

# LECTURES ON CLASS FIELD THEORY

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§8.1. Let  $K$  be a local field and  $\pi \in K$  a chosen uniformizer. Given  $\pi$ , we can construct a formal  $\mathcal{O}_K$ -module  $E$ . If we want to explicitly equip  $E$  with a formal coordinate, we need to choose  $\xi \in LT_\pi$ .

Consider  $E_{\text{tors}} \subset M_{\bar{K}}$ , the group of torsion points of  $E(\bar{K})$ . As explained last time, we have  $E_{\text{tors}} \cong K/\mathcal{O}_K$  as  $\mathcal{O}_K$ -modules, and by looking at the Galois action, we obtain a character

$$\begin{array}{ccc} \chi : \text{Gal}(\bar{K}/K) & \longrightarrow & \mathcal{O}_K^\times \\ \downarrow & \nearrow & \\ \text{Gal}(K^{(\pi)}/K) & \xrightarrow{\cong} & \end{array}$$

(this defines an abelian extension  $K^{(\pi)}$  of  $K$ )

Remark.  $K^{(\pi)}$  is totally ramified over  $K$ .

Later we will see that  $K^{(\pi)}K^{\text{nr}}$  is a maximal abelian extension of  $K$ .

§8.2. At this point we would like to investigate how  $K^{(\pi)}$  depends on the choice of  $\pi$ .

Recall: If  $L$  is any finite extension of  $K$  contained in  $K^{(\pi)}$ , then  $\pi \in N_{L/K}(L)$ .

We will see that the Lubin-Tate groups for different choices of  $\pi$  become isomorphic over  $\hat{\mathcal{O}}_{K^{nr}}$ . Let  $\pi, \pi' \in \mathcal{O}_K$  be two uniformizers, and write  $\pi' = u\pi$ ,  $u \in \mathcal{O}_K^\times$ .

Let  $f_\pi(t)$  be a L-T  $\pi$ -formal power series.

For any  $a \in \mathcal{O}_K$ , we get  $[a]_{f_\pi} \in \mathcal{O}_K[[t]]$ , the formal power series representing the action of  $a$  on  $E_{f_\pi}^{(f_\pi)}$ .

In particular, we can look at

$$[u]_{f_\pi} = ut + \dots$$

Goal: We would like to find

$$\varphi(t) \in \hat{\mathcal{O}}_{K^{nr}}[[t]]$$

with the following properties: if

$$\sigma \in \text{Gal}(K^{nr}/K)$$

is the Frobenius automorphism, then

$$\sigma\varphi = \varphi \circ [u]_{f_\pi}$$

Lemma. (1) Such a  $\varphi$  exists.

$$(2) \forall a \in \mathcal{O}_K, \varphi \circ [a]_{\mathfrak{f}_\pi} \circ \varphi^{-1} \in \mathcal{O}_K[[t]]$$

$$(3) \varphi \circ [\pi u]_{\mathfrak{f}_\pi} \circ \varphi^{-1} \in L^T \pi u$$

Proof. We can construct  $\varphi$  term-by-term. Then we see that

$$\text{Write } \varphi(t) = \varepsilon t + \dots$$

we must have  $\sigma_\varepsilon = \varepsilon \cdot u$ . As  $u$  is a unit,

we can find an  $\varepsilon \in \hat{\mathcal{O}}_{K, \text{nr}}$  satisfying this identity by approximating it modulo higher and higher powers of  $\mathfrak{m}_{K, \text{nr}}$ . Similarly, we can determine the higher order coefficients of  $\varphi$ . This proves (1).

[Remark. An alternate proof of (1) can be obtained by reducing it to the question of surjectivity of the Lang map for the group of formal power series, viewed as a group scheme over the residue field of  $\mathcal{O}_K$ .]

Part (2) follows immediately from the equation that defines  $\varphi$ .

