a proof of Theorem 1.15.

It is trivial to verify properties (3) and (1) in the definition of a triangulated category for the category $\text{Ho}(\mathcal{C})$.

What about rotation invariance of distinguished triangles? Use the lemma above to reduce to triangles of the form

$$
M \xrightarrow{u} N \xrightarrow{} \text{Cone}(u) \xrightarrow{} M[1],
$$

where $u$ is a split embedding, and prove property (2) in this case.
For property (3), the first part follows by construction. For the second part, given a commutative diagram in $\text{Ho}(C)$,

\[
\begin{array}{cccc}
M & \xrightarrow{f} & N & \xrightarrow{\text{Cone}(f)} & M[1] \\
\downarrow{g} & & \downarrow{h} & & \\
M' & \xrightarrow{f'} & N' & \xrightarrow{\text{Cone}(f')} & N'[1]
\end{array}
\]

Lift $f, f', g, h$ to honest DG morphisms in $C$. The resulting square in $E$ will fail to commute by a coboundary. The resulting square in $\text{Ho}^0(C, N')$. If we write this coboundary as the differential of some DG element of $\text{Hom}_E(M, N')$, we get a DG morphism $\text{Cone}(f) \to \text{Cone}(f')$ which completes the diagram above.

For the octahedron axiom, start with any commutative triangle

\[
\begin{array}{ccc}
M & \xrightarrow{} & N \\
& \searrow & \nearrow \\
& L &
\end{array}
\]

and use the key lemma twice to replace it with a commutative triangle

\[
\begin{array}{ccc}
M & \xrightarrow{} & N' \\
& \searrow & \nearrow \\
& L'' &
\end{array}
\]

where all the maps are split embeddings. Then use the second isomorphism theorem (for complexes of abelian groups).