For part (3), the only non-obvious thing to check is that  $\varphi \circ [\pi u]_{\mathfrak{f}_\pi} \circ \varphi^{-1} \equiv t^q \pmod{\mathfrak{m}_K}$  where  $q = |\mathcal{O}_K/\mathfrak{m}_K|$ . Now

$$\begin{aligned} \varphi \circ [\pi u]_{\mathfrak{f}_\pi} \circ \varphi^{-1} &\equiv \varphi \circ [u]_{\mathfrak{f}_\pi} \circ \text{Fr}_q \circ \varphi^{-1} \\ &\equiv \varphi \circ [u]_{\mathfrak{f}_\pi} \circ (\sigma_\varphi)^{-1} \circ \text{Fr}_q \equiv \text{Fr}_q. \end{aligned}$$

**§8.3. Corollary.** The formal power series  $\varphi \in \hat{\mathcal{O}}_{K^{nr}}[[t]]$  constructed in §8.2 gives an isomorphism of formal  $\mathcal{O}_K$ -modules, defined over  $\hat{\mathcal{O}}_{K^{nr}}$ ,

$$E(\mathbb{F}_\pi) \xrightarrow{\cong} E \quad (\varphi \circ [\pi u]_{\mathbb{F}_\pi} \circ \varphi^{-1})$$

(This easily follows from Lemma 8.2(3).)

**§8.4. Corollary.** The compositum  $K^{nr} K^{(\pi)}$  does not depend on the choice of the uniformizer  $\pi \in K$ .

(This is left as an exercise using Cor. 8.3.)

**§8.5. Exercise** (Sasha is 90% sure that the statement is correct). Suppose we have a formal (1-dimensional)  $\mathcal{O}_K$ -module  $E$  over  $\hat{\mathcal{O}}_{K^{nr}}$ . Let us call it a Lubin-Tate module if for any uniformizer  $\pi \in \mathfrak{m}_K$ , we have

$$[\pi] \left( \underbrace{R/\mathfrak{m}_K R}_{SII} \right) = (R/\mathfrak{m}_K R)^q$$

$$\overline{\mathbb{F}}[[t]]$$

where  $R = \hat{\mathcal{O}}_{K^{nr}}[[t]]$  (so  $E \cong \text{Spf}(R)$  as a formal scheme).

Statement of the exercise;

- (a) Any two Lubin-Tate  $\mathcal{O}_K$ -modules over  $\hat{\mathcal{O}}_{K^{nr}}$  (defined as above) are isomorphic.
- (b) Pick a uniformizer  $\pi \in \mathcal{O}_K$ . Then  $E$  can be uniquely descended to  $\mathcal{O}_K$  in such a way that the result is a Lubin-Tate group with respect to  $\pi$  (defined as in the previous lecture).

**§8.6.** Let us write  $L = K^{nr} \cdot K^{(\pi)}$  where the notation is as in Corollary 8.4. We would like to show that  $L$  is a maximal abelian extension of  $K$  and describe the corresponding action of  $K^\times$  on  $L$  in Lubin-Tate terms.

First of all, we have

$$\text{Gal}(L/K) \cong \underbrace{\text{Gal}(K^{nr}/K)}_{\hat{\mathbb{Z}}} \times \underbrace{\text{Gal}(K^{(\pi)}/K)}_{\cong \mathcal{O}_K}$$

Also,  $K^\times = \pi \mathbb{Z} \times \mathcal{O}_K^\times$

We define a homomorphism

$$r_\pi : K^\times \longrightarrow \text{Gal}(L/K)$$

to be the obvious inclusion  $\pi \mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$  on

the first factor and the map

$$\begin{array}{ccc} \mathcal{O}_K^\times & \longrightarrow & \mathcal{O}_K^\times \\ a & \longmapsto & a^{-1} \end{array}$$

on the second factor.

§ 8.7. Theorem. (a)  $L = K^{ab}$

(b)  $r_\pi$  does not depend on  $\pi$  and is equal to the canonical isomorphism of local CFT

Proof. The key thing is to check that the homomorphism  $r_\pi$  is independent of the choice of  $\pi$ . Suppose  $\pi, \pi' \in K$  are two uniformizers. We have

$$K^{(\pi')} \hookrightarrow L = K^{nr} \cdot K^{(\pi)}$$

by Corollary 8.4. We would like to understand how the element of  $\text{Gal}(L/K)$ , which acts by the Frobenius  $\sigma$  on  $K^{nr}$  and by the identity on  $K^{(\pi)}$ , acts on  $K^{(\pi')}$ . Let us write  $\pi' = \pi u$ ,  $u \in \mathcal{O}_K^\times$ .

Let  $\varphi \in \hat{\mathcal{O}}_{K^{nr}}[[t]]$  be the element constructed in § 8.2. Then  $\varphi$ , in particular, yields an isomorphism

$$E_{\text{tors}}^{(\mathbb{F}_\pi)} \xrightarrow{\sim} E_{\text{tors}}^{(g\pi u)}$$

where  $g_{\pi u} = \varphi \circ [\pi u]_{\mathbb{F}_\pi} \circ \varphi^{-1}$ .

Let us choose  $\alpha \in E_{\text{tors}}^{(\mathbb{F}_\pi)}$  and compute

$$(\sigma, \text{id})(\varphi(\alpha)) = ?$$

Recall that  $\sigma_\varphi = \varphi \circ [u]_{\mathbb{F}_\pi}$ .

Thus

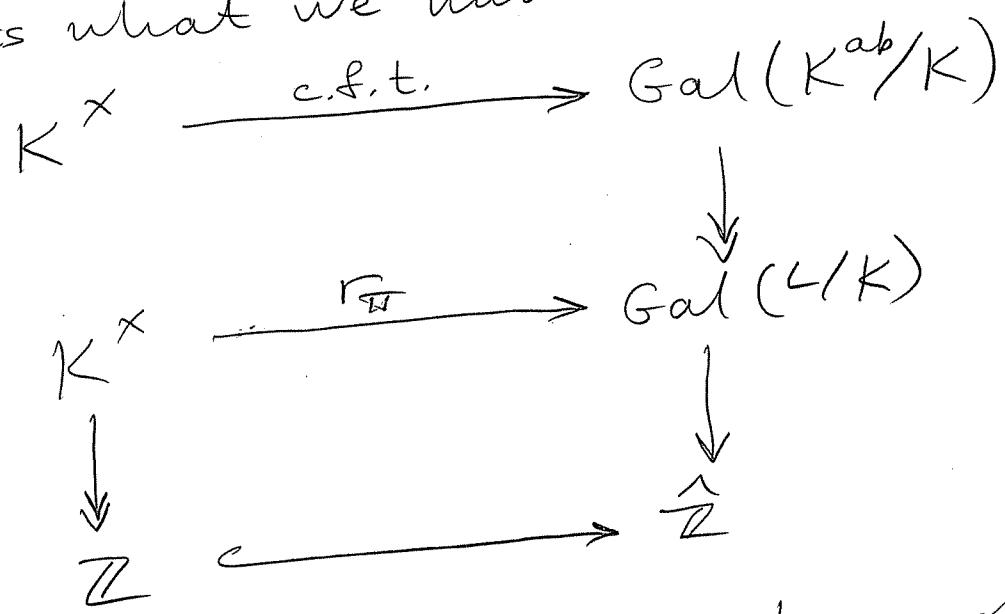
$$\begin{aligned}
 (\sigma, \text{id})(\varphi(\alpha)) &= \varphi([u]_{\mathbb{F}_\pi}(\alpha)) \\
 &= [u]_{\mathbb{G}_{\pi u}}(\varphi(\alpha))
 \end{aligned}$$

Hence the action of the Frobenius on  $K(\pi u)$  coincides with the action of  $[u]_{\mathbb{G}_{\pi u}}$ .

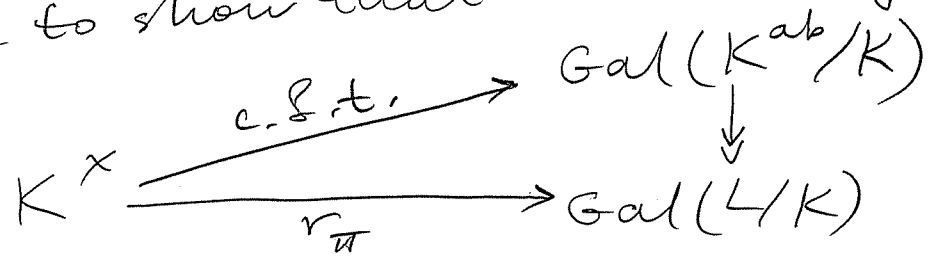
Check: This is precisely the statement that  $r_\pi$  is independent of  $\pi$ .

**§ 8.8.**

We claim that the rest of the theorem follows easily from the last claim. Here is what we have so far:



We need to show that the triangle



commutes (this will imply Theorem 8.7).

Now in this triangle, both compositions are equal on  $\pi$ , because  $\pi$  is in the image of the norm map for any finite extension of  $K$  contained in  $K^{(\pi)}$ .

Because of the previous result, we see that the two maps  $K^\times \longrightarrow \text{Gal}(L/K)$  coincide on the set of all uniformizers in  $K$ , and hence on the subgroup of  $K^\times$  generated by them. But this subgroup is dense. All the other details are left as an easy exercise.

### Introduction to geometric local CFT

**§8.9.** In this story we need to restrict our attention to local function fields. The results are formulated in terms of geometry over the residue fields. We would like to explain the de Rham version, i.e., for  $\mathbb{C}((t))$ .

Recall that in this version of local CFT we are interested in connections on the formal punctured disc. The nontrivial side of the story comes from connections with irregular singularities.

Since we are working with CFT, our "Galois modules" should be 1-dimensional, i.e., we will work with connections of line bundles.

Roughly speaking, we will identify the Picard groupoid of such connections with the Picard groupoid of extensions (in the de Rham sense) of  $K^\times$  by  $G_m, k$

§8.10. More precise description:

$X = \text{Spec } k((t))$ , where  $k$  is a field of characteristic 0

Galois side:

{line bundles with a connection on  $X$ }

the strict Picard groupoid defined by the complex

$$K^X \xrightarrow{d \log} \omega_X$$

(where  $K = k((t))$  and  $\omega_X = k((t))dt$ )

Automorphic side:

Let us view  $K^X$  as a group ind-scheme over  $k$ , and let  $G_m$  be the multiplicative group over  $k$ .

We have the strict Picard groupoid  $\text{Ext}_{dR}(K^X, G_m) = \left\{ \begin{array}{l} \text{extensions of } K^X \text{ by } G_m \\ \text{in the de Rham sense} \end{array} \right\}$

What does this mean?

§8.11. Extensions in the de Rham sense.

Given any (commutative) extension of commutative group schemes, we say that it is an extension in the de Rham sense if it is equipped with an infinitesimal trivialization, i.e., a trivialization of the corresponding extension of formal groups. (In fact, it is enough to trivialize the corresponding extension of Lie algebras.)

Another viewpoint:  $\text{Ext}_{\mathbb{A}^1 R}^t(K^X, \mathbb{G}_m)$  is the Picard groupoid of multiplicative line bundles with flat connections on  $K^X$  (i.e., 1-dimensional character sheaves on  $K^X$ ).

**§8.12. Theorem.** The strict Picard groupoids  $[K^X \xrightarrow{d \log} \omega_X]$  and  $\text{Ext}_{\mathbb{A}^1 R}^t(K^X, \mathbb{G}_m)$  are naturally equivalent.

**§8.13. A categorical reformulation.**

The complex  $[K^X \xrightarrow{d \log} \omega_X]$  really defines a strict Picard stack rather than merely a groupoid, because both  $K^X$  and  $\omega_X$  can be viewed as group ind-schemes over  $k$ .

Instead of looking at the groupoid of "sections" of this stack over  $k$ , we can look at all  $\mathcal{O}$ -modules on this stack.

On the automorphic side, we replace the groupoid  $\text{Ext}_{\mathbb{A}^1 R}^t(K^X, \mathbb{G}_m)$  by the category of all  $\mathcal{D}$ -modules on the ind-scheme  $K^X$ .

It turns out that these two categories are equivalent, and one can deduce Theorem 8.12 from (the proof of) this result.

§8.14.

Observation. There is a

natural identification  $\omega_X \cong \text{Lie}(K^X)^*$ ,

where  $*$  denotes passage to the dual vector space, in the topological sense.

Proof.

$\text{Lie}(K^X) = k((t))$ , and we have the

nondegenerate Tate pairing

$$\text{Res} : \omega_X \times k((t)) \longrightarrow k$$

$$(\nu, f) \longmapsto \text{Res}_0(f \cdot \nu).$$

§8.15.

Next step: we claim that

$$\text{Hom}_{\text{group ind-schemes}}(K^X, G_m) \cong K^X$$

denote this by  $\text{Char}(K^X)$

Such an identification was found by Contou-Carrère (it is not at all obvious!) one of the last students of Grothendieck

This isomorphism is one of the first fundamental results of local geometric CFT, and it holds for local function fields in any characteristic.

§8.16.

The isomorphism in §8.15 comes from the Contou-Carrère symbol

$$K^X \times K^X \longrightarrow G_m$$

It will be defined below.

§8.17. Let us first describe  $K^\times$  more explicitly as a group ind-scheme over  $k$ . For every  $k$ -algebra  $R$ , we have

$$K^\times(R) = R((t))^\times.$$

Thus we see that  $K^\times$  has a huge formal subgroup.

Exercise.  $f \in R((t))$  is invertible  $\iff$   $f$  has a non-nilpotent coefficient, and the lowest degree non-nilpotent coefficient of  $f$  is a unit in  $R$ .

(Typical example:  $1 + \epsilon t^{-1}$  is invertible in  $R((t))$  if and only if  $\epsilon$  is nilpotent.)

Moral:  $K^\times$  contains a countable direct sum of copies of  $\hat{G}_a$  (the formal completion of  $G_a$ ).

Also,  $K^\times$  visibly contains a countable direct product of copies of  $G_a$ . Since  $\text{char}(G_a) = \hat{G}_a$ , we see that  $K^\times$  has at least a chance of being self-dual.

References for the Cantou-Carrère pairing:

- A note by C-C in Comptes Rend. on the 1980s
- A short explanation in Deligne's article on the tame symbol
- Beilinson-Bloch-Esnault (something about  $\epsilon$ -factors)

§8.18. The self-duality of  $K^\times$  implies the result stated in §8.13, because of a general principle discovered by G. Laumon.

Namely, if  $G$  is any commutative group ind-scheme over a field  $k$  of char. 0, then  $\mathcal{D}$ -modules on  $G$  are the same as  $\mathcal{O}$ -modules on the quotient stack  $G/\hat{G}$ , where  $\hat{G}$  is the formal completion of  $G$  at  $1 \in G$ , acting on  $G$  by translations. Also, since we are in char. 0, the exponential map identifies  $\hat{\mathfrak{g}} \xrightarrow{\cong} \hat{G}$ , where  $\mathfrak{g} = \text{Lie}(G)$ .

Thus we get  $\mathcal{D}_{K^\times\text{-mod}} \cong \left\{ \begin{array}{l} \mathcal{O}\text{-modules} \\ \text{on } K^\times/\hat{K} \end{array} \right\}$ ,

where we are using  $\exp: \hat{K} \longrightarrow K^\times$ .

The identification  $\omega_X \cong K^*$  (dual vector space) yields an identification of the Cartier dual of  $\omega_X$  with the formal completion  $\hat{K}$ .

In fact, one can check that the Tate pairing and the Cartier-Carrère symbol are compatible in the sense that they identify the strict Picard stack  $[\hat{K} \xrightarrow{\exp} K^\times]$  with the Cartier dual of  $[K^\times \xrightarrow{d\log} \omega_X]$ .

As proved by Laumon, this induces an equivalence between the categories of  $\mathcal{O}$ -modules on these two stacks. [I think that the reference is a paper by Laumon called "Generalized Fourier transform" (in French), available on the arXiv.